A boolean algebraic approach to semiproper iterations II

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Generalized stationarity

**Definition**

Let $X$ be an uncountable set. A set $C$ is a *club* on $\mathcal{P}(X)$ iff there is a function $f_C : X^{<\omega} \to X$ such that $C$ is the set of elements of $\mathcal{P}(X)$ closed under $f_C$, i.e.

$$C = \{ Y \in \mathcal{P}(X) : f_C[Y]^{<\omega} \subseteq Y \}$$

A set $S$ is *stationary* on $\mathcal{P}(X)$ iff it intersects every club on $\mathcal{P}(X)$.

**Example**

The set $\{X\}$ is always stationary since every club contains $X$. Also $\mathcal{P}(X) \setminus \{X\}$ and $[X]^\kappa$ are stationary for any $\kappa \leq |X|$ (following the proof of the well-known downwards Löwenheim-Skolem Theorem). Notice that every element of a club $C$ must contain $f_C(\emptyset)$, a fixed element of $X$. 

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Remark

The reference to the support set $X$ for clubs or stationary sets may be omitted, since every set $S$ can be club or stationary only on $\bigcup S$.

Given any first-order structure $M$, from the set $M$ we can define a Skolem function $f_M : M^{<\omega} \to M$ (i.e., a function coding solutions for all existential first-order formulas over $M$). Then the set $C$ of all elementary submodels of $M$ contains a club (the one corresponding to $f_M$). Henceforth, every set $S$ stationary on $X$ must contain an elementary submodel of any first-order structure on $X$. 
Definition

The *club filter* on $X$ is

$$\text{CF}_X = \{ C \subset \mathcal{P}(X) : C \text{ contains a club} \}.$$  

Similarly, the *non-stationary ideal* on $X$ is

$$\text{NS}_X = \{ A \subset \mathcal{P}(X) : A \text{ not stationary} \}.$$  

Lemma

CF$_X$ is a $\sigma$-complete filter on $\mathcal{P}(X)$, and the stationary sets are exactly the CF$_X$-positive sets.
Definition

Given a family \( \{ S_a \subseteq \mathcal{P}(X) : a \in X \} \), the diagonal union of the family is \( \nabla_{a \in X} S_a = \{ z \in \mathcal{P}(X) : \exists a \in z \ z \in S_a \} \), and the diagonal intersection of the family is \( \Delta_{a \in X} S_a = \{ z \in \mathcal{P}(X) : \forall a \in z \ z \in S_a \} \).

Lemma (Fodor)

\( \text{CF}_X \) is normal, i.e. is closed under diagonal intersection. Equivalently, every function \( f : \mathcal{P}(X) \to X \) that is regressive on a \( \text{CF}_X \)-positive set is constant on a \( \text{CF}_X \)-positive set.
From now on we shall be interested just in stationary subsets of $[X]^{<\aleph_0}$ for suitable uncountable sets $X$. 
Definition

Let $\mathbb{B}$ be a complete boolean algebra, $M < H_\theta$ be countable with $\theta \gg |\mathbb{B}|$, and $\text{PD}(\mathbb{B})$ be the collection of predense subsets of $\mathbb{B}$ of size at most $\omega_1$. The boolean value

$$sg(\mathbb{B}, M) = \bigwedge \left\{ \bigvee (D \cap M) : D \in \text{PD}(\mathbb{B}) \cap M \right\}$$

is the degree of semigenericity of $M$ with respect to $\mathbb{B}$. 
Proposition

Let $B$, $M$, $\text{PD}(B)$ be as in the previous definition, and let $A(B)$ be the collection of maximal antichains of $B$ of size at most $\omega_1$. Then

$$\text{sg}(B, M) = \bigwedge \{ \bigvee (A \cap M) : A \in A(B) \cap M \}$$

Proposition

Let $B$ be a complete boolean algebra and $M < H_\theta$ for some $\theta \gg |B|$. Then for all $b \in M \cap B$

$$\text{sg}(B \upharpoonright b, M) = \text{sg}(B, M) \wedge b.$$
**Definition**

Let $\mathcal{B}$ be a complete boolean algebra. $\mathcal{B}$ is semiproper (SP) iff for club many $M < H_\theta$ in $[H_\theta]^{\aleph_0}$ whenever $b$ is in $\mathcal{B}^+ \cap M$, we have that $sg(\mathcal{B}, M) \land b > 0_\mathcal{B}$.

Similarly, $i : \mathcal{B} \to \mathcal{C}$ is semiproper (SP) iff $\mathcal{B} \in SP$ and for club many $M \in [H_\theta]^{\aleph_0}$, whenever $c$ is in $\mathcal{C}^+ \cap M$ we have that

$$\pi(c \land sg(\mathcal{C}, M)) = \pi(c) \land sg(\mathcal{B}, M).$$
Shelah’s semiproperness

Definition

(Shelah) Let $P$ be a partial order, and fix $M < H_\theta$. Then $q$ is a $M$-semigeneric condition for $P$ iff for every $\dot{\alpha} \in V^P \cap M$ such that $1_P \Vdash \dot{\alpha} < \check{\omega}_1$,

$$q \Vdash \dot{\alpha} < M \cap \omega_1.$$ 

$P$ is semiprproper in the sense of Shelah if there exists a club $C$ of elementary substructures of $H_\theta$ such that for every countable $M \in C$, there exist a $M$-semigeneric condition below every element of $P \cap M$. 
Proposition

Let $\mathcal{B}$ be a complete boolean algebra, and fix $M < H_\theta$. Then

$$sg(\mathcal{B}, M) = \bigvee \{ q \in \mathcal{B} : q \text{ is a } M\text{-semigeneric condition} \}$$

Proposition

$P$ is semiproper in the sense of Shelah iff $RO(P)$ is semiproper.
Proposition

Let $B$ be a semiproper complete boolean algebra, and let $\dot{C}$ be such that

$$[(\dot{C} \in SP) = 1_B],$$

then $D = B \ast \dot{C}$ and $i_{B \ast \dot{C}}$ are both semiproper.
Lemma

Let $\mathcal{B}, \mathcal{C}_0, \mathcal{C}_1$ be semiproper complete boolean algebras, and let $G$ be any $V$-generic filter for $\mathcal{B}$. Let $i_0, i_1, j$ form a commutative diagram of regular embeddings as in the following picture:

Moreover assume that $\mathcal{C}_0/i_0[G]$ is semiproper in $V[G]$ and

$$\left[ \mathcal{C}_1/j[G_{C_0}] \text{ is semiproper} \right]_{C_0} = 1_{C_0}.$$

Then in $V[G]$, $j/G : \mathcal{C}_0/G \to \mathcal{C}_1/G$ is a semiproper embedding.
Corollary

Assume \( F = \{ i_{\alpha \beta} : \alpha \leq \beta < \lambda \} \) is such that

\[
[B_{\alpha+1}/i[G_{B_\alpha}] \text{ is semiproper}]_{B_\alpha} = 1_{B_\alpha}
\]

for all \( \alpha < \lambda \).

Let \( G_\alpha \) be \( V \)-generic for \( B_\alpha \). Then

\[
F/G_\alpha = \{ i_{\eta \beta}/G_\alpha : \alpha \leq \eta \leq \beta < \lambda \}
\]

is in \( V[G_\alpha] \) an iteration system made of semiproper embeddings.
Definition

An iteration system $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ is semiproper iff $i_{\alpha\beta}$ is semiproper for all $\alpha \leq \beta < \lambda$.

An iteration system $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ is RCS iff $B_\alpha = \text{RO}(\text{RCS}(\mathcal{F} \upharpoonright \alpha))$ for all $\alpha < \lambda$. 

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Lemma

Let $\mathcal{F} = \{i_{nm} : n \leq m < \omega\}$ be a semiproper iteration system. Then $T(\mathcal{F})$ and the corresponding $i_{n\omega}$ are also semiproper.
Lemma

Let \( F = \{ i_{\alpha \beta} : B_\alpha \rightarrow B_\beta : \alpha \leq \beta < \omega_1 \} \) be an RCS and semiproper iteration system. Then \( C(F) \) and the corresponding \( i_{\alpha \omega_1} \) are semiproper.
Lemma

Let $\mathcal{F} = \left\{ i_{\alpha\beta} : B_\alpha \to B_\beta : \alpha \leq \beta < \lambda \right\}$ be an RCS and semiproper iteration system such that $C(\mathcal{F})$ is $< \lambda$-cc. Then $C(\mathcal{F})$ and the corresponding $i_{\alpha\lambda}$ are semiproper.
Theorem

Let $\mathcal{F} = \{ i_{\alpha\beta} : \mathcal{B}_\alpha \to \mathcal{B}_\beta : \alpha \leq \beta < \lambda \}$ be an RCS iteration system, such that for all $\alpha < \beta < \lambda$,

$$\left[ \mathcal{B}_\beta / i_{\alpha\beta}[\dot{G}_\alpha] \text{ is semipr} \right] = 1_{\mathcal{B}_\alpha}$$

and for all $\alpha$ there is a $\beta > \alpha$ such that $\mathcal{B}_\beta \models |\mathcal{B}_\alpha| \leq \omega_1$. Then $\text{RCS}(\mathcal{F})$ and the corresponding $i_{\alpha\lambda}$ are semipr.
Fact

Let $\mathcal{F} = \{i_{\alpha\beta} : \alpha \leq \beta < \lambda\}$ be a semiproper iteration system, $f$ be in $T(\mathcal{F})$. Then

$$\mathcal{F} \upharpoonright f = \{(i_{\alpha\beta})_f(\beta) : B_\alpha \upharpoonright f(\alpha) \to B_\beta \upharpoonright f(\beta) : \alpha \leq \beta < \lambda\}$$

is a semiproper iteration system and its associated retractions are the restriction of the original retractions.
Lemma

Let $\mathcal{F} = \left\{ i_{\alpha\beta} : B_\alpha \to B_\beta : \alpha \leq \beta < \lambda \right\}$ be an RCS and semiproper iteration system. Let $M \prec H_\theta$ be countable, $g \in M$ be any condition in $\text{RCS}(\mathcal{F})$, $\dot{\alpha} \in M$ be a name for a countable ordinal, $\delta \in M$ be an ordinal smaller than $\lambda$.

Then there exists a condition $g' \in \text{RCS}(\mathcal{F}) \cap M$ below $g$ with $g'(\delta) = g(\delta)$ and $g' \wedge i_\delta(\text{sg}(B_\delta, M))$ forces that $\dot{\alpha} < M \cap \omega_1$. If $\lambda = \omega_1$, then the support of $g' \wedge i_\delta(\text{sg}(B_\delta, M))$ is contained in $M \cap \omega_1$. 