Notes from the Prague Set Theory seminar

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1.1 Ch. Brech: Application of PID to Banach spaces

1.2 Definition. *For a Banach space X a family of pairs $\langle x_\alpha, f_\alpha : \alpha \in \Gamma\rangle \subseteq X \times X^*$ is a* bi-orthogonal *system if $f_\alpha(x_\beta) = \delta_{\alpha\beta}$.*

1.3 Note. *The span of the first (or second) coordinates need not be the full space (dual space).*

Considering

1.4 Fact. *Separable infinite dimensional Banach spaces have infinite bi-orthogonal systems. (in fact even satisfying the stronger condition mentioned in the above note).*

it is natural to ask

1.5 Question. *Suppose X is a Banach space of density κ. Is it true that it has a bi-orthogonal system of size κ?*

1.6 Example (Kunen, '80)**.** *Assuming CH, there is a scattered, compact, Hausdorff, non-metrizable space K of weight ω_1 such that all finite powers are hereditarily separable.*

1.7 Example (Todorčević, '89)**.** *Assuming $\mathfrak{b} = \omega_1$ there is a space as in 1.6*

Both of the above examples yield a nonseparable Banach space with no uncountable bi-orthogonal system, viz $X = C(K)$. X will be Lindelof in the weak topology and, if $\langle x_\alpha, f_\alpha : \alpha < \omega_1\rangle$ is a bi-orthogonal system, then $\{x_\alpha : \alpha < \omega_1\}$ will be discrete in the weak topology — a contradiction. (To see this note that the basis of the weak topology consists of sets of the form

$$V(x, f_0, \ldots, f_n, \varepsilon) = \{y \in X : (\forall i \leq n)(|f_i(y) - f_i(x)| < \varepsilon)\}$$

1.8 Theorem (Todorčević, 2006)**.** *Assuming PID and $\mathfrak{p} > \omega_1$ then every nonseparable Banach space has an uncountable bi-orthogonal system.*

See 1.53 for the definition of the P-ideal dichotomy (PID).

1.9 Question (Todorčević)**.** *Assume PID. Is $\mathfrak{b} = \omega_1$ equivalent to the existence of a nonseparable Banach space with no uncountable bi-orthogonal system.*

1.10 Definition. *An* Asplund *Banach space is a space whose separable spaces have separable duals.*

1.11 Fact. *If Y is scattered then $C(Y)$ is Asplund.*

1.12 Definition. *For a Banach space X a family of pairs $\langle x_\alpha, f_\alpha : \alpha \in \Gamma\rangle \subseteq X \times X^*$ is a ε-*bi-orthogonal *system if $|f_\alpha(x_\beta)| < \varepsilon$ for each $\alpha \neq \beta$.*

1.13 Theorem (Brech)**.** *Assume PID and $\mathfrak{b} > \omega_1$ then every nonseparable Asplund space has an uncountable ε-bi-orthogonal system for each $0 < \varepsilon < 1$.*

Proof. Hahn-Banach extension theorem means we can extend bi-orthogonal from subspaces. In particular we may assume $d(X) = \omega_1$. Write $X = \bigcup_{\alpha<\omega_1} X_\alpha$ as an increasing union of separable closed subspaces. By induction construct $x_\alpha \in X_{\alpha+1} \setminus X_\alpha$ and $h_\alpha \in X^*$ such that $h_\alpha(x_\alpha) = \mathbf{1}$, $||h_\alpha|| = 1$ and $h_\alpha \restriction X_\alpha = 0$. Let $S = \{h_\alpha - h_\beta : \beta \neq \alpha\}$. The proof now splits into three successive claims. First, fix a $D \subseteq X$ dense, $\mathbb{Q}$-linear subspace of X of size ω_1.

1.14 Claim. *There is an uncountable subset $\{f_\alpha : \alpha < \omega_1\} \subseteq S$ such that for each $x \in D$ the sequence $\langle f_\alpha(x) : \alpha < \omega_1\rangle \in c_0(\omega_1)(= (\{t : (\forall\varepsilon > 0)(|\{\alpha < \omega_1 : t(\alpha) \geq \varepsilon\}| < \omega)\}$.*

Proof of claim. Apply PID to the ideal

$$\mathcal{I} = \{A \in [S]^{\omega_1} : (\forall x \in D)(\langle f_\alpha(x) : \alpha \in A\rangle \in c_0(A))\}$$

(which is a P-ideal under $\mathfrak{b} > \omega_1$) to find the uncountable subset of S.

■

1.15 Claim. *There is an uncountable* $\Gamma \in [\omega_1]^{\omega_1}$ *such that*

$$(\forall x \in D)(\langle f_\alpha(x) : \alpha \in \Gamma\rangle \in l_1(\Gamma)),$$

where

$$l_1(\Gamma) = \left\{ t \in {}^\Gamma X : \sum_{\alpha\in\Gamma} |h(\alpha)| < \infty \right\}$$

Proof of claim. Let

$$\mathcal{I} = \left\{ A \in [\omega_1]^\omega : (\forall x \in D)\left(\sum_{\alpha\in A} |f_\alpha(x) < \infty\right) \right\}$$

First show, using $\mathfrak{b} > \omega_1$, that $\mathcal{I}$ is a P-ideal. Now apply PID to $\mathcal{I}$. The first possibility gives $\Gamma = K$ so it is sufficient to show that the second possibility is impossible :-) But that is easy, since any sequence in c_0 always contains an infinite summable subsequence.

■

The following claim now finishes the proof.

1.16 Claim. *There is a sequence* $\langle \alpha_\xi : \xi \in \omega_1\rangle \subseteq \Gamma$ *and* $\langle x_\xi : \xi \in \omega_1\rangle \subseteq X$ *such that* $\langle x_\xi, f_\xi : \xi \in \omega_1\rangle$ *is an* ε*-bi-orthogonal system.*

□

1.17 J. Lopez-Abad: Families of finite sets

First a remark concerning Christina's talk.

1.18 Definition. *An* Auerbach *base for a finite dimensional normed space* X *is a sequence* $\langle x_i, f_i : i < dim(X)\rangle \subseteq X \times X^*$ *which generatex* X *and such that* $f_i(x_j) = \delta_{ij}$ *and, crucially,* $||f_i|| = ||x_i|| = 1$ *for* $i < dim(X)$.

1.19 Fact. *Every finite dimensional normed space has an Auerbach basis.*

1.20 Theorem (Pelczynski). *If* X *is an infinite dimensional separable Banach space then it has, for each* $\varepsilon > 0$ *a* Markushevich basis *(i.e. a sequence* $\langle x_i, f_i : i < \omega\rangle \subseteq X \times X^*$, $f_i(x_j) = \delta_{ij}$*, such that the first coordinates span a dense subspace of* X*) such that* $1 - \varepsilon < ||f_i||, ||x_i|| < 1 + \varepsilon$*).*

Now we will consider families of finite subsets of S (typically, $S = \mathbb{N}$) and the two relations $\subseteq, \sqsubseteq$.

1.21 Notation. *Given* $M \in [S]^\omega$ *and a family* $\mathcal{F} \subseteq [S]^{<\omega}$ *we let*

$$\mathcal{F} \restriction M = \mathcal{F} \cap \mathcal{P}(M)$$

(the restriction*) and*

$$\mathcal{F}[M] = \{s \cap M : s \in \mathcal{F}\}$$

(the trace*)*

Let us consider when $\mathcal{F} \restriction M$ or $\mathcal{F}[M]$ are $\subseteq$ or a $\sqsubseteq$ antichains. When are they $\subseteq$ or $\sqsubseteq$ hereditary.

1.22 Observation. *The family* $\mathcal{F}$ *is compact iff every sequence in* $\mathcal{F}$ *has a* Δ*-system subsequence with root in* $\mathcal{F}$.

1.23 Definition. *The family* $\mathcal{F}$ *is* precompact *iff every sequence in* $\mathcal{F}$ *has a* Δ*-system subsequence.*

1.24 Note. $\mathcal{F}$ *is precompact iff its closure (in* 2^ω*) is compact.*

1.25 Example. *Given* $l < \omega$ *the family* $[\omega]^l$ *is a precompact antichail while its closure,* $[\omega]^{\leq l}$*, is hereditary and compact.*

1.26 Theorem (Ramsey). *For any coloring* $\chi : [\omega]^l \to n$ *there is an infinite* $M \subseteq \omega$ *such that* $\chi \restriction [M]^l$ *is constant.*

1.27 Definition. *A family* $\mathcal{F}$ *is* Ramsey *if for any coloring* $\chi : \mathcal{F} \to n$ *there is an infinite* M *such that* $|\{i < n : \chi^{-1}\{i\} \cap \mathcal{F} \restriction M\}| \leq 1$

1.28 Theorem (Nash-Williams). *For a family* $\mathcal{F}$ *the following are equivalent: 1.* $\mathcal{F}$ *is Ramsey 2. there is an infinite* M *such that* $\mathcal{F} \restriction M$ *is a* $\sqsubseteq$*-antichain (it is* thin*) 3. there is an infinite* M *such that* $\mathcal{F} \restriction M$ *is a* $\subseteq$*-antichain (it is a* Sperner system*)*

1.29 Definition (Nash-Williams). *Let* $\mathcal{F} \subseteq [M]^{<\omega}$*. It is called a* barrier *on* M *if it is Sperner and is unavoidable, i.e. every infinite subset of* M *has an initial part in* $\mathcal{F}$*. It is called* block *on* M *if it is a thin unavoidable family.*

1.30 Proposition. *For every thin (Sperner) family* $\mathcal{F}$ *there is an infinite* M *such that* $\mathcal{F} \restriction M = \emptyset$ *or* $\mathcal{F} \restriction M$ *is a barrier.*

1.31 Fact. *If* $\mathcal{F}$ *is a barrier on* M *then for every coloring* $\chi : \mathcal{F} \to n$ *there is an infinite* $N \subseteq M$ *such that* $\chi \restriction (\mathcal{F} \restriction N)$ *is constant.*

1.32 Fact. *If $\mathcal{F}$ is precompact then there is an infinite set such that its trace $\mathcal{F}[M]$ is the closure of a barrier.*

1.33 Observation. *If $\mathcal{B}$ is a barrier, then its topological closure is equal to its $\subseteq$-closure and its $\sqsubseteq$-closure.*

In particular, every precompact family has a $\subseteq$-hereditary trace. For example, if $\mathcal{F} = [\omega]^n$ then $\mathcal{F}[2\mathbb{N}]$ is hereditary.

1.34 Theorem. *Suppose $\mathcal{F}$ is arbitrary then there exists an infinite M such that 1.$[M]^{<\omega} = \overline{\mathcal{F}[M]}^{\subseteq}$ or 2. $\mathcal{F}[M]$ is the closure of a barrier on M.*

1.35 Theorem (Erdös-Rado)**.** *For each $n, l < \omega$ and a coloring $\chi : [\omega]^l \to n$ there is $I \subseteq l$ and an infinite M such that for each $s = \{n_1 < \cdots < n_l\}, t = \{m_1 < \cdots < m_l\} \in [M]^l$ $c(s) = c(t)$ iff $(\forall i \in I)(n_i = m_i)$.*

How do coloring of barriers behave?

1.36 Theorem (Pudlák-Rödl)**.** *For each barrier $\mathcal{B}$ on M and a coloring $\chi : \mathcal{B} \to X$ there is an infinite M and a barrier $\mathcal{C}$ on M, a $\varphi : \mathcal{B} \restriction M \to \mathcal{C}$ and a $\hat{\chi} : \mathcal{C} \to X$ such that $\hat{\chi} \circ \varphi = \chi$, $\hat{chi}$ is 1-1 and $\varphi(s) \subseteq s$.*

1.37 Definition. *Let $\{m_n : n < \omega\} = M$ be an increasing enumeration. A* 0-uniform family *on M is $\{\emptyset\}$. An* $\alpha+1$-uniform family *on M if*

$$\mathcal{F}_{\{m_n\}} = \{s : m_n < s \;\&\; s \cup \{m_n\} \in \mathcal{F}\}$$

is α-uniform in $\{m_l : l > n\}$. If α is limit, then a family $\mathcal{F}$ is α-uniform if there is an increasing sequence α_n converging to α such that $\mathcal{F}_{\{m_n\}}$ is α_n-uniform.

1.38 Theorem (Pudlák-Rödl)**.** *TFAE 1. There is an infinite M such that $\mathcal{B} \restriction M$ is a barrier on M 2. There is an infinite M such that $\mathcal{B} \restriction M$ is uniform on M*

1.39 Theorem. *If $\varphi : \mathcal{B} \to Fin$ where $\mathcal{B}$ is a barrier and $\varphi[\mathcal{B}]$ is precompact then there is an infinite M such that for each $s \in \mathcal{B} \restriction M$ we have $\varphi(s) \cap M \subseteq s$.*

1.40 Corollary. *If $\varphi : \mathcal{B} \to c_0$ is a mapping of a barrier into c_0 whose image is relatively compact in the weak topology then for each $\varepsilon > 0$ there is an infinite M such that $\forall s \in \mathcal{B} \restriction M$ we have*

$$\sum_{n \in M \setminus s} |\varphi(s)|_n < \varepsilon$$

1.41 Example. *The set $\mathcal{S} = \{s : |s| = \min s\}$ is ω-uniform, precompact and large in the following sense.*

1.42 Definition. *A family $\mathcal{F}$ is* large *in M iff for each $N \subseteq M$ and for all $n < \omega$ there is $s \in \mathcal{F}$ such that $|s \cap N| \geq n$. It is λ-filling (for $\lambda \in [0, 1]$) if for each $s \subseteq M$ there is $t \in \mathcal{F} \restriction s$ such that $|t| \geq \lambda \cdot |s|$.*

1.43 Theorem. *For a cardinal κ the following are equivalent: 1. There is a compact, hereditary, large family on κ. 2. κ is not ω-Erdös. 3. some statement about Banach spaces*

1.44 Definition. *A cardinal κ is ω-Erdös iff for any coloring $\chi : [\kappa]^{<\omega} \to 2$ there is an infinite set $A \subseteq \kappa$ such that on $[A]^{<\omega}$ the coloring χ only depends on the cardinality.*

The following is Fremlin's DU-problem.

1.45 Question. *Does there exist a 1/2-filling compact (or precompact) family on ω_1?*

1.46 Note. *Taking the Cantor set 2^ω instead of ω_1 and wanting the family to be Borel (or Analytic?) the answer is no (see Dodos, P. and Kanellopoulos, K:* On filling families of finite subsets of the Cantor set *, Math. Proc. Cam. Phil. Soc. 145 (2008), pp 165-175, doi:*10.1017/S0305004108001096 *)*

1.47 A. Avilés: Tukey classification of orthogonals

The following theorem can be found in Avilés, A., Plebanek, G., Rodriguez, J.: *Measurability in$C(2^\kappa)$ and Kunen cardinals* , Israel Journal of Math. 195 (2012), pp 1-30, doi:10.1007/s11856-012-0122-0 .

1.48 Theorem (Avilés, Plebanek, Rodriguez)**.** *Assume Analytic Determinacy. If $\mathcal{I}$ is an analytic family of subsets of ω then the orthogonal*

$$\mathcal{I}^\perp = \{A \subseteq \omega : (\forall B \in \mathcal{I})(|A \cap B| < \omega)\}$$

(which is co-analytic) is Tukey-equivalent (see 1.52) to either 1. $\{0\}$, 2.ω, 3. ω^ω, 4. $K(\mathbb{Q})$ (i.e. compact subsets of the rationals ordered by inclusion) or finite subsets 5. $\mathbb{R}$.

1.49 Theorem (Todorčević)**.** *Let $\mathcal{I}$ and $\mathcal{J}$ be two orthogonal (i.e. $\mathcal{J} \subseteq \mathcal{I}^\perp$ analytic ideals on ω. Then either $\mathcal{I}$ and $\mathcal{J}$ are countably separated (i.e. there is a countable $\mathcal{C} \subseteq P(\omega)$, such that for any disjoint $a \in \mathcal{I}, b \in \mathcal{J}$ there is a $c \in \mathcal{C}$ such that $a \subseteq c$ and $b \subseteq \omega \setminus c$). or there is a 1-1 function $u : 2^{<\omega} \to \omega$ such that 0-chains go to elements of $\mathcal{I}$ and 1-chains go to elements of $\mathcal{J}$ (where an i-chain is a sequence $\{x_0, x_1, \ldots\} \subseteq 2^{<\omega}$ such that for all n we have $x_{n+1} = x_n^\frown(i, k_1, \ldots, k_{p_n})$ for some $k_1, \ldots, k_{p_n}$).*

1.50 Note. *In the above theorem, analytic determinacy gives the same result for Σ^1_2 ideals.*

Applying the theorem of Todorčević to the ideal $\mathcal{I}$ and $\mathcal{I}^\perp$, the second option gives $[\mathbb{R}]^{<\omega}$.

1.51 Theorem (Avilés, Todorčević)**.** *$\mathcal{I}, \mathcal{J}$ are countably separated iff there is a metrizable compactification K of ω and a decomposition $K \setminus \omega = U \cup V$ such that $(\forall a \in \mathcal{I})(\overline{a}^K \subseteq U)$ and $(\forall b \in \mathcal{J})(\overline{b}^K \subseteq V)$*

This theorem directly reduces 1.47 to Fremlin's characterization:

1.52 Theorem (Fremlin)**.** *If $\mathcal{E}$ is a co-analytic metric space then $K(E)$ is Tukey-equivalent to 1,2,3 or 4.*

References

1.53 Definition. *Recall that a poset P is* Tukey-reducible *to Q if there is a cofinal function $f : Q \to P$. A poset P of size (or cofinality) at most ω_1 is said to have* Tukey-type $[\omega_1]^{<\omega}$ *if there is $P' \in [P]^{\omega_1}$ such that any infinite part of P' is unbounded (in P).*

1.54 Definition. PID *is the following statement. Whenever $\mathcal{I} \subseteq [\omega_1]^{\leq\omega}$ is an ideal then there are exactly two possibilities:*

1. *$(\exists K \in [\omega_1]^{\omega_1})([K]^{\leq\omega} \subseteq \mathcal{I})$ or*
2. *ω_1 can be partitioned into countably many sets $\{A_n : n < \omega\}$ such that for each $I \in \mathcal{I}$ and $n < \omega$ $I \cap A_n = \emptyset$.*