The consistency of a club–guessing failure at the successor of a regular cardinal

David Asperó

University of East Anglia

Philip Welch's 60th birthday

Bristol, 23 March 2014

Club–Guessing principles are well–studied weakenings of Jensen's $\diamondsuit_\kappa$, for suitable cardinal κ, where the guessing device is a club–sequence $(C_\delta : \delta \in S)$ for $S \subseteq \kappa$ (i.e., $C_\delta \subseteq \delta$ is a club for all δ) and the relevant guessing applies to clubs $C \subseteq \kappa$ rather than arbitrary subsets of κ.

Club–Guessing, and related principles, at the level of ω_1 can be easily manipulated by forcing. On the other hand, instances of Club–Guessing at higher regular cardinals are often ZFC theorems.

Well–known example (Shelah): For every infinite regular cardinal κ and every stationary $S \subseteq S^{\kappa^{++}}_{\kappa} := \{\alpha \in \kappa^{++} : \text{cf}(\alpha) = \kappa\}$ there is a club–sequence $(C_\delta : \delta \in S)$ such that

- each C_δ is a club of δ of order type κ, and
- for every club $E \subseteq \kappa^{++}$ there is some $\delta \in S$ such that $C_\delta \subseteq E$.

Notation: $X(\delta)$ is the δ-th member of X if X is a set of ordinals.

Another (not so well–known) example:

Theorem

(Shelah, Claim 3.3 in Colouring and non-productivity of $\aleph_2$–c.c., Ann. Pure and Applied Logic, vol. 84, 2 (1997), 153–174)

Let $\kappa \geq \omega_1$ be a regular cardinal. Then for every stationary $S \subseteq S^{\kappa^+}_{\kappa}$ there is a club–sequence $\langle C_\delta \mid \delta \in S \rangle$ such that for all $\delta \in S$,

- $\mathrm{ot}(C_\delta) = \kappa$, *and*
- $\mathrm{cf}(C_\delta(\alpha + 1)) = \kappa$ *for all* $\alpha < \kappa$,

and such that for every club $D \subseteq \kappa^+$ there is some $\delta \in S$ (equivalently, stationary many $\delta \in S$) such that

$$\{\alpha < \kappa \mid C_\delta(\alpha + 1) \in D\}$$

is stationary.

See also D. Soukup and L. Soukup, *Club guessing for dummies* for a nicely written proof of the above.

Question (Shelah, Question 5.4 in *On what I do not understand (and have something to say): Part I*, Fundamenta Math., vol. 166, 1–2 (2000), 1–82.)

Is it true in ZFC that for every regular cardinal $\kappa \geq \omega_1$ there is a club–sequence $\vec{C} = \langle C_\delta \mid \delta \in S^{\kappa^+}_{\kappa} \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ and such that for every club $D \subseteq \kappa^+$ there is some δ such that

$$\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$

is stationary?

See also D. Soukup and L. Soukup, *Club guessing for dummies* for a nicely written proof of the above.

Question (Shelah, Question 5.4 in *On what I do not understand (and have something to say): Part I*, Fundamenta Math., vol. 166, 1–2 (2000), 1–82.)

Is it true in ZFC that for every regular cardinal $\kappa \geq \omega_1$ there is a club–sequence $\vec{C} = \langle C_\delta \mid \delta \in S^{\kappa^+}_{\kappa} \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ and such that for every club $D \subseteq \kappa^+$ there is some δ such that

$$\{\alpha < \kappa \mid \{C_\delta(\alpha + 1), C_\delta(\alpha + 2)\} \subseteq D\}$$

is stationary?

According to Shelah in the above paper, if there is a club–sequence as in the above question on $S^{\kappa^+}_{\kappa}$ and GCH holds, then there is a κ^+–Souslin tree. In particular, an affirmative answer to above question would yield an affirmative answer to the following well–known open question.

Question

Does GCH imply that there is an ω_2–Souslin tree?

Theorem

It is consistent that there is a regular cardinal $\kappa \geq \omega_1$ such that there is no club–sequence $\vec{C} = \langle C_\delta \mid \delta \in S^{\kappa^+}_\kappa \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ and such that for every club $D \subseteq \kappa^+$ there is some δ such that

$$\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$

is stationary.

This theorem answers Shelah's question negatively.

Proof by variation of method of finite–support iterations with symmetric systems as side conditions (see next).

Finite–support iterations with symmetric systems as side conditions

Definition

(Shelah) A partial order $\mathbb{P}$ is *proper* if for every (equiv. for some) regular θ such that $\mathbb{P} \in H(\theta)$ there is a club $E \subseteq [H(\theta)]^{\aleph_0}$ such that for every $N \in E$ and every $p \in \mathbb{P} \cap N$ there is $q \leq_{\mathbb{P}} p$ such that q forces $D \cap \dot{G} \cap N \neq \emptyset$ for every dense set $D \subseteq \mathbb{P}$ (equiv., every maximal antichain $D \subseteq \mathbb{P}$), $D \in N$. We say that q *is* $(N, \mathbb{P})$*–generic*.

Proper forcing is nice:

- Every proper forcing preserves $\aleph_1$.
- The class of proper forcings is quite large: It contains c.c.c., σ–closed, the forcing for threading a $\square_\kappa$–sequence, ...
- Every countable support iteration of proper forcings is proper.

Notation: Given a class Γ of partial orders and a cardinal κ, $\mathsf{FA}(\Gamma)_\kappa$ means: For every $\mathbb{P} \in \Gamma$ and every collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$, if $|\mathcal{D}| \leq \kappa$, then there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

(Baumgartner–Shelah) From the above it follows: Starting from a supercompact cardinal, one can easily perform a countable support iteration and obtain a model of $\mathrm{FA}(\{\mathbb{P} : \mathbb{P} \text{ proper}\})_{\aleph_1}$, a.k.a. PFA, which is a very powerful forcing axiom.

General fact: For any ordinal α of cofinality ω_1, any countable support iteration of non-trivial forcings of length α always adds Cohen subsets of ω_1 over all intermediate models. Hence, $2^{\aleph_0} \leq \aleph_2$ hold in any extension by a countable support iteration of length κ with $\mathrm{cf}(\kappa) \geq \omega_1$.

In fact, $\mathrm{FA}(\{\mathbb{P} : \mathbb{P} \text{ proper}\})_{\aleph_2}$ is false, and PFA even implies $2^{\aleph_0} = \aleph_2$.

Suppose we want to build a model of some consequence of PFA (or something we should be able to force by iterating instances of proper forcing) together with $2^{\aleph_0} \geq \aleph_3$.

Countable iterations won't do by the above general fact.

Bigger supports don't work either: The preservation lemma for properness doesn't work for them.

Finite supports don't work either; in fact, any finite support iteration of non–c.c.c. forcings of length ω collapses ω_1.

A solution: Use finite supports, together with countable elementary substructures of some $H(\theta)$ as side conditions affecting the whole iteration or initial segments of the iteration in order to ensure properness (the idea of using countable structures as side conditions in order to “force” a non–proper forcing to become proper is old (Todorčević, 1980’s, implicit in work of Baumgartner), but this was not done in the context of actual iterations).

Typically we will want our iteration to have the $\aleph_2$–c.c. (after all we are interested in $2^{\aleph_0}$ arbitrarily large). The natural approach of using finite $\in$–chains of structures won’t work, though, since we have too many structures. We will replace $\in$–chains of structures by “matrices” of structures with suitable symmetry properties. If we start with CH and consider only iterands with the $\aleph_2$–c.c., we might succeed.

General template of the constructions: Start with CH, let κ be regular with $2^{<\kappa} = \kappa$, and let $\Phi : \kappa \longrightarrow H(\kappa)$ such that $\Phi^{-1}(a) \subseteq \kappa$ unbounded for all $a \in H(\kappa)$. Let $(\mathcal{P}_\alpha \,:\, \alpha \leq \kappa)$ such that for all $\alpha \leq \kappa$, a condition in $\mathcal{P}_\alpha$ is a pair $q = (F, \Delta)$ such that:

(1) F is a finite function with $\mathrm{dom}(F) \subseteq \alpha$.

(2) Δ is a finite set of pairs (N, γ), with $(N, \in, \Phi) \prec (H(\kappa), \in, \Phi)$, $|N| = \aleph_0$, and γ is an ordinal with $\gamma \leq \alpha$ and $\gamma \leq \sup(N \cap \kappa)$.

(3) $\{N \,:\, (N, \gamma) \in \Delta \text{ for some } \gamma\}$ is a finite "symmetric" system of structures.

(4) For all $\beta < \alpha$,

$$q_\beta = (F \restriction \beta, \{(N, \min\{\beta, \gamma\}) : (N, \gamma) \in \Delta)\}$$

is a $\mathcal{P}_\beta$–condition.

(5) For every $\xi \in \mathrm{dom}(F)$:

(5.1) If $\Phi(\xi)$ is a $\mathcal{P}_\xi$–name for a condition in Γ – where Γ is a suitable subclass of proper forcing notions with the $\aleph_2$–c.c. (fixed for the construction) – then $q_\xi \Vdash_\xi F(\xi) \in \Phi(\xi)$; otherwise, $F(\xi) = 0$.

(5.2) For every $(N, \gamma) \in \Delta$, if $\Phi(\xi)$ is as above and $\xi \in N \cap \gamma$, then $q|_\xi \Vdash_\xi F(\xi)$ is $(N[\dot{G}_\xi], \Phi(\xi))$–generic.

Given $\mathcal{P}_\alpha$–conditions q^0, q^1, $q^1 \leq_\alpha q^0$ iff

(a) for all $\beta < \alpha$, $q^1|_\beta \leq_\beta q^0|_\beta$,

(b) $\Delta_{q^0} \subseteq \Delta_{q^1}$, and

(c) if $\alpha = \alpha_0 + 1$ and $\Phi(\alpha_0)$ is as in (5.1), then $q^1|_{\alpha_0} \Vdash_{\alpha_0} F_{q^1}(\alpha_0) \leq_{\Phi(\alpha_0)} F_{q^0}(\alpha_0)$.

- It is immediate to see that $\mathcal{P}_\beta$ is a complete suborder of $\mathcal{P}_\alpha$ if $\beta < \alpha$: Thanks to the use of the markers γ in $(N, \gamma) \in \Delta$.
- Using CH and standard Δ–system arguments it is not difficult to see that all $\mathcal{P}_\alpha$ have the $\aleph_2$–c.c.
- Properness of $\mathcal{P}_\alpha$: We define a sequence $(\mathcal{M}_\alpha)_{\alpha \le \kappa}$ of clubs of $[H(\kappa)]^{\aleph_0}$ (e.g. let $(\theta_\alpha)_{\alpha \le \kappa}$ be a sufficiently fast increasing sequence of regular cardinals above 2^κ and let $\mathcal{M}_\alpha$ be the set of $N^* \cap H(\kappa)$ such that $|N^*| = \aleph_0$, $N^* \prec (H(\theta_\alpha), \in)$, and $\Phi, (\theta_\beta)_{\beta < \alpha} \in N^*$).

  The proof of properness is by induction on α. We prove that if $N^* \prec H(\theta)$ countable, $N = N^* \cap H(\kappa) \in \mathcal{M}_{\alpha+1}$ and $(N, \min\{\alpha, \sup(N \cap \kappa)\}) \in \Delta_q$, then q is $(N^*, \mathcal{P}_\alpha)$–generic. It is crucial that supports are finite for the induction to run. At limit stages α of cofinality ω_1 we use the symmetry of $\{N : (N, \gamma) \in \Delta_q \text{ for some } \gamma\}$ for $q \in \mathcal{P}_\alpha$.

An application: A generalisation of Martin's Axiom

Recall: $\mathrm{MA}_\kappa = \mathrm{FA}(\{\mathcal{P} : \mathcal{P} \text{ c.c.c.}\})_\kappa$. Consistent by Solovay–Tennenbaum by iterating c.c.c. forcing with finite supports.

For every κ, MA_κ consistent with any form of Club Guessing at ω_1 simply because MA_κ can be forced by a c.c.c. forcing and every club of ω_1 in a c.c.c. extension includes a club from the ground model.

On the other hand, MA_{ω_1} is consistent also with $\neg\,\mathrm{WCG}$ and other strong failures of Club–Guessing at ω_1 (where $\neg\,\mathrm{WCG}$ means "For every club–sequence $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ with $\mathrm{ot}(C_\delta) = \omega$ for all δ there is a club $D \subseteq \omega_1$ such that for all $\delta \in D$, $C_\delta \cap D$ is finite"): Since PFA implies these failures of Club–Guessing and PFA implies MA_{ω_1}.

A natural question at this point: Does any axiom MA_κ (for $\kappa > \omega_1$) imply any nontrivial form of Club–Guessing at ω_1? (Traditional forcing techniques don't seem to work for this.)

Definition

(A–Mota) A partial order $\mathcal{P}$ is $\aleph_{1.5}$–c.c. iff there is θ such that $\mathcal{P} \in H(\theta)$ and there is a club $E \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq E$ and every $p \in \mathbb{P}$, if $p \in N$ for some $N \in \mathcal{N}$ with $\delta_N = \min\{\delta_M : M \in \mathcal{N}\}$, then there $q \leq_{\mathcal{P}} p$ such that q is $(N, \mathcal{P})$–generic for all $N \in \mathcal{N}$.

(Notation: $\delta_N = N \cap \omega_1$ if $N \cap \omega_1$ is an ordinal.)

Obvious: Every c.c.c. poset is $\aleph_{1.5}$–c.c. and every $\aleph_{1.5}$–c.c. poset is proper.

Easy exercise: Every $\aleph_{1.5}$–c.c. poset is $\aleph_2$–c.c.

A natural question at this point: Does any axiom MA_κ (for $\kappa > \omega_1$) imply any nontrivial form of Club–Guessing at ω_1? (Traditional forcing techniques don't seem to work for this.)

Definition

(A–Mota) A partial order $\mathcal{P}$ is $\aleph_{1.5}$–c.c. iff there is θ such that $\mathcal{P} \in H(\theta)$ and there is a club $E \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq E$ and every $p \in \mathbb{P}$, if $p \in N$ for some $N \in \mathcal{N}$ with $\delta_N = \min\{\delta_M : M \in \mathcal{N}\}$, then there $q \leq_{\mathcal{P}} p$ such that q is $(N, \mathcal{P})$–generic for all $N \in \mathcal{N}$.

(Notation: $\delta_N = N \cap \omega_1$ if $N \cap \omega_1$ is an ordinal.)

Obvious: Every c.c.c. poset is $\aleph_{1.5}$–c.c. and every $\aleph_{1.5}$–c.c. poset is proper.

Easy exercise: Every $\aleph_{1.5}$–c.c. poset is $\aleph_2$–c.c.

Definition

(A–Mota) Given a cardinal κ, $\text{MA}^{1.5}_{\kappa}$ is
$\text{FA}(\{\mathcal{P} : \mathcal{P} \text{ is } \aleph_{1.5}\text{–c.c.}\})_{\kappa}$.

Theorem

(A–Mota) (CH) Let $\kappa \geq \omega_2$ be a regular cardinal such that $\mu^{\aleph_0} < \kappa$ for all $\mu < \kappa$ and $\diamondsuit(\{\alpha < \kappa : \text{cf}(\alpha) \geq \omega_1\})$ holds. Then there is a proper forcing notion $\mathcal{P}$ of size κ with the $\aleph_2$–c.c. such that the following statements hold in the generic extension by $\mathcal{P}$:

(1) $2^{\aleph_0} = \kappa$

(2) $\text{MA}^{1.5}_{<2^{\aleph_0}}$

Proof is by finite support iteration with symmetric systems of countable structures as side conditions.

Some applications of $\mathrm{MA}_\kappa^{1.5}$:

1. MA_κ
2. The product of any two $\aleph_{1.5}$–c.c. posets is $\aleph_2$–c.c.
3. $\neg\mho$
4. For every $\tau < \omega_1$ and every $\mathcal{C} \subseteq \mathcal{P}(\omega_1)$, if $|\mathcal{C}| \leq \kappa$ and $\mathrm{ot}(X) \leq \tau$ for all $X \in \mathcal{C}$, then there is a club $D \subseteq \omega_1$ such that $D \cap X$ is finite for all $X \in \mathcal{C}$.

In particular, (4) answers, in a very strong sense, the above question on MA vs. Club–Guessing at ω_1. Also, we don't know how to obtain (4) by any other method.

Recent work of Neeman on higher analogs of PFA using finite iterations with "side conditions of different types": Consistency of $\mathsf{FA}(\{\mathbb{P} : \mathbb{P} \text{ relaxed 2–size proper forcing}\})_{\aleph_2}$.

$\aleph_{1.5} \subseteq$ relaxed 2–size proper. On the other hand, $2^{\aleph_0} = \aleph_3$ in Neeman's model; in fact,
$\mathsf{FA}(\{\mathbb{P} : \mathbb{P} \text{ relaxed 2–size proper forcing}\})_{\aleph_3}$ is false.

The consistency of a club–guessing failure at κ^+

Theorem

Let $\omega_1 \leq \kappa < \kappa^{++} \leq \theta$ be regular cardinals such that $2^{<\kappa} = \kappa$, $2^\kappa = \kappa^+$ and $2^{<\theta} = \theta$. Then there is a partial order $\mathcal{P}$ with the following properties.

1. *$\mathcal{P}$ is $<\kappa$–closed and κ^{++}–c.c.*
2. *There is some $\Phi \in H(\theta^+)$ such that $\mathcal{P}$ is proper with respect to all $N \prec H((2^\theta)^+)$ such that $\mathcal{P}, \Phi \in N$, $|N| = \kappa$ and ${}^{<\kappa}N \subseteq N$.*
3. *$\mathcal{P}$ forces that for every club–sequence $\langle C_\delta \mid \delta \in S^{\kappa^+}_\kappa \rangle$ with $\mathrm{ot}(C_\delta) = \kappa$ for all δ there is a club $D \subseteq \kappa^+$ such that*
$$\{\alpha < \kappa \mid \{C_\delta(\alpha+1), C_\delta(\alpha+2)\} \subseteq D\}$$
is bounded in κ for all δ.
4. *$\mathcal{P}$ forces $2^\mu = \theta$ for every $\mu \in [\kappa, \theta)$.*

Proof is

- by $<\kappa$–support iteration of natural forcing for adding, by $<\kappa$–approximations, clubs of κ^+ killing the relevant club–guessing of club–sequences picked by bookkeeping function, and
- using symmetric $<\kappa$–systems of structures N such that $|N| = \kappa$ and ${}^{<\kappa}N \subseteq N$.

The proof of relevant form of properness is not by induction. It is a direct construction.

Adding many Baumgartner clubs

Cohen's forcing $2^{<\omega}$ for adding a real is perhaps the simplest non-trivial forcing notion one can think of (and the first to be discovered).

A simple and nicely behaved forcing for adding any arbitrary number of Cohen reals: $\mathrm{Add}(\omega, X)$ = the partial order of finite functions $p \subseteq X \times 2^{<\omega}$, ordered by reverse inclusion.

For every $\mathrm{Add}(\omega, X)$–generic G and every $\alpha \in X$, $r_\alpha^G := \bigcup\{p(\alpha) \mid p \in G\}$ is a Cohen real over V and $r_\alpha^G \neq r_{\alpha'}^G$ for $\alpha \neq \alpha'$ in X.

Also:

- $\mathrm{Add}(\omega, X)$ has the c.c.c.,
- It is homogeneous (in the sense that given any p, $p' \in \mathrm{Add}(\omega, X)$ there are $q \leq p$ and $q' \leq p'$ such that $\mathrm{Add}(\omega, X) \restriction q \cong \mathrm{Add}(\omega, X) \restriction q'$).
- It can be naturally represented as the product of $\mathrm{Add}(\omega\, X_0)$ and $\mathrm{Add}(\omega, X_1)$ for any partition (X_0, X_1) of X. In particular, for every G as above and all $\alpha \neq \alpha'$ in X, $r^G_{\alpha'}$ is Cohen generic generic over $V[r^G_\alpha]$.

Cohen forcing and $\mathrm{Add}(\omega, \theta)$ have of course been extensively studied for more than 50 years now. For example, $\mathrm{Add}(\omega, \theta)$ is the forcing that Cohen used to prove the consistency of $\neg\,\mathrm{CH}$ (by forcing over L).

A prominent $\aleph_{1.5}$–c.c. forcing: Baumgartner's forcing $\mathbb{B}$ for adding a club of ω_1 with finite conditions: $p \in \mathbb{B}$ iff $p \subseteq \omega_1 \times \omega_1$ is a finite function that can be extended to a strictly increasing and continuous function $F : \omega_1 \longrightarrow \omega_1$. Extension: Reverse inclusion.

If G is $\mathbb{B}$–generic, $F = \bigcup G$ is the enumerating function of a new club of ω_1.

- $|\mathbb{B}| = \aleph_1$.
- $\mathbb{B}$ is $\aleph_{1.5}$–c.c.: If $p \in \mathbb{B}$ and $N_0, \ldots N_m$ are elementary substructures with $\mathrm{range}(p) \subseteq \delta_{N_j}$ for all j, then $p \cup \{(\delta_{N_0}, \delta_{N_0}), \ldots (\delta_{N_m}, \delta_{N_m})\}$ is $(N_j, \mathbb{B})$–generic for all j.
- $\mathbb{B}$ is absolute between models agreeing on ω_1.

Also, Zapletal proved:

- If $x \in \mathbb{R}$, $x^\sharp$ exists, and $\mathcal{P} \in L[x]$ is a non–atomic partial order on ω_1^V, then $\mathcal{P}$ is forcing–equivalent to the disjoint sum of some number of copies of forcings in

$$\{2^{<\omega}, \mathrm{Add}(\omega, \omega_1), \mathbb{B}, \mathrm{Coll}(\omega, \omega_1)\}$$

- (PFA) $\mathbb{B}$ is a minimal nowhere c.c.c. poset (i.e., not c.c.c. below any condition), in the sense that every nowhere c.c.c. poset adds a generic for $\mathbb{B}$.
- If $P = \{p_\alpha \mid \alpha < \omega_1\}$ is a proper nowhere c.c.c. forcing notion adding a club $C \subseteq \omega_1$ such that for all $\alpha \in C$, $\dot{G} \cap \{p_\beta \mid \beta < \alpha\}$ is generic for $\{p_\beta \mid \beta < \alpha\}$ (where $\dot{G}$ denotes the generic filter), then $\mathrm{RO}(P) = \mathrm{RO}(\mathbb{B})$

Given a set X of ordinals, there is a forcing, which I will denote by $\mathrm{Add}_{\mathbb{B}}(X)$, which is quite simple to define and which has the following properties.

(1) For every $\mathrm{Add}_{\mathbb{B}}(X)$–generic G and every $\alpha \in X$ one can naturally extract a Baumgartner club C^G_α from G. Furthermore, $C^G_\alpha \neq C^G_{\alpha'}$ for all distinct α, α' in X.

(2) $\mathrm{Add}_{\mathbb{B}}(X)$ is proper and has the $\aleph_2$–c.c. In fact it is $\aleph_{1.5+\epsilon}$–c.c.

(3) $\mathrm{Add}_{\mathbb{B}}(X)$ is homogeneous (in the above sense).

(4) For every partition (X_0, X_1) of X, $\mathrm{Add}_{\mathbb{B}}(X)$ can be naturally represented as the product $\mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$. In particular, if G is as in (1) and $\alpha \neq \alpha'$ are in X, then $C^G_{\alpha'}$ is $\mathbb{B}$–generic over $V[C^G_\alpha]$.

Why is something non–trivial necessary?

Fact

Both the finite support product of $\aleph_0$ copies of $\mathbb{B}$ and the countable support product of $\aleph_0$ copies of $\mathbb{B}$ collapse ω_1 (in fact, the countable support product of $\aleph_0$ copies of Cohen forcing collapses ω_1).

Why is something non–trivial necessary?

Fact

Both the finite support product of $\aleph_0$ *copies of* $\mathbb{B}$ *and the countable support product of* $\aleph_0$ *copies of* $\mathbb{B}$ *collapse* ω_1 *(in fact, the countable support product of* $\aleph_0$ *copies of Cohen forcing collapses* ω_1*).*

Definition: Let X be a set of ordinals. $\mathrm{Add}_{\mathbb{B}}(X)$ is the following forcing notion: Conditions in $\mathrm{Add}_{\mathbb{B}}(X)$ are pairs of the form $p = (f, \mathcal{N})$ with the following properties.

(1) f is a function with $\mathrm{dom}(f) \in [X]^{<\omega}$ and such that $f(\alpha) \in \mathbb{B}$ for every $\alpha \in \mathrm{dom}(f)$.

(2) $\mathcal{N}$ is a finite function with $\mathrm{dom}(\mathcal{N}) \subseteq X$ such that for every $\delta \in \mathrm{dom}(\mathcal{N})$,

  (2.1) δ is a countable indecomposable ordinal,

  (2.2) $\mathcal{N}(\delta)$ is a countable subset of X,

  (2.3) $\delta \in \mathrm{dom}(f(\alpha))$ and $f(\alpha)(\delta) = \delta$ for all $\alpha \in \mathrm{dom}(f) \cap \mathcal{N}(\delta)$, and

  (2.4) $\mathrm{ot}(\mathcal{N}(\delta')) < \delta$ for every $\delta' \in \mathrm{dom}(\mathcal{F}) \restriction \delta$.

Given $\mathrm{Add}_{\mathbb{B}}(X)$ conditions $(f_0, \mathcal{N}_0)$, $(f_1, \mathcal{N}_1)$, $(f_1, \mathcal{N}_1)$ extends $(f_0, \mathcal{N}_0)$ iff

- $\mathrm{dom}(f_0) \subseteq \mathrm{dom}(f_1)$ and $f_0(\alpha) \subseteq f_1(\alpha)$ for every $\alpha \in \mathrm{dom}(f_0)$, and
- $\mathrm{dom}(\mathcal{N}_0) \subseteq \mathrm{dom}(\mathcal{N}_1)$ and $\mathcal{N}_0(\delta) \subseteq \mathcal{N}_1(\delta)$ for every $\delta \in \mathrm{dom}(\mathcal{F}_0)$.

The definition of $\mathrm{Add}_{\mathbb{B}}(X)$ is a streamlined version of the previous finite–support iterations with symmetric systems.

Given $\alpha \in X$ and G generic for $\mathrm{Add}_{\mathbb{B}}(X)$, let $F_G^\alpha = \{f(\alpha) : (f, \mathcal{F}) \in G \text{ for some } \mathcal{F}\}$.

Fact

Given $\alpha \in X$ and a generic filter G for $\mathrm{Add}_{\mathbb{B}}(X)$, F_G^α is a generic filter for $\mathbb{B}$.

Given a condition $p = (f, \mathcal{N})$ in $\mathrm{Add}_{\mathbb{B}}(X)$ and $Y \subseteq X$, let

$$p \upharpoonright Y = (f \upharpoonright Y, \{\langle \delta, N \cap Y \rangle \mid \langle \delta, N \rangle \in \mathcal{N},\ \delta \in \mathrm{dom}(\mathcal{N})\})$$

Also: Given two functions $\mathcal{F}$ and $\mathcal{G}$, $\mathcal{F} \oplus \mathcal{G}$ is the function $\mathcal{H}$ with domain $\mathrm{dom}(\mathcal{F}) \cup \mathrm{dom}(\mathcal{G})$ such that

- $\mathcal{H}(x) = \mathcal{F}(x)$ for every $x \in \mathrm{dom}(\mathcal{F}) \setminus \mathrm{dom}(\mathcal{G})$,
- $\mathcal{H}(x) = \mathcal{G}(x)$ for every $x \in \mathrm{dom}(\mathcal{G}) \setminus \mathrm{dom}(\mathcal{F})$, and
- $\mathcal{H}(x) = \mathcal{F}(x) \cup \mathcal{G}(x)$ for every $x \in \mathrm{dom}(\mathcal{F}) \cap \mathrm{dom}(\mathcal{G})$.

Lemma

Let X_0, X_1 be disjoint sets of ordinals. Then, the function sending a pair $((f_0, \mathcal{F}_0), (f_1, \mathcal{F}_1)) \in \mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$ to $(f_0 \cup f_1, \mathcal{F}_0 \oplus \mathcal{F}_1)$ is an isomorphism between $\mathrm{Add}_{\mathbb{B}}(X_0) \times \mathrm{Add}_{\mathbb{B}}(X_1)$ and $\mathrm{Add}_{\mathbb{B}}(X_0 \cup X_1)$. The inverse of this function is the function sending $p \in \mathrm{Add}_{\mathbb{B}}(X)$ to $(p \upharpoonright X_0, p \upharpoonright X_1)$.

Lemma

$\mathrm{Add}_{\mathbb{B}}(X)$ *has the* $\aleph_2$*-c.c.*

Proof using the usual Δ–system lemma (for uncountable collections of finite sets). No need of CH.

Lemma
$\mathrm{Add}_{\mathbb{B}}(X)$ *is proper.*

Proof: Let $M \prec H(\lambda)$ countable, $X \in M$, $p = (f, \mathcal{F}) \in \mathrm{Add}_{\mathbb{B}}(X) \cap M$, and let $N = M \cap X$. Let f^* be s.t. $\mathrm{dom}(f^*) = \mathrm{dom}(f)$ and $f^*(\alpha) = f(\alpha) \cap \{\langle \delta_N, \delta_N \rangle\}$ for every $\alpha \in \mathrm{dom}(f)$ and let $p^* = (f^*, \mathcal{F} \cup \{\langle \delta_M, N \rangle\})$. Then p^* is a condition in $Add^B(X)$.

We prove by induction on γ that if $\gamma \in N$, then $p^* \restriction (X \cap \gamma)$ is $(M, \mathrm{Add}_{\mathbb{B}}(X \cap \gamma))$–generic:

Cases $\gamma = 0$, $\gamma = \gamma_0 + 1$, and $\mathrm{cf}(\gamma) \neq \emptyset$ easy or not so interesting.

Case $\mathrm{cf}(\gamma) = \omega_1$: Let $A \in M$ a maximal antichain of $\mathrm{Add}_{\mathbb{B}}(X \cap \gamma)$, $p' = (f', \mathcal{F}') \leq p^* \upharpoonright (X \cap \gamma)$, $p' \leq r \in A$. Want to see p' compatible with $q \in A \cap M$.

Let $\sigma \in \gamma \cap M$ be such that $\mathrm{dom}(f' \upharpoonright \sup(M \cap \gamma)) < \sigma$ and let $(\gamma_\nu)_{\nu<\omega_1} \in M$ be a strictly increasing and continuous sequence of ordinals converging to γ. Let D be the set of all conditions in $\mathrm{Add}_{\mathbb{B}}(X \cap \gamma)$ extending some condition in A. Let G be $\mathrm{Add}_{\mathbb{B}}(X \cap \sigma)$–generic with $p' \upharpoonright (X \cap \sigma) \in G$, and let $C \in M[G]$ be the club of $\nu \in \omega_1 = \omega_1^{V[G]}$ – where the equality holds by induction hypothesis – such that for every $\nu' < \nu$ there is some $q = (g, \mathcal{G}) \in D$ such that

(i) $q \upharpoonright (X \cap \sigma) \in G$,

(ii) $\mathrm{dom}(g) \setminus \sigma \subseteq [\gamma_{\nu'}, \gamma_\nu)$,

(iii) $\mathrm{ot}(\mathcal{G}(\delta')) < \delta$ for every $\delta \in \mathrm{dom}(\mathcal{F}' \upharpoonright \delta_M)$ and every $\delta' \in \mathrm{dom}(\mathcal{G} \upharpoonright \delta)$, and

(iv) $\mathrm{ot}(\mathcal{F}'(\delta')) < \delta$ for every $\delta \in \mathrm{dom}(\mathcal{G})$ and every $\delta' \in \mathrm{dom}(\mathcal{F}' \upharpoonright \delta)$.

$C \in M[G]$, and clearly closed by definition.

C unbounded: For every $\xi \in \delta_M$ there is some $q = (g, \mathcal{G}) \in D$ s.t. $\mathrm{dom}(g) \setminus \sigma \subseteq [\gamma_\xi, \gamma)$. This is witnessed by p'.

Since $\mathrm{ot}(C \cap \delta_M) = \delta_M$ and $\mathrm{ot}(\mathcal{F}'(\delta)) < \delta_M$ for every $\delta \in \mathrm{dom}(\mathcal{F}' \restriction \delta_M)$, we may find $\nu \in C \cap \delta_M$ and $\nu' < \nu$ s.t. $[\nu', \nu) \cap \bigcup\{\mathcal{F}'(\delta) \mid \delta \in \mathrm{dom}(\mathcal{F}' \restriction \delta_M)\} = \emptyset$.

Let $q = (g, \mathcal{G}) \in D \cap M[G]$ be such that (i)–(iv) above hold for q with this choice of ν and ν', and note that $q \in M$ again since $M[G] \cap V = M$ by induction hypothesis.
Let $(\delta_i)_{i<n}$ be an enumeration of $\mathrm{dom}(\mathcal{F}') \setminus \delta_M$ and let g' be the function with domain $\mathrm{dom}(g) \setminus \sigma$ such that $g'(\alpha) = g(\alpha) \cup \{\langle \delta_i, \delta_i \rangle \mid i < n\}$ for all $\alpha \in \mathrm{dom}(g')$. We may find a condition $(h, \mathcal{H})$ extending $p' \restriction (X \cap \sigma)$ and $q \restriction (X \cap \sigma)$. Then

$$(h \cup g', \mathcal{H} \oplus \mathcal{F}' \oplus \mathcal{G})$$

is a common extension of p' and q. $\square$

Similarly: It can be shown that $\mathrm{Add}_{\mathbb{B}}(X)$ is $\aleph_{1.5+\epsilon}$–c.c.: I.e., for some regular θ such that $\mathrm{Add}_{\mathbb{B}}(X) \in H(\theta)$ there is a club $E \subseteq [H(\theta)]^{\aleph_0}$ such that for every finite $\mathcal{N} \subseteq E$, if $\mathrm{ot}(N \cap X) < \delta_{N'}$ wenever N, N', $\delta_N < \delta_{N'}$, and $p \in \mathrm{Add}_{\mathbb{B}}(X) \cap N$ for some $N \in \mathcal{N}$ with δ_N minimal, then there is $p' \leq p$ which is $(N, \mathrm{Add}_{\mathbb{B}}(X))$–generic for all $N \in \mathcal{N}$.

$\mathrm{MA}_{\kappa}^{1.5+\epsilon}$, for arbitrary κ, implies $\mathrm{MA}_{\kappa}^{1.5}$ and can be shown consistent by exactly the same proof as for $\mathrm{MA}_{\kappa}^{1.5}$.

Applications:

Proposition

If $\mathrm{ot}(X) \geq \omega_2$*, then* $\mathrm{Add}_{\mathbb{B}}(X)$ *forces* $\mathfrak{b}(\omega_1) = \aleph_2$*.*

$\mathfrak{b}(\omega_1)$: Minimum κ s.t. there is $\mathcal{F} \subseteq {}^{\omega_1}\omega_1$ such that no $g : \omega_1 \longrightarrow \omega_1$ dominates each $f \in \mathcal{F}$ (mod. countable).

Proof.

Identical to the proof that $\mathrm{Add}(X)$ forces $\mathfrak{b} = \aleph_1$ if $\mathrm{ot}(X) \geq \omega_1$. □

Consider the following weak form of club–Guessing at ω_1:

KA (Kunen's Axiom): There is a club–sequence $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ with $\mathrm{ot}(C_\delta) = \omega$ for all δ s.t. for every club $D \subseteq \omega_1$ there is some δ with $D \cap [C_\delta(n), C_\delta(n+1)) \neq \emptyset$ for co–finitely many $n < \omega$.

Definition

Let $\mathcal{C} \subseteq \mathcal{P}(\omega_1)$ be such that $\mathrm{ot}(X) = \omega$ for all $X \in \mathcal{C}$. $\mathcal{C}$ is a KA–*set* if for every club $D \subseteq \omega_1$ there is some $X \in \mathcal{C}$ such that $D \cap [X(n), X(n+1)) \neq \emptyset$ for co–finitely many $n < \omega$.

Given a cardinal κ, KA_κ means: There is a KA–set of size at most κ.

Proposition

A Baumgartner club destroys all KA*–sets from the ground model. Hence,* $\mathrm{Add}_{\mathbb{B}}(X)$ *forces* $\neg\,\mathrm{KA}_\lambda$ *for every* λ *such that* $\lambda^{\aleph_1} < |X|$.

Consider the following weak form of club–Guessing at ω_1:

KA (Kunen's Axiom): There is a club–sequence $(C_\delta : \delta \in \mathrm{Lim}(\omega_1))$ with $\mathrm{ot}(C_\delta) = \omega$ for all δ s.t. for every club $D \subseteq \omega_1$ there is some δ with $D \cap [C_\delta(n), C_\delta(n+1)) \neq \emptyset$ for co–finitely many $n < \omega$.

Definition

Let $\mathcal{C} \subseteq \mathcal{P}(\omega_1)$ be such that $\mathrm{ot}(X) = \omega$ for all $X \in \mathcal{C}$. $\mathcal{C}$ is a KA–*set* if for every club $D \subseteq \omega_1$ there is some $X \in \mathcal{C}$ such that $D \cap [X(n), X(n+1)) \neq \emptyset$ for co–finitely many $n < \omega$.

Given a cardinal κ, KA_κ means: There is a KA–set of size at most κ.

Proposition

A Baumgartner club destroys all KA–sets from the ground model. Hence, $\mathrm{Add}_{\mathbb{B}}(X)$ forces $\neg\,\mathrm{KA}_\lambda$ for every λ such that $\lambda^{\aleph_1} < |X|$.

Easy to get $\mathfrak{b}(\omega_1) = \aleph_2 + 2^{\aleph_0} = 2^{\aleph_1}$ large by traditional means:

Add Cohen subsets of ω_1 and then do c.c.c. forcing. But the first step forces $\diamondsuit$, and the second step preserves Club Guessing.

Baumgartner measurability:

Definition

κ is *Baumgartner measurable* iff there is a κ–complete ideal I on κ such that $\mathcal{P}(\kappa)/I$ is forcing–equivalent to $\mathrm{Add}_{\mathbb{B}}(2^{\kappa})$.

Proposition

If κ is a measurable cardinal, then $\mathrm{Add}_{\mathbb{B}}(\kappa)$ forces that $\kappa = 2^{\aleph_1}$ is Baumgartner measurable.

Given a poset $\mathcal{P}$, $\mathfrak{m}(\mathcal{P})$ = Minimum κ such that $\mathrm{FA}(\{\mathcal{P}\})_\kappa$ fails.

Natural topology on the club filter $\mathcal{C}_{\omega_1}$ on ω_1 such that, if $\mathfrak{m}(\mathbb{B}) > \aleph_1$, $\mathfrak{m}(\mathbb{B}) = \mathbf{cov}(\mathcal{C}_{\omega_1})$ (= minimum size of a collection of nowhere dense subsets, in this topology, of $\mathcal{C}_{\omega_1}$ whose union is $\mathcal{C}_{\omega_1}$): This topology has as basis $\{\{C \in \mathcal{C}_{\omega_1} : p \subseteq \widetilde{C}\} : p \in \mathbb{B}\}$, where $\widetilde{C}$ is the enumerating function of C.

The same thing is of course true for $\mathfrak{m}(\text{Cohen})$ vis–à–vis $\mathbf{cov}(\text{Baire space})$. And of course a similar translation always holds for any poset $\mathcal{P}$ for the right topology on $\{F : F \text{ filter on } \mathcal{P}\}$. What is nice in the case of $\mathbb{B}$ and Cohen forcing is the appealing appearance of the topological side of the characterisation.

$\mathfrak{m}(\mathbb{B}) > \aleph_1$ is necessary: $\mathrm{KA} \Longrightarrow \mathbf{cov}(\mathcal{C}_{\omega_1}) = \aleph_0$.

Vague question: What is the right analog of random forcing (the ideal of Lebesgue null sets) for the club filter on ω_1?

Collapsing exactly $\aleph_3$

The starting point is the following result of U. Abraham (*On forcing without the continuum hypothesis*, J. of Symbolic Logic, vol. 48, 3 (1983), 658–661):

Theorem

(Abraham) (ZFC) There is a poset $\mathcal{P}$ collapsing ω_2 and preserving all other cardinals.

Abraham's forcing is built as follows: Let $A \subseteq \omega_2$ such that $\omega_2^{L[A]} = \omega_2^V$ (and then of course also $\omega_1^{L[A]} = \omega_1^V$). Then

$$\mathcal{P} = \mathrm{Add}(\omega_1) * \mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$$

- $\mathcal{P}$ collapses ω_2 and has a dense subset of size $\aleph_2$.
- Preservation of ω_1: If G is $\mathrm{Add}(\omega_1)$–generic, $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$ is σ–closed in $L[A][G]$, but certainly not in general in $V[G]$. However, $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$ is σ–distributive in $V[G]$. Given a $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][\dot{G}]}$–condition p and a $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][G]}$–name $\dot{F}$ in $V[G]$ for a function $\dot{F} : \omega \longrightarrow \mathrm{Ord}$, we may find a condition $p' \leq p$ in $\mathrm{Coll}(\omega_1, \omega_2)^{L[A][G]}$ deciding all of $\dot{F}$. We use the Cohen reals added by G in order to guide this construction.

Question

(in Abraham's paper) Can this be extended to higher cardinals? In particular, is there a forcing collapsing exactly $\aleph_3$?

Theorem

(ZFC) There is a poset $\mathcal{P}$ collapsing ω_3 and preserving all other cardinals.

Proof: Let $A \subseteq \omega_3$ such that $\omega_3^{L[A]} = \omega_3^V$. Then

$$\mathcal{P} = \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]} * \mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$$

works.

Proof uses following standard lemma.

Lemma

(Covering Lemma) For every $X \in L[A]$, $i \leq 3$, and every club D of $[X]^{\aleph_i}$ in $L[A]$, D is a stationary subset of $[X]^{\aleph_i}$ in V.

It follows that $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ is proper in V.

Also, as before, $\mathcal{P}$ collapses ω_3 and has a dense set of size $\aleph_3$.

Now let G be $\mathrm{Add}_{\mathbb{B}}(\omega_1)$–generic. $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$ is $<\omega_2$–closed in $L[A][G]$ but not necessarily in $V[G]$. But $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$ is $<\omega_2$–distributive in $V[G]$:

Let $\dot{F} \in V[G]$ be a $\text{Coll}(\omega_2, \omega_3)^{L[A][G]}$–name for a function into the ordinals with domain ω_1, and let $p \in \text{Coll}(\omega_2, \omega_3)^{L[A][G]}$. Let also $(N_\nu)_{\nu<\omega_1}$ be, in V, a continuous $\in$–chain of countable elementary substructures of some large enough $H(\theta)^V$ containing everything relevant.

By the Covering Lemma we may assume, for $N = \bigcup_{\nu<\omega_1} N_\nu$, that $N \cap H(\omega_3)^{L[A]} = M \cap H(\omega_3)^{L[A]}$ for some elementary substructure $M \in L[A]$ of some $H(\chi)^{L[A]}$, M of size $\aleph_1$ in $L[A]$. We may also assume that $M = \bigcup_{\nu<\omega_1} M_\nu$ for a continuous $\in$–chain $(M_\nu)_{\nu<\omega_1} \in L[A]$ of countable $M_\nu \prec H(\chi)^{L[A]}$. Let also $f : \omega_1 \to M$ be a surjection in $L[A]$ such that for every ν, $f \restriction \delta_{M_\nu}$ is a surjection from δ_{M_ν} onto M_ν and such that every member of M_ν is $f(\xi)$ for cofinally many $\xi < \delta_{M_\nu}$. We may further assume $f \restriction \delta_{M_\nu} \in M_{\nu+1}$ for all $\nu < \omega_1$. Finally, we may assume $N_0 \cap H(\omega_3)^{L[A]} = M_0 \cap H(\omega_3)^{L[A]}$.

We build in $L[A][G]$ a decreasing sequence $(p^\nu)_{\nu<\omega_1}$ of conditions in $\mathrm{Coll}(\omega_2,\omega_3)^{L[A][G]}$ extending p together with an increasing sequence $(\gamma^\nu)_{\nu<\omega_1}$ of countable ordinals, as follows:

1. $p^0 = p$ and $\gamma_0 = 0$
2. If ν is a non-zero limit ordinal and both $(p^{\nu'})_{\nu'<\nu}$ and $(\gamma^{\nu'})_{\nu'<\nu}$ have been defined, then $p^\nu = \bigcup_{\nu'<\nu} p^{\nu'}$ and $\gamma^\nu = sup_{\nu'<\nu}\gamma^{\nu'}$.
3. Suppose $\nu < \omega_1$ is an ordinal and p^ν and γ^ν have been defined. Let $p^{\nu+1}$ be $\dot{p}_G$ if $\dot{p}$ is an $\mathrm{Add}_{\mathbb{B}}(\omega_1)$–name, $F^G(\delta_{M_{\gamma^\nu}})(0) = \gamma$, for $\gamma > \gamma^\nu$, $F^G(\delta_{M_{\gamma^\nu}})(1) = \gamma + \xi$, $\xi < \delta_{M_\gamma}$, $f(\xi) = \dot{p}$, and $\dot{p}_G$ is a condition in $\mathrm{Coll}(\omega_2,\omega_3)$ extending p^ν. In this case let also $\gamma^{\nu+1} = \gamma$. Otherwise let $p^{\nu+1} = p^\nu$ and $\gamma^{\nu+1} = \gamma^\nu + 1$.

Note: the construction of $(p^{\nu'})_{\nu'\leq\nu+1}$ and $(\gamma^{\nu'})_{\nu'\leq\nu+1}$ takes place in $M_{\gamma^{\nu+1}+1}[G]$ and these sequences are definable in that model from $f \restriction \delta_{M_{\gamma^{\nu+1}}}$ and $(M_{\nu'})_{\nu'\leq\gamma^{\nu+1}}$.
In the end we let p^* be $\bigcup_{\nu<\omega_1} p^\nu$. Note $p^* \in L[A][G]$.

Claim

p^* decides $\dot{F}_G(i)$ for every $i < \omega_1$.

Proof: Proof uses the fact that $\mathrm{Add}_{\mathbb{B}}(\omega_1)$ can be naturally seen as a product:

Let $\dot{p} \in V$ be some $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name for a $\mathrm{Coll}(\omega_2, \omega_3)$–condition extending $\dot{p}^*$ and deciding the value of $\dot{F}(i)$. Let $(f, \mathcal{N})$ be a condition in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$. By the Covering Lemma we may find a countable $P \prec H(\theta)$, for some large enough θ, containing all relevant objects – which includes $(\dot{p}^\nu)_{\nu<\omega_1}$ and $\dot{p}$ – and such that $X := P \cap L_{\omega_3}[A] \in L[A]$.

Let f' be the function with domain $\mathrm{dom}(f)$ sending α to $f(\alpha) \cup \{(\delta_X, \delta_X)\}$ and let $\mathcal{N}' = \mathcal{N} \cup \{(\delta_X, \delta_X)\}$. Then $(f', \mathcal{N}')$ is an extension of $(f, \mathcal{N})$ which is $(P, \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]})$–generic. Hence, it forces that there is an $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$–name $\dot{q}$ (in V) such that $(f', \mathcal{N}')$ forces the following.

(a) There is some $(g, \mathcal{M}) \in \dot{G} \cap \mathrm{Add}_{\mathbb{B}}(\delta_X)$ such that $(g, \mathcal{M})$ forces that $\dot{q}$ is a condition in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$ deciding $\dot{F}(i)$.

(b) For every $\nu < \delta_X$ there is some $(g, \mathcal{M}) \in \dot{G} \cap \mathrm{Add}_{\mathbb{B}}(\delta_X)$ such that $(g, \mathcal{M})$ forces that $\dot{q}$ extends $\dot{p}^{\nu}$ in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$.

Let now $\delta < \omega_1$, $\delta > \delta_X$, be such that $\delta_{N_\delta} = \delta$ and $N_\delta \cap L_{\omega_3}[A] = M_\delta \cap L_{\omega_3}[A]$. Let also $Y = \omega_1 \setminus \{\delta_X\}$ and let $\varphi : \mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]} \longrightarrow \mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$, $\varphi \in N_\delta \cap L[A]$, be an isomorphism which is the identity on $\mathrm{Add}_{\mathbb{B}}(\delta_X)^{L[A]}$. Let f'' be the function with domain $\mathrm{dom}(f')$ sending $\alpha \in \mathrm{dom}(f')$ to $f'(\alpha) \cup \{(\delta, \delta)\}$ and let $\mathcal{N}'' = \mathcal{N}' \cup \{(\delta, \delta \setminus \{\delta_X\})\}$.

Then $(f'', \mathcal{N}'')$ is a condition in $\text{Add}_{\mathbb{B}}(Y)^{L[A]}$ and there is an $\text{Add}_{\mathbb{B}}(Y)^{L[A]}$–name $\tilde{q}$ in N_δ such that $(f'', \mathcal{N}'')$ forces the following in $\text{Add}_{\mathbb{B}}(Y)^{L[A]}$.

(c) There is some $(g, \mathcal{M}) \in \dot{G} \cap \text{Add}_{\mathbb{B}}(\delta_X)$ such that $(g, \mathcal{M})$ forces that $\tilde{q}$ is a condition in $\text{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$ deciding $\hat{\varphi}(\dot{F})(i)$.

(d) For every $\nu < \delta_X$ there is some $(g, \mathcal{M}) \in \dot{G} \cap \text{Add}_{\mathbb{B}}(\delta_X)$ such that $(g, \mathcal{M})$ forces that $\dot{q}$ extends $\hat{\varphi}(\dot{p}^\nu)$ in $\text{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$.

Since $(f'', \mathcal{N}'')$ is $(N_\nu, \mathrm{Add}_{\mathbb{B}}(Y)^{L[A]})$–generic, we can fix an extension $(\bar{f}, \bar{\mathcal{N}})$ of $(f'', \mathcal{N}'')$ and an $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$–name $\bar{q} \in N_\delta \cap L_{\omega_3}[A]$ for a $\mathrm{Coll}(\omega_2, \omega_3)$–condition such that $(\bar{f}, \bar{\mathcal{N}})$ forces the following in $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$.

(e) There is some $(g, \mathcal{M}) \in \dot{G} \cap \mathrm{Add}_{\mathbb{B}}(\delta_X)$ such that $(g, \mathcal{M})$ forces that $\bar{q}$ decides $\hat{\varphi}(\dot{F})(i)$.

(f) For every $\nu < \delta_X$ there is some $(g, \mathcal{M}) \in \dot{G} \cap \mathrm{Add}_{\mathbb{B}}(\delta_X)$ such that $(g, \mathcal{M})$ forces that $\bar{q}$ extends $\hat{\varphi}(\dot{p}^\nu)$ in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$.

Let now $(h, \bar{\mathcal{N}})$ be a condition in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ extending $(\bar{f}, \bar{\mathcal{N}})$ and such that $h(\delta_X)(0) = \delta$ and $h(\delta_X)(1) = \delta + \xi$ for some $\xi < \delta$ such that $f(\xi) = \hat{\psi}(\bar{q})$, where $\psi = \varphi^{-1}$.

Now, since φ is the identity on $\mathrm{Add}_{\mathbb{B}}(\delta_X)^{L[A]}$ and since $(\bar{f}, \bar{\mathcal{N}})$ forces (e) in $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$, $(h, \bar{\mathcal{N}})$ forces in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ that

(g) there is some $(g, \mathcal{M}) \in \dot{G} \cap \mathrm{Add}_{\mathbb{B}}(\delta_X)$ such that $(g, \mathcal{M})$ forces that $f(\xi)$ decides $\hat{\psi}(\hat{\varphi}(\dot{F}))(i)$ $(= \dot{F}(i))$.

Similarly, again by the above and since $(\bar{f}, \bar{\mathcal{N}})$ forces (f) in $\mathrm{Add}_{\mathbb{B}}(Y)^{L[A]}$, we have that $(h, \bar{\mathcal{N}})$ forces in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$ that

(h) for every $\nu < \delta_X$ there is some $(g, \mathcal{M}) \in \dot{G} \cap \mathrm{Add}_{\mathbb{B}}(\delta_X)$ such that $(g, \mathcal{M})$ forces that $f(\xi)$ extends $\hat{\psi}(\hat{\varphi}(\dot{p}^{\nu}))$ $(= \dot{p}^{\nu})$ in $\mathrm{Coll}(\omega_2, \omega_3)^{L[A][\dot{G}]}$.

Hence, $(h, \bar{\mathcal{N}})$ forces, in $\mathrm{Add}_{\mathbb{B}}(\omega_1)^{L[A]}$, that $\dot{p}_{\delta_X+1} = f(\xi)$ and $\dot{p}_{\delta_X+1}$ decides $\dot{F}(i)$. It follows that $(h, \bar{\mathcal{N}})$ forces that $\dot{p}^*$ decides $\dot{F}(i)$. $\square$

The above claim finishes the proof of the Theorem. □