REPORT ON 'AXIOMATIC APPROACHES TO FORCING TECHNIQUES IN SET THEORY'

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1. The General Overview of the Program

While the Baire Category Theorem is a relatively simple mathematical result, it has played a central role on several levels in modern set theory. While its facility for building complex mathematical structures was hinted at in the work of Banach in the 1920s, its importance within set theory became especially apparent after Cohen's invention of the method of forcing in the early 1960s.

One interpretation of the Baire Category Theorem is that no compact topological space can be covered by countably many nowhere dense subsets. Shortly after the invention of forcing, the importance of the following question emerged: Can this assertion can be strengthened by replacing "countably many" by "at most θ" for some uncountable cardinal θ? The answer is negative, but it is possible if some restrictions are placed on the class of compact spaces. For instance if one restricts to the class of compact spaces with no uncountable family of pairwise disjoint open sets, then one obtains the statement of MA_θ, where MA abbreviates Martin's Axiom. In every reasonable class of compact spaces, however, the corresponding assertion about Baire category is not a theorem of ZFC.

For broader classes of compact spaces, it becomes much more natural to formulate the corresponding Baire category statement in terms of partial orders, also known as forcings. The existence of a point not contained in a collection of nowhere dense sets in a compact space translates into the existence of a filter which intersects all members of a collection of dense sets in a given forcing. Such a filter is said to be generic with respect to this collection of dense sets and the collection of dense sets is said to admit a generic filter. We say that the forcing axiom for a class of partial orders holds if whenever Q is in the class, any family of $\aleph_1$ many dense sets in Q admits a generic filter. This is just the Baire category assertion cast into the language of forcings.

Forcings which do not preserve the stationarity of a subset of ω_1 necessarily contain a family of $\aleph_1$ many dense sets which does admit a generic filter. Remarkably, this is the only obstruction to having a consistent forcing axiom.

Theorem (Foreman, Magidor, Shelah). [8] *The forcing axiom for partial orders which preserve stationary subsets of ω_1 is consistent relative to the existence of a supercompact cardinal.*

The forcing axiom specified in the above theorem is known as Martin's Maximum (MM). It is also known that MM implies $|\mathbb{R}| = \aleph_2$ and in particular that $\aleph_1$ in the formulation of MM can not be increased to $\aleph_2$ [8].

Our understanding of forcing and forcing axioms has evolved dramatically over the last decade with both the development of new techniques and the realization of the potential for future applications. This pursuit is timely in that it represents both the beginning of a deeper interaction with areas such as analysis and an area in which young researchers are increasingly active. The aim of the workshop was to train new researchers in the area and to deepen our understanding of the applications of forcing both to other areas of mathematics and to set theory itself.

While an emphasis was be placed on applications of forcing axioms, the program also dealt with other aspects of forcing, such as those relevant to proving results in ZFC. The goal was in part of explore new possibilities for applications in areas of mathematics such as analysis, ergodic theory, and group theory.

The study of forcing axioms takes many forms. On one hand, the initial interest in these axioms arose from their ability to settle a large number of statements arising in mathematics outside of set theory and foundations. This trend has continued up to the present.

Theorem (Moore). [9] *Assume MM. Every uncountable linear order contains an isomorphic copy of one of the following: X, ω_1, $-\omega_1$, C, $-C$. Here X is any set of reals of cardinality $\aleph_1$ and C is any Countryman line.*

Theorem (Todorcevic). [17] *Assume MM. If B is an infinite dimensional Banach space of density at most $\aleph_1$ then B has an infinite dimensional separable quotient.*

While direct applications of MM generally require specialized knowledge of set theory, there are an increasing number of combinatorial principles that follow from MM which are at the same time powerful and approachable by the non specialist. Both applying them and isolating new principles of this type is an important theme in set theoretic research. Two such principles are the P-Ideal Dichotomy and Todorcevic's formulation of the Open Coloring Axiom (see [13], [15]):

PID: If X is a set and $\mathcal{I}$ is an ideal of countable subsets of X which is countably directed under mod finite containment, then either

(1) there is an uncountable $Z \subseteq X$, all of whose countable subsets are in $\mathcal{I}$ or
(2) X can be covered by countably many sets none of which contain an infinite subset in $\mathcal{I}$.

OCA: If G is a graph on a separable metric space whose edge set is topologically open, then either

(1) G contains an uncountable complete subgraph or
(2) G admits a vertex coloring with countably many colors.

We will now describe some ways in which these principles have been applied. Recall that the Calkin Algebra is the quotient of the bounded operators on a separable Hilbert space modulo the compact operators.

Theorem (Farah). [7] *Assume OCA. All automorphisms of the Calkin algebra are inner.*

Von Neumann asked whether the countable chain condition and weak distributivity were sufficient conditions to ensure the existence of a strictly positive measure on a complete Boolean algebra. Maharam broke this problem into two halves: whether these conditions implied the existence of a strictly positive continuous submeasure and whether the existence of such a submeasure implies the existence of a strictly positive measure. The PID played an important role in the solution to the first of these problems.

Theorem (Balcar, Jech, Pazak). [2] *Assume PID. If a complete Boolean algebra satisfies the countable chain condition and is weakly distributive, then it supports a strictly positive continuous submeasure.*

The latter problem was solved negatively by Talagrand [11].

In some cases, applications of forcing axioms yield ZFC theorems as bi-products. For instance, Todorcevic modified the proof of the above result to prove the following.

Theorem (Todorcevic). [16] *If a complete Boolean algebra satisfies the σ-bounded chain condition and is weakly distributive, then it supports a strictly positive continuous submeasure.*

Todorcevic's structural analysis of Rosenthal compacta is another example [14].

In addition to exploring further applications of forcing axioms, an emphasis in the workshop was also placed on exploring uses of the method of forcing itself outside of set theory. The workshop included participants both from within set theory as well as other areas of mathematics where the organizers felt there was potential for future applications of the method of forcing.

2. Activities and progress made during the workshop

2.1. **General structure.** The activities of the workshop were organized into several different components. The first was a three part tutorial by Moore on forcing and forcing axioms which provided a foundation for the rest of the meeting. The aim of the tutorial was to present forcing to those working outside of set theory who had minimal knowledge of forcing. An emphasis was placed on uses of forcing to prove results within the standard axioms of mathematics. This then laid the framework for several semi-expository talks which covered more specialized material but in a way which was meant to be accessible to the entire audience. This was then complemented by a relatively small number of research talks, often by junior researchers. Those coming from outside the subject presented expository talks aimed at presenting problems from other areas of mathematics to a set theoretic audience. Finally, there were a series of lengthy informal evening discussions to help facilitate further interaction.

2.2. **Summary of the talks.** The program began with a lecture by Slawomir Solecki explaining his approach to inductive proofs of various finite Ramsey-theoretic statements (see [10]). The approach uses the algebraic structure of a partial semigroup in order to reduce a given finite Ramsey statement to a simple pigeonhole principle. This is quite analogous to the scheme in infinite Ramsey theory which uses combinatorial forcing in order to make the same sort of reduction (see [19]).

There were few other talks by researchers in other areas of mathematics exposing problems where forcing technique might be relevant. Typical such talks were given by Richard Haydon and Marc Sapir, who are experts in the Banach space geometry and geometric group theory, respectively. For example, Haydon introduced the audience to a well-known problem about characterizing injective Banach spaces which calls for examples that could be obtained using the method of forcing. On the other hand, Sapir gave an overview of asymptotic cone constructions and the possible role set theory might play in addressing open problems which relate to them.

The talks by Marcin Sabok and Benjamin Weiss concerned Borel equivalence relations and dynamical systems, respectively. Sabok's talk concerned dichotomy theorems for Borel equivalence relations which were both inspired by and proved using the method of forcing. The talk by Weiss explained what he calls *generic dynamics*, the study of dynamical systems modulo the ideal of meager sets. This is naturally related to Cohen forcing. He began by recalling the difference between generic dynamics and the better known measurable dynamics. His talk contained an exposition of a problem of Dennis Sullivan which asked whether every action of a countable group on a Polish space is equivalent modulo a meager set to an action of $\mathbb{Z}$. The problem was motivated by Marczewski-Szpilrajn problem concerning whether the sets in the Banach-Tarski paradox can be Baire measurable, a problem solved later by Dougherty and Foreman [5] [6]. However Weiss's lecture opened up a renewed interest in the problems of generic dynamics and their exact relationship with the method of forcing.

The talks by Christina Brech, Piotr Koszmider and Jordi Lopez-Abad were about direct applications of the method of forcing to problems from the geometry of Banach spaces. The talks by Brech and Koszmider examined forcing extensions which do not have universal object in a class of Banach spaces of a given density [3]. The technique of utilizing the homogeneity of a forcing played a prominent role in their results. For example Brech's talk contained a proof that after adding $\aleph_2$ Cohen reals to a model of CH the function space $C([0, \omega_2])$ is not isomorphic to a subspace of ℓ_∞/c_0, supplementing the classical result of Kunen who showed that in the Cohen model the quotient algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ does not contain a chain of order type ω_2. The talk of Lopez-Abad was about the general version of the unconditional basic sequence problem and its relationship to large cardinals. It also described a specific forcing extension which satisfies that every Banach space of density at least $\aleph_\omega$ has an infinite unconditional basic sequence [4].

The talk by Antonio Aviles, while not directly related to forcing, did mention some applications of dichotomies about analytic gaps in $\mathcal{P}(\omega)/\mathrm{Fin}$ originally inspired by the Open Coloring Axiom and the theory of forcing axioms. The analysis of such graphs has received renewed attention in part because of the work of Aviles and Todorcevic [1].

Itay Neeman gave us an extensive overview of his most recent results on the new generation of Forcing Axioms that work at the level of $\aleph_2$ rather than $\aleph_1$. It involved some deep and intricate analysis of the side condition methods with models of two different cardinalities. He managed to state his Forcing axiom and hint towards its consequences. Compared to our current understanding of MM, forcing axioms at this higher level are still at the very early stages of their development. Still, this represents one of the most exciting developments within pure set theory and the extent to which forcing can be used to influence the combinatorics of ω_2. In his talk Miguel Mota presented a related approach to producing models of forcing axioms on both levels $\aleph_1$ and $\aleph_2$ in which the continuum is larger than $\aleph_2$, long an obstacle within the subject.

We were particularly pleased by two fine talks by two of the youngest participants of the workshop, Vera Fischer and Teruyuki Yorioka. For example, Fischer was introducing to us the template forcing originally invented by Shelah in order to solve a problem about cardinal characteristics of the continuum and further perfected by Brendle and others. In particular, Fischer was presenting the intricate definitions of the template forcing having in mind her own recent application of the method. Yorioka presented his new preservation results for statements of the form $\mathfrak{x} = \omega_1$ where $\mathfrak{x}$ is some standard characteristics of the continuum. This touches an old problem asking if any other cardinal invariant of the continuum could admit the oscillation theory similar to the oscillation theory of bounding cardinal $\mathfrak{b}$ introduced by Todorcevic in the early 1980's [12] [13].

One of the main features of the formal part of the Workshop were the three tutorials given by Justin Moore and one by Stevo Todorcevic about applying the technique of Forcing for proving theorems rather than consistency results. This is a subject that is growing in recent years and a feature of the technique of forcing that is particularly relevant for applications. In fact one of the main goals of this particular workshop is to make the researchers from other areas aware of this. The tutorial of Moore started with listing strong analogies between the technique of Forcing and probability theory. He gave us several proofs in order to exemplify this sort of applications of Forcing. For example, particularly well received his presentation of the proof of the difficult Ramsey-theoretic statement for trees due to Halpern and Läuchli that uses the forcing for adding many Cohen reals. Moore moreover prepared 35 pages of lecture notes which he distributed prior to the workshop. In addition to containing further details on the examples presented in the tutorials, it contained several other examples of how forcing can be used to prove results within the standard axioms of mathematics. Currently there are plans to further flesh out the notes and publish them. Todorcevic's part of the tutorial was devoted to an application of Forcing in order to show that some standard large cardinal axioms of set theory imply that universally meager metric spaces must be σ-discrete [18].

In addition to the scheduled talks, the workshop contained lengthy evening discussion sessions, typically lasting two hours or more. These took place every day except Friday and started after dinner. In a given day, participants of the workshop would use this opportunity to discuss the lectures of the day. This typically but not always meant that they would ask the lecturers for clarifications and further information. For example, after Haydon's talk, Aviles gave an additional informal lecture on the injective Banach space problem in which he mentioned a rich array of related questions. These discussion sessions were particularly useful as a support to the tutorial lectures where not only the presented material was discussed but also further results were mentioned that could not be fitted in during the tutorial in the given day. It was a pleasant experience to see that the participants from other areas of mathematics were quite actively following the workshop and especially the tutorials whose original plan was exactly to reach such audience. Another success of the workshop is its problem session where essentially everybody participated. The result was a rich array of problems coming from different areas of mathematics and we sincerely hope that our workshop will be directly responsible for the solution of some of these problems in the near future.

3. Schedule of the Workshop

Monday

7:00–8:45	Breakfast
8:45–9:00	Introduction and Welcome by BIRS Station Manager, TCPL
9:00-9:10	opening remarks (Justin Moore)
9:10-10:10	Slawomir Solecki (TCPL, 202)
10:10-10:40	Coffee Break (TCPL lobby)
10:40-11:40	Marcin Sabok (TCPL, 202)
11:30–13:00	Lunch
13:00–14:00	Guided Tour of The Banff Centre
15:00–15:30	Coffee, TCPL
15:30-16:30	Justin Moore (TCPL, 201)
16:40-17:40	Antonio Aviles (TCPL, 201)
17:40–19:30	Dinner
20:00–22:00	Discussion/problem session (TCPL, 202)

Tuesday

7:00–9:00	Breakfast
9:00-10:00	Mark Sapir (TCPL, 202)
10:00-10:30	Coffee Break (TCPL lobby)
10:30-11:30	Richard Haydon (TCPL, 202)
11:30	Group Photo; meet on the front steps of Corbett Hall
11:30–13:00	Lunch
14:00-15:00	Justin Moore (TCPL, 202)
15:00–15:30	Coffee, TCPL
15:30-16:30	Justin Moore (TCPL, 202)
16:40-17:40	Stevo Todorcevic (TCPL, 202)
17:40–19:30	Dinner
20:00–22:00	Discussion/problem session (TCPL, 202)

Wednesday

7:00–9:00	Breakfast
9:00-10:00	Itay Neeman (TCPL, 202)
10:00-10:30	Coffee Break (TCPL lobby)
10:30-11:30	Christina Brech (TCPL, 202)
11:30–13:30	Lunch
	Free Afternoon
17:30–19:30	Dinner
20:00–22:00	Discussion/problem session (TCPL, 202)

Thursday

7:00–9:00 Breakfast

9:00-10:00 Benjamin Weiss (TCPL, 202)
10:00-10:30 Coffee Break (TCPL lobby)
10:30-11:30 Piotr Koszmider (TCPL, 202)

11:30–13:00 Lunch

15:00–15:30 Coffee, TCPL

15:30-16:30 Jordi Lopez-Abad (TCPL, 202)
16:40-17:40 Vera Fischer (TCPL, 202)

17:40–19:30 Dinner

20:00–22:00 Discussion/problem session (TCPL, 202)

Friday

7:00–9:00 Breakfast

9:00-9:40 Teruyuki Yorioka (TCPL, 202)
9:40-10:20 Coffee Break (TCPL lobby)
10:20-11:00 Miguel Angel Mota (TCPL, 202)

11:30–13:00 Lunch

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