Template iterations and maximal cofinitary groups

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\begin{itemize}
\item (Shelah) $\text{con}(\delta < \alpha)$
\item (Brendle) $\text{con}(\text{cof}(\alpha) = \omega)$
\end{itemize}
Theorem (V.F., A. Törnquist)

Assume CH. Let $\lambda$ be a singular cardinal of countable cofinality. Then there is a ccc generic extension in which $\alpha_g = \lambda$. 
cofin($S_\infty$) is the set of cofinitary permutations in $S_\infty$, i.e. permutations $\sigma \in S_\infty$ which have finitely many fixed points.

A mapping $\rho : A \to S_\infty$ induces a cofinitary representation of $\mathbb{F}_A$ if the canonical extension of $\rho$ to a homomorphism $\hat{\rho} : \mathbb{F}_A \to S_\infty$ is such that $\text{im}(\hat{\rho}) \subseteq \{1\} \cup \text{cofin}(S_\infty)$. 
Evaluations

Let $A$ be a set, $s \subseteq A \times \omega \times \omega$. For $a \in A$, let $s_a = \{(n, m) \in \omega \times \omega : (a, n, m) \in s\}$. For a word $w \in W_A$, define $e_w[s] \subseteq \omega \times \omega$ recursively as follows:

- if $w = a$ then $(n, m) \in e_w[s]$ iff $(n, m) \in s_a$,
- if $w = a^{-1}$ then $(n, m) \in e_w[s]$ iff $(m, n) \in s_a$, and
- if $w = a^i u$ for some $u \in W_A$ and $i \in \{1, -1\}$ without cancelation then

$$(n, m) \in e_w[s] \iff (\exists k) e_{a^i}[s](k, m) \land e_u[s](n, k).$$
If $s_a$ is a partial injection for all $a$, then $e_w[s]$ is a partial injection.

We refer to $e_w[s]$ as the evaluation of $w$ given $s$.

By definition we let $e_{\emptyset}[s, \rho]$ be the identity in $S_\infty$. 
Let $A$, $X$ be disjoint and let $\rho : X \to S_\infty$ be a function. For a word $w \in W_{A \cup X}$ and $s \subseteq A \times \omega \times \omega$, define

$$(n, m) \in e_w[s, \rho] \iff (n, m) \in e_w[s \cup \{(x, k, l) : \rho(x)(k) = l\}].$$

If $s_a$ is a partial injection for $a$, then $e_w[s, \rho]$ is also a partial injection, referred to as the evaluation of $w$ given $s$ and $\rho$. 
Forcing M.c.g.'s

Let $A, X$ be disjoint non-empty sets and let $\rho : X \to S_\infty$ induce a cofinitary representation. Then $\mathbb{Q}_{A,\rho}$ is the poset of all $(s, F)$ where $s \subseteq A \times \omega \times \omega$ is finite, $s_a$ is a finite injection for all $a$ and $F \subseteq \hat{W}_{A \cup X}$ is finite. Define $(s, F) \leq_{\mathbb{Q}_{A,\rho}} (t, E)$ iff

- $s \supseteq t$, $F \supseteq E$ and,
- for all $n \in \omega$ and $w \in E$, if $e_w[s, \rho](n) = n$ then already $e_w[t, \rho](n) \downarrow$ and $e_w[t, \rho](n) = n$.

If $X = \emptyset$ then we write $\mathbb{Q}_A$ for $\mathbb{Q}_{A,\rho}$.

**Lemma**

The forcing notion $\mathbb{Q}_{A,\rho}$ is Knaster.
Lemma

Let $A$ and $X$ be disjoint set and $\rho : X \to S_\infty$ a function inducing a cofinitary representation of $F_X$. Then

- ("Domain extension") For any $(s, F) \in Q_{A,\rho}$, $a \in A$ and $n \in \omega$ such that $n \notin \text{dom}(s_a)$ there are cofinitely many $m \in \omega$ s.t. $(s \cup \{(a, n, m)\}, F) \leq (s, F)$.

- ("Range extension") For any $(s, F) \in Q_{A,\rho}$, $a \in A$ and $m \in \omega$ such that $m \notin \text{ran}(s_a)$ there are cofinitely many $n \in \omega$ s.t. $(s \cup \{(a, n, m)\}, F) \leq (s, F)$. 
Proposition

Let $G$ be $\mathcal{Q}_{A,\rho}$-generic. Then $\rho_G : A \cup X \to S_\infty$, where $\rho_G|X = \rho$ and for all $a \in A$

$$\rho_G(a) = \bigcup \{s_a : (\exists F \in \hat{W}_{A \cup X}) (s, F) \in G\},$$

induces a cofinitary representation $\hat{\rho}_G : \mathbb{F}_{A \cup X} \to S_\infty$ extending $\hat{\rho}$.

Proof:

Let $a \in A$, $n \in \omega$. Then $D_{a,n} = \{(s, F) \in \mathcal{Q}_{A,\rho} : (\exists m)(a, n, m) \in s\}$ and $R_{a,n} = \{(s, F) \in \mathcal{Q}_{A,\rho} : (\exists m)(a, m, n) \in s\}$ are dense, and so $\rho_G : A \cup X \to S_\infty$ is indeed a function.
It remains to see that $\rho_G$ induces a cofinitary representation. Let $w \in W_{A\cup X}$. There are $w' \in \widehat{W}_{A\cup X}$, $u \in W_{A\cup X}$ such that $w = u^{-1}w'u$. Since $D_{w'} = \{(s, F) \in Q_{A, \rho} : w' \in F\}$ is dense

$\exists (s, F) \in G$ such that $w' \in F$. Suppose then $e_{w'}[\rho_G](n) = n$. Then there is some $(t, E) \leq Q_{A, \rho} (s, F)$ in $G$ forcing this. But then $e_{w'}[t, \rho](n) = n$ and so by definition $e_{w'}[s, \rho](n) = n$. Thus

$$\text{fix}(e_{w'}[\rho_G]) = \text{fix}(e_{w'}[s, \rho]),$$

which is finite. Finally, $\text{fix}(e_w[\rho_G]) = e_u[\rho_G]^{-1}(\text{fix}(e_{w'}[\rho_G]))$, so $\text{fix}(e_w[\rho_G])$ is finite.
Notation:
For \( s \subseteq A \times \omega \times \omega \) and \( A_0 \subseteq A \), write \( s \upharpoonright A_0 \) for \( s \cap A_0 \times \omega \times \omega \). For a condition \( p = (s, F) \in \mathcal{Q}_{A, \rho} \) we will write \( p \upharpoonright A_0 \) for \( (s \upharpoonright A_0, F) \), and \( p \upharpoonright\upharpoonright A_0 \) ("strong restriction") for \( (s \upharpoonright A_0, F \cap \widehat{\mathcal{W}}_{A_0 \cup B}) \).
Lemma: Strong Embedding

Let $B, C \subseteq D$, $B \cap C = A$ be given set and $p \in \mathcal{Q}_{B,\rho}$. Then there is a condition $p_0 \in \mathcal{Q}_{A,\rho}$ such that $p_0 \leq p \upharpoonright A$ and whenever $q_0 \leq \mathcal{Q}_{C,\rho} p_0$, then $q_0$ is compatible in $\mathcal{Q}_{D,\rho}$ with $p$.

We say that $\mathcal{Q}_{B,\rho}$ has the strong embedding property and $q_0$ is called a strong reduction of $p$. If $C = A$, $B = D$ then the above gives in particular that $\mathcal{Q}_{A,\rho}$ is a complete suborder of $\mathcal{Q}_{B,\rho}$.
Lemma: Quotients

Let $A_0 \cap A_1 = \emptyset$, $A = A_0 \cup A_1$. Let $G$ be $\mathbb{Q}_{A,\rho}$-generic, $H = G \cap \mathbb{Q}_{A_0,\rho}$. Then $K = \{ p \upharpoonright A_1 : p \in G \}$ is $\mathbb{Q}_{A_1,\rho_H}$-generic over $V[H]$ and $\rho_G = (\rho_H)_K$.

Proof:

Let $D \subseteq \mathbb{Q}_{A_1,\rho_H}$ be dense, $D \in V[H]$. Define $D' = \{ p \in \mathbb{Q}_{A,\rho} : p \upharpoonright A_0 \forces_{\mathbb{Q}_{A_0,\rho}} p \upharpoonright A_1 \in \dot{D} \}$ and let $p_0 \in H$ forces “$D$ is dense”. We claim that $D'$ is dense below $p_0$ (in $\mathbb{Q}_{A,\rho}$.)
Let \((s, F) = p \leq_{QA, \rho} p_0\). There is \(p_0 \leq_{QA_0, \rho} p \upharpoonright A_0\) such that for any \(p_1 \leq_{QA_0, \rho} p_0\), \(p_1\) is compatible with \(p\). Thus we can find \(q = (s_0, F_0) \in QA_1, \rho_H\) and \((t, E) \leq_{QA_0, \rho} p_0\) such that

\[
(t, E) \Vdash_{QA_0, \rho} \dot{q} \in \dot{D} \land \dot{q} \leq_{QA_1, \rho_H} \dot{p} \upharpoonright A_1.
\]

But then \((s_0 \cup t, F_0) \leq_{QA, \rho} (s \upharpoonright A_1 \cup t, F)\), and so \((s_0 \cup t, F_0 \cup E) \leq_{QA, \rho} (s, F)\). Since clearly \((s_0 \cup t, F_0 \cup E) \in D'\), this shows that \(D'\) is dense below \(p_0\). Now, since \(p_0 \in G\) it follows that there is \(q' \in D' \cap G\). In \(V[H]\) it then holds that \(q' \upharpoonright A_1 \in D\), which shows that \(K \cap D \neq \emptyset\). \(\Box\)
Theorem

Let $|A| > \aleph_0$ and $G$ be a $\mathbb{Q}_{A,\rho}$-generic over $V$. Then $\text{im}(\rho_G)$ is a maximal cofinitary group in $V[G]$.

Proof

Let $z \notin X \cup A$, where $\rho : X \to S_\infty$. Suppose there in $V[G]$ there is $\sigma \in \text{cofin}(S_\infty)$ such that $\rho'_G : A \cup X \cup \{z\} \to S_\infty$ defined by $\rho'_G \restriction X \cup A = \rho_G$, $\rho'_G(z) = \sigma$ induces a cofinitary representation.

Let $\dot{\sigma}$ be a name for $\sigma$. Then there is $A_0 \subseteq A$ countable so that $\dot{\sigma}$ is a $\mathbb{Q}_{A_0,\rho}$-name and so $\sigma \in V[H]$, where $H = G \cap \mathbb{Q}_{A_0,\rho}$.
Let \( a_1 \in A \setminus A_0 \) and let \( K \) be defined as in the previous Lemma. Note that for every \( N \in \omega \)

\[ D_{\sigma,N} = \{(s, F) \in \mathcal{Q}_{A_1, \rho_H} : (\exists n \geq N)s_{a_1}(n) = \sigma(n)\} \]

is dense in \( \mathcal{Q}_{A_1, \rho_H} \) and so in \( V[H][K] \)

\[ \exists \infty n((\rho_H)_K(a_1)(n) = \sigma(n)). \]

However \( (\rho_H)_K = \rho_G \), which contradicts that \( \rho'_G \) induces a cofinitary representation.
Definition: \( \mathbb{L} \)

\( \mathbb{L} \) consists of pairs \((\sigma, \phi)\) such that \( \sigma \in \omega(\omega([\omega]), \phi \in \omega(\omega([\omega])) \)

such that \( \sigma \subseteq \phi, \ \forall i < |\sigma|(|\sigma(i)| = i) \) and \( \forall i \in \omega(\|\phi(i)\| \leq |\sigma|) \).

The extension relation is defined as follows: \( (\sigma, \phi) \leq (\tau, \psi) \) if and only if \( \sigma \) end-extends \( \tau \) and \( \forall i \in \omega (\psi(i) \subseteq \phi(i)) \).

- A slalom is a function \( \phi : \omega \rightarrow [\omega]^{\omega} \) such that
  \( \forall n \in \omega (|\phi(n)| \leq n) \). A slalom localizes a real \( f \in \omega \omega \) if there is \( m \in \omega \) such that \( \forall n \geq m (f(n) \in \phi(n)) \).

- \( \mathbb{L} \) adds a slalom which localizes all ground model reals.
add(\mathcal{N}) is the least cardinality of a family \( F \subseteq \omega^\omega \) such that no slalom localizes all members of \( F \).

cof(\mathcal{N}) is the least cardinality of a family \( \Phi \) of slaloms such that every real is localized by some \( \phi \in \Phi \).

\( \alpha_g \geq \text{non}(\mathcal{M}) \).

In our intended forcing construction cofinally often we will force with the partial order \( \mathbb{L} \), which using the above characterization will provide a lower bound for \( \alpha_g \).
Definition: $\sigma$-Suslin

Let $(\mathcal{S}, \leq_{\mathcal{S}})$ be a Suslin forcing notion, $\mathcal{S} \subseteq <\omega \times \omega$. We say that $\mathcal{S}$ is $n$-Suslin if whenever $(s, f) \leq_{\mathcal{S}} (t, g)$ and $(t, h)$ is a condition in $\mathcal{S}$ such that

$$h|n \cdot |s| = g|n \cdot |s|$$

then $(s, f)$ and $(t, h)$ are compatible. A forcing notion is called $\sigma$-Suslin, if it is $n$-Suslin for some $n$. 
If $\mathcal{S}$ is $n$-Suslin and $m \geq n$, then $\mathcal{S}$ is also $m$-Suslin.

Every $\sigma$-Suslin forcing notion is $\sigma$-linked and so has the Knaster property.

Hechler forcing $\mathbb{H}$ is 1-Suslin, localization $\mathbb{L}$ is 2-Suslin.
Definition: Nice name for a real

Let $\mathcal{B}$ be a partial order and for each $n \geq 1$ let $\mathcal{B}_n$ be a maximal antichain. We will say that the set $\{(b, s(b))\}_{b \in \mathcal{B}_n, n \geq 1}$ is a nice name for a real if

1. whenever $n \geq 1$, $b \in \mathcal{B}_n$ then $s(b) \in {}^n\omega$

2. whenever $m > n \geq 1$, $b \in \mathcal{B}_n$, $b' \in \mathcal{B}_m$ and $b, b'$ are compatible, then $s(b)$ is an initial segment of $s(b')$.

We can assume that all names for reals are nice and abusing notation we will write $\dot{f} = \{(b, s(b))\}_{b \in \mathcal{B}_n, n \in \omega}$. 
Lemma: Canonical Projection of a name for a real

Let $A$ be a complete suborder of $B$. Let $\dot{f} = \{(b, s(b))\}_{b \in B_n, n \geq 1}$ be a $B$-nice name for a real. Then there is $\dot{g} = \{(a, s(a))\}_{a \in A_n, n \geq 1}$, an $A$-nice name for a real, such that for all $n \geq 1$, for all $a \in A_n$, there is $b \in B_n$ such that $a$ is a reduction of $b$ and $s(a) = s(b)$.

Whenever $\dot{f}, \dot{g}$ are as above, we will say that $\dot{g}$ is a canonical projection of $\dot{f}$. Furthermore if $\dot{f}$ is a $B$-nice name for a real below some $y \in B$ and $x \in A$ is a reduction of $y$, then there is a $A$-nice name for a real below $x$ which is canonical projection of $\dot{f}$.
Definition: Good Suslin
Let $\mathbb{S}$ be a Suslin forcing notion, $\mathbb{S} \subseteq <\omega \omega \times \omega \omega$. Then $\mathbb{S}$ is said to be good if whenever $\mathbb{A}$ is a complete suborder of $\mathbb{B}$, $x \in \mathbb{A}$ is a reduction of $y \in \mathbb{B}$ and $\dot{f}$ is a nice name for a real below $y$ such that $y \models_{\mathbb{B}} (\check{s}, \dot{f}) \in \dot{\mathbb{S}}$ for some $s \in <\omega \omega$, there is a canonical projection $\dot{g}$ of $\dot{f}$ below $x$ such that $x \models (\check{s}, \dot{g}) \in \dot{\mathbb{S}}$. 
\( \mathcal{D} \) and \( \mathcal{L} \) are good \( \sigma \)-Suslin forcing notions.
Let \((L, \leq)\) be a linearly ordered set, \(x \in L\). Then

\[ L_x := \{ y \in L : y < x \} . \]

If \(L_0 \subseteq L\) and \(A \subseteq L\), then define the \(L_0\)-closure of \(A\) as follows:

\[ \text{cl}_{L_0}(A) = A \cup \bigcup_{x \in A} L_x \cap L_0 . \]
Definition: Template

A template is a tuple $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$ where $L = L_0 \cup L_1$, $L_0 \cap L_1 = \emptyset$, $(L, \leq)$ is a linear order, $\mathcal{I} \subseteq \mathcal{P}(L)$, such that

- $\mathcal{I}$ is closed under finite intersections and unions, $\emptyset, L \in \mathcal{I}$.
- If $x, y \in L, y \in L_1$ and $x < y$ then $\exists A \in \mathcal{I}(A \subseteq L_y \land x \in A)$.
- If $A \in \mathcal{I}, x \in L_1 \setminus A$, then $A \cap L_x \in \mathcal{I}$.
- $\{A \cap L_1 : A \in \mathcal{I}\}$ is well-founded when ordered by inclusion.
- All $A \in \mathcal{I}$ are $L_0$-closed.
Define $D_p : \mathcal{I} \to \mathbb{ON}$ by letting $D_p(A) = 0$ for $A \subseteq L_0$ and

$$D_p(A) = \sup \{D_p(B) + 1 : B \in \mathcal{I} \wedge B \cap L_1 \subset A \cap L_1 \}.$$ 

Let $R_k(\mathcal{T}) = D_p(L)$.

For $A \subseteq L$ let $\mathcal{I}|A = \{A \cap B : B \in \mathcal{I}\}$ and

$$\mathcal{T}_A = ((A, \leq), \mathcal{I}|A, L_0 \cap A, L_1 \cap A).$$

If $A \in \mathcal{I}$ then $R_k(\mathcal{T}_A) = D_p(A)$.

For $x \in L$ let $\mathcal{I}_x = \{B \in \mathcal{I} : B \subseteq L_x\}.$
Definition: Iterating a good $\sigma$-Suslin poset along a template and adding a m.c.g.

Let $Q = Q_{L_0}$ be the poset for adding a m.c.g. with $L_0$-generators, $S$ good $\sigma$-Suslin. Then $P(T, Q, S)$ is defined recursively:

- If $Rk(T) = 0$, then $P(T, Q, S) = Q_{L_0}$.

Assume $P(T, Q, S)$ has been defined for all templates of rank $< \kappa$, let $Rk(T) = \kappa$ and for all $B \in I$ of rank strictly smaller than $\kappa$ let $P_B = P(T_B, Q, S)$. Then:
cont.:

- $\mathbb{P}(T, Q, S)$ consists of all $P = (p, F^p)$ where $p$ is a finite partial function with $\text{dom}(p) \subseteq L$, $P \upharpoonright L_0 := (p \upharpoonright L_0, F^p) \in Q$ and if $x_p \overset{\text{def}}{=} \max\{\text{dom}(p) \cap L_1\}$ is defined then $\exists B \in \mathcal{I}_{x_p}$ such that $P \upharpoonright L_{x_p} := (p \upharpoonright L_{x_p}, F^p \cap \dot{\mathcal{W}}_{L_{x_p} \cap L_0}) \in \mathbb{P}_B$, $(P \upharpoonright L_{x_p}, p(x_p)) \in \mathbb{P}_B \ast \dot{S}$ where $p(x_p)$ is of the form $(\dot{s}_x^p, \dot{f}_x^p)$, for $s_x^p \in <\omega \omega$, $\dot{f}_x^p$ is a $\mathbb{P}_B$ name for a real.
cont.: Define \( Q \leq_P P \) iff \( \text{dom}(p) \subseteq \text{dom}(q) \), \( Q \upharpoonright L_0 \leq_Q P \upharpoonright L_0 \), and if \( x_p \) is defined then either

- \( x_p < x_q \) and \( \exists B \in \mathcal{I}_{x_q} \) such that \( P \upharpoonright L_{x_q}, Q \upharpoonright L_{x_q} \in \mathbb{P}_B \) and \( Q \upharpoonright L_{x_q} \leq_{\mathbb{P}_B} P \upharpoonright L_{x_q} \), or
- \( x_p = x_q \) and \( \exists B \in \mathcal{I}_{x_q} \) witnessing \( P, Q \in \mathbb{P} \), and such that

\[
(Q \upharpoonright L_{x_q}, q(x_q)) \leq_{\mathbb{P}_B \ast \dot{\mathcal{S}}} (P \upharpoonright L_{x_p}, p(x_p)).
\]
Completeness of Embeddings Lemma

Let $\mathcal{T} = ((L, \leq), \mathcal{I}, L_0, L_1)$, let $\mathcal{Q} = \mathcal{Q}_{L_0}$ be the poset for adding m.c.g. with $L_0$-generators, $\mathbb{S}$ be good $\sigma$-Suslin.

Let $B \in \mathcal{I}$, $A \subset B$ be closed. Then $\mathbb{P}_B$ is a poset, $\mathbb{P}_A \subset \mathbb{P}_B$, every $P = (p, F^p) \in \mathbb{P}_B$ has a canonical reduction $P_0 = (p_0, F^{p_0}) \in \mathbb{P}_A$ such that

- $\text{dom}(p_0) = \text{dom}(p) \cap A$, $F^{p_0} = F^p$,
- $s^p_{x_0} = s^p_x$ for all $x \in \text{dom}(p_0) \cap L_1$,
- $(p_0\restriction L_0, F^{p_0})$ is a strong $\mathcal{Q}_A$-reduction of $(p \restriction L_0, F^p)$.

and whenever $D \in \mathcal{I}$, $B, C \subseteq D$, $C$ is closed, $C \cap B = A$ and $Q_0 \leq \mathbb{P}_C P_0$, then $Q_0$ and $P$ are compatible in $\mathbb{P}_D$. 
If $A = C$, $D = B$ then $\mathbb{P}_A$ is a complete suborder of $\mathbb{P}_B$. 
Transitivity:

If \( \text{Rk}(T) = 0 \), then since \( \mathbb{P} = \mathcal{Q}_{L_0} \) clear. So assume the Lemma for all templates of rank \(< \alpha\), and let \( \text{Rk}(T) = \alpha \). Fix \( P_0, P_1, P_2 \in \mathbb{P} \) such that \( P_1 \leq \mathbb{P} P_0 \) and \( P_2 \leq \mathbb{P} P_1 \), and assume that \( x_{P_0} \) is defined.

Fix witnesses \( B_1 \in \mathcal{I}_{x_{P_1}} \) and \( B_2 \in \mathcal{I}_{x_{P_2}} \) to \( P_1 \leq \mathbb{P} P_0 \) and \( P_2 \leq \mathbb{P} P_1 \). Since \( D_{\mathbb{P}}(B_1 \cup B_2) < \alpha \), by inductive hypothesis

\[
\mathbb{P}_{B_1}, \mathbb{P}_{B_2} \triangleleft \mathbb{P}_{B_1 \cup B_2},
\]

and so we have \( P_i \upharpoonright L_{x_{P_2}} \in \mathbb{P}_{B_1 \cup B_2} \) for \( 0 \leq i \leq 2 \), and

\[
P_2 \upharpoonright L_{x_{P_2}} \leq \mathbb{P}_{B_1 \cup B_2} P_1 \upharpoonright L_{x_{P_2}} \leq \mathbb{P}_{B_1 \cup B_2} P_0 \upharpoonright L_{x_{P_2}}.
\]

Thus by inductive hypothesis \( P_2 \upharpoonright L_{x_{P_2}} \leq \mathbb{P}_{B_1 \cup B_2} P_0 \upharpoonright L_{x_{P_2}} \).
If $x_{p_0} < x_{p_2}$ then by definition $P_2 \leq_{P} P_0$. So assume that $x_{p_0} = x_{p_2}$. Then $p_i(x_{p_2})$ is a $P_{B_1 \cup B_2}$-name for $0 \leq i \leq 2$. Since $P_{B_1}, P_{B_2} < P_{B_1 \cup B_2}$ we must have that

- $P_1 \upharpoonright L_{x_{p_2}} \models_{P_{B_1 \cup B_2}} p_1(x_{p_2}) \leq \dot{\mathcal{S}} p_0(x_{p_2})$, and
- $P_2 \upharpoonright L_{x_{p_2}} \models_{P_{B_1 \cup B_2}} p_2(x_{p_2}) \leq \dot{\mathcal{S}} p_1(x_{p_2})$ and so
- $P_2 \upharpoonright L_{x_{p_2}} \models_{P_{B_1 \cup B_2}} p_2(x_{p_2}) \leq \dot{\mathcal{S}} p_0(x_{p_2})$.

Thus $(P_2 \upharpoonright L_{x_{p_2}}, p_2(x_{p_2})) \leq_{P_{B_1 \cup B_2} \ast \dot{\mathcal{S}}} (P_0 \upharpoonright L_{x_{p_2}}, p_0(x_{p_2}))$ as required.
\( \mathbb{P}_A \subseteq \mathbb{P}_B \):

Let \( \mathcal{I} \) be of rank \( \alpha \). Let \( A \subseteq B \) be closed, \( B \in \mathcal{I} \). Let \( R \in \mathbb{P}_A \) and let \( x = x_r \). By definition there is \( \check{A} \in (\mathcal{I}|A)_x \) such that

\[
R \upharpoonright L_x \in \mathbb{P}_{\check{A}} \text{ and } \dot{f}^r_x \text{ is a } \mathbb{P}_{\check{A}}\text{-name.}
\]

By the properties of \( \mathcal{I} \) there is \( \check{B} \in \mathcal{I}_{B,x} \) such that \( \check{A} = \check{B} \cap A \). Then \( \text{Rk}(\mathcal{T}_{\check{B}}) < \alpha \) and so by inductive hypothesis \( \mathbb{P}_{\check{A}} \preceq \mathbb{P}_{\check{B}} \).

Therefore

\[
R \upharpoonright L_x \in \mathbb{P}_{\check{B}} \text{ and } \dot{f}^r_x \text{ is a } \mathbb{P}_{\check{B}}\text{-name.}
\]

That is \( R \in \mathbb{P}_B \).
Definition of \( p_0(P, A, B) \)

Let \( \mathcal{I} \) be of rank \( \alpha \). Let \( A \subset B \) be closed, \( B \in \mathcal{I} \). Let \( P = (p, F^p) \in \mathbb{P}_B \). We have to construct \( P_0 = p_0(P, A, B) \). By definition there is \( \bar{B} \in \mathcal{I}_{B,x} \) such that \( \bar{P} = P \upharpoonright L_x = (p \upharpoonright L_x, F^p \cap \widehat{W}_{L_x \cap L_0}) \in \mathbb{P}_B \). Let \( \bar{A} = \bar{B} \cap A \). Then by inductive hypothesis there is \( \bar{P}_0 = p_0(\bar{P}, \bar{A}, \bar{B}) = (\bar{p}_0, F^{\bar{p}_0}) \). Define \( P_0 = (p_0, F^{p_0}) \) as follows:

- \( p_0 \upharpoonright L_x = \bar{p}_0, \ p_0 \upharpoonright L \setminus L_x = p \upharpoonright L \setminus L_x \),
- If \( x \notin A \) let \( p_0(x) = p(x) \), and
- if \( x \in A \) let \( p_0(x) \) be a canonical projection of \( p(x) \) below \( \bar{P}_0 \) (since \( S \) is a good Suslin, such projection exists).
- \( F^{p_0} = F^p \cap \widehat{W}_A \).
Examples:

- Let \( L = L_1 = \mu = L_\mu \) be an ordinal, \( L_0 = \emptyset \), \( \mathcal{I} = \{ L_\alpha = \alpha : \alpha \leq \mu \} = \mu + 1 \). Then \( P(T, D) \) is the \( \mu \)-stage finite support iteration of Hechler forcing \( D \).

- Let \( L = L_1 \) be arbitrary (and so \( L_0 = \emptyset \)). Let \( \mathcal{I} = [L]^{<\omega} \cup \{ L \} \). Then \( (L, \mathcal{I}) \) is a template and \( P(T, D) \) adds a family of functions in \( \omega^\omega \) which is canonically order isomorphic to \( L \).
Lemma

- $\mathbb{P}(\mathcal{T}, \mathcal{Q}, \mathcal{S})$ is Knaster.
- Let $x \in L_1$, $A \in \mathcal{I}_x$. Then the two-step iteration $\mathbb{P}_A \ast \mathcal{S}$ completely embeds into $\mathbb{P}$.
- For any $p \in \mathbb{P}(\mathcal{T}, \mathcal{Q}, \mathcal{S})$ there is countable $A \subseteq L$ such that $p \in \mathbb{P}_{\text{cl}(A)}$. If $\tau$ is a $\mathbb{P}$-name for a real then there is a countable $A \subseteq L$ such that $\tau$ is a $\mathbb{P}_{\text{cl}(A)}$-name.
Lemma
Let $P = P(\mathcal{T}, Q_{L_0}, L)$ and let $\lambda_0$ be a regular uncountable cardinal such that $\lambda_0 \subseteq L_1$ (as an order), $\lambda_0$ is cofinal in $L$, and $L_\alpha \in \mathcal{I}$ for all $\alpha < \lambda_0$. Then in $V^P$, $\text{non}(\mathcal{M}) = \lambda_0$ and so $a_g \geq \lambda_0$. Furthermore if $L$ is of uncountable cofinality and $L_0$ is cofinal in $L$, then $P$ adds a maximal cofinitary group of size $|L_0|$.
Proof: \( \text{non}(\mathcal{M}) = \lambda_0 \)

Let \( G \) be \( \mathbb{P} \)-generic and let \( \phi_\alpha \) be the slalom added in coordinate \( \alpha < \lambda_0 \). Since \( \lambda_0 \) is regular, uncountable and is cofinal in \( L \), the family \( \langle \phi_\alpha : \alpha < \lambda_0 \rangle \) localizes all reals \( V[G] \) (indeed any real must appear in some \( V[G \cap \mathbb{P}_{L_\alpha}] \) for some \( \alpha < \lambda_0 \).) Thus \( \text{cof}(\mathcal{N}) \leq \lambda_0 \). On the other hand, if \( F \subseteq \omega^\omega \) is a family of size \( < \lambda_0 \) in \( V[G] \), then there must be some \( \alpha < \lambda_0 \) such that all reals of \( F \) already are in \( V[G \cap \mathbb{P}_{L_\alpha}] \), and so \( \phi_\alpha \) localizes all reals in \( F \). Thus \( \text{add}(\mathcal{N}) \geq \lambda_0 \). Therefore \( \text{non}(\mathcal{M}) = \lambda_0 \) and so \( \alpha_g \geq \lambda_0 \). \( \square \)
Proof: $\mathbb{P}$ adds a m.c.g. of size $|L_0|$

Let $G$ be $\mathbb{P}$-generic, $\rho_G : L_0 \to S_\infty$ be defined as follows: for $x \in L_0$ let $\rho_G(x) = \bigcup \{ s_p^x : p \in G \land p \upharpoonright L_0 = (s^p, F^p) \}$. Note that $\rho_G = \rho_{G_0}$ where $G_0 = G \cap \mathbb{P}_{L_0}$ and so it induced a cofinitary representation of $\mathbb{P}_{L_0}$. We claim that $\text{im}(\rho_G)$ is a m.c.g.

Otherwise, there are $\sigma \in \text{cofin}(S_\infty)$ and $b_0 \notin L_0$ such that $\rho'_G : L_0 \cup \{ b_0 \} \to S_\infty$, where $\rho'_G \upharpoonright L_0 = \rho_G$ and $\rho'_G(b_0) = \sigma$, induces a cofinitary representation. Let $\dot{\sigma}$ be a $\mathbb{P}$-name for $\sigma$. Then for some countable $A \subseteq L$, $\dot{\sigma}$ is a $\mathbb{P}_{\text{cl}(A)}$-name. Since $\lambda_0$ is regular, uncountable and cofinal in $L$, there is $\alpha \in \lambda_0$ such that $\text{cl}(A) \subseteq L_\alpha$ and so $\mathbb{P}_{\text{cl}(A)} \preceq \mathbb{P}_{L_\alpha}$. Let $H = G \cap \mathbb{P}_{L_\alpha}$ and let $x \in L_0 \setminus L_\alpha$. 

Vera Fischer
Claim

In $V[H]$ the set $D_{\sigma,N}$ consisting of all $p \in \mathbb{P}/H$ such that for some $n \geq N(s^p_x(n) = \sigma(n))$ where $p \upharpoonright L_0 = (s^p, F^p)$ is dense.

Proof:
Let $p_0 \in \mathbb{P}/H$. Thus $p \upharpoonright L_0 \cap L_\alpha \in H_0 := G \cap \mathbb{P}_{L_0 \cap L_\alpha}$. The set $D^0_{\sigma,N,x} = \{p \in (\mathbb{Q}_{L_0}/\mathbb{Q}_{L_\alpha \cap L_0}) : (\exists n \geq N)s^p_x(n) = \sigma(n)\}$ is dense in $V[H_0]$ and so $\exists(t, E) \leq (s^{p_0} \upharpoonright L_0 \setminus L_x, F^{p_0})$ such that $(t, E) \in D^0_{\sigma,N,x}$ i.e. $t_x(n) = \sigma(n)$ for some $n \geq N$. Define $p_1 \in \mathbb{P}/H$ as follows: $p_1 \upharpoonright L_\alpha = p_0 \upharpoonright L_\alpha$, $p_1 \upharpoonright (L_0 \setminus L_\alpha) = (t, E)$, $p_1 \upharpoonright L_1 \setminus L_\alpha = p_0 \upharpoonright L_1 \setminus L_\alpha$. Then in $V[H]$, $p_1 \leq p_0$ and $p_1 \in D_{\sigma,n}$. \qed
Then in $V[G]$ there are infinitely many $n$ such that $\sigma(n) = \sigma_x(n)$, contradicting the fact that $\rho'_G$ induces a cofinitary representation.
An isomorphism of names argument provides that in $V^P$ there are no mcg of size $< \lambda$ and so $V^P \models a_g = \lambda$. 
Theorem

It is consistent with the usual axioms of set theory that the minimal size of a maximal cofinitary group is of countable cofinality.
Theorem (V.F., A. Törnquist)

Assume CH. Let \( \lambda \) be a singular cardinal of countable cofinality and let \( \bar{a} \in \{ a, a_p, a_g, a_e \} \). Then there are a good \( \sigma \)-Suslin poset \( S_{\bar{a}} \) and a finite function poset with the strong embedding property \( Q_{\bar{a}} \), which is Knaster (and so \( P(\mathcal{T}_0, Q_{\bar{a}}, S_{\bar{a}}) \) is Knaster) such that \( V^P(\mathcal{T}_0, Q_{\bar{a}}, S_{\bar{a}}) \models \bar{a} = \lambda \). Then in particular \( V^P(\mathcal{T}_0, Q_{\bar{a}}, S_{\bar{a}}) \models \text{cof}(\bar{a}) = \omega \).
Thank you!