Martin’s conjecture, uniformity, and countable Borel equivalence relations

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Caltech
Turing equivalence and the Turing jump

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Given $x \in 2^\omega$, let $x'$ be the halting problem relative to $x$, or the Turing jump of $x$. That is, $x'$ is the set of $n$ so that the $n$th program halts relative to $x$. By relativizing the proof of the insolubility of the halting problem, we have $x \nleq_T x'$.

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Martin’s cone theorem

A Turing cone is a set of the form $\{ x : x \geq_T y \}$ for some $y$. One way of viewing Turing reducibility restricted to this cone is thinking of relativizing ordinary Turing reducibility on $2^\omega$ so that every program can use $y$ as an oracle.
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**Theorem (Martin 69)**

Assume \( \text{AD} \). Suppose \( A \subseteq 2^\omega \) is Turing invariant. Then either \( A \) contains a Turing cone, or \( 2^\omega \setminus A \) contains a Turing cone.

This theorem yields a measure or ultrafilter on the sets of Turing degrees, where a set of degrees has measure 1 iff it contains a Turing cone. This is called Martin measure. It is a notion of largeness that preserves all the structure and behavior of computability that recursion theorists like to study (those that relativize).
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An example application

Given two Borel equivalence relations $E$ and $F$ on Polish spaces $X$ and $Y$, say that $E$ is **Borel reducible** to $F$ if there is a Borel function $f : X \to Y$ so that $x E y \iff f(x) F f(y)$. Say $E$ is **countable** if it has countable classes, and say that a countable Borel equivalence relation $E$ is is **universal** if for every countable Borel equivalence relation $F$, $F \leq_B E$. 

Recall $x, y \in 2^\omega$ are arithmetically equivalent if there is an $n$ such that $x(n) \geq_T y$ and $y(n) \geq_T x$, where $x(n)$ is the $n$th iterate of the Turing jump.

**Theorem (Slaman-Steel)** Arithmetical equivalence is a universal countable Borel equivalence relation.

**Corollary (M., answering Jackson-Kechris-Louveau)** If $E$ is a universal countable Borel equivalence relation on $X$, and $B \subseteq X$ is $E$-invariant, then either $E \upharpoonright B$ or $E \upharpoonright (X \setminus B)$ is universal.
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Many natural questions about Martin measure are open.

- Open: for what (not necessarily well-orderable) cardinalities $X$ can we decompose the Turing degrees into $X$ many Martin measure 0 pieces?

- Open: If $X$ is a Polish space, $E$ is a equivalence relation with countable classes, and $\mu$ is a nonatomic Borel probability measure on $X$ which is ergodic with respect to $E$, then $\mu$ induces an ultrafilter $U$ on $X/E$. Can $U$ be Rudin-Kiesler reducible to Martin measure? (I.e. are randomness and information orthogonal?)
Martin’s conjecture

A function $f : 2^\omega \to 2^\omega$ is **Turing invariant** if $x \equiv_T y$ implies $f(x) \equiv_T f(y)$.

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Inspired by the wellfoundedness of the Wadge hierarchy, and an old question of Sacks, Martin conjectured a classification of all Turing invariant functions. The ones that are not constant should be precisely the transfinite iterates of the Turing jump.
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- $\mathcal{M}$ is prewellordered by setting $f \leq g$ if $f(x) \leq_T g(x)$ on a Turing cone.

For the rest of the talk, I’ll work just with the Borel functions, where the conjecture localizes as follows:

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If $f$ is a Borel Turing invariant function, then either there exists a constant $y \in 2^\omega$ such that $f(x) \equiv_T y$ on a Turing cone, or there exists an $\alpha < \omega_1$ so that $f(x) \equiv_T x(\alpha)$ on a Turing cone, where $x(\alpha)$ is the $\alpha$th iterate of the Turing jump of $x$. 
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- $\mathcal{M}$ is prewellordered by setting $f \leq g$ if $f(x) \leq_T g(x)$ on a Turing cone.
- The identity function has rank 0, and if $f$ has rank $\alpha$, then the Turing jump $x \mapsto f(x)'$ of $f$ has rank $\alpha + 1$. 

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Implications of Martin’s conjecture

If Martin’s conjecture is true, it has many consequences for the theory of countable Borel equivalence relations. For example,

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- M. (2011): If Martin’s conjecture is true, then if $E$ is a universal countable Borel equivalence relation on $X$, and $\mu$ is a Borel probability measure on $X$, then there exists a $\mu$-conull Borel set $A \subseteq X$ such that $E \restriction A$ is not universal.
Uniformity

An important property of the Turing jump is that it is uniform: if \( x \equiv_T y \) and we know what programs are used to compute \( x \) from \( y \) and vice versa, then we can say what programs to use show \( x' \equiv_T y' \), independently of what \( x \) and \( y \) are.
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Formally, say that a function \( f : 2^\omega \to 2^\omega \) is **uniformly Turing invariant** if there is a function \( u : \omega^2 \to \omega^2 \) such that \( x \equiv_T y \) via \((i,j)\) implies \( f(x) \equiv_T f(y) \) via \( u(i,j) \).
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Slaman and Steel have shown Martin's conjecture reduces to a question of uniformity:

**Theorem (Steel (1982) and Slaman and Steel (1988))**

Martin's conjecture is true iff every Borel Turing invariant function is Turing equivalent to a Borel uniformly Turing invariant function on a Turing cone.
Rigidity

Similar sorts of rigidity where all homomorphisms between two equivalence relations are uniform have been proved (for example in ergodic theory and operator algebras) for orbit equivalence relations of certain group actions. However, the tools used to prove these theorems fail badly when used in this setting.
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For example,

**Theorem (Thomas (2009))**

\[ \equiv_T \text{ is not generated by a free group action.} \]
An improvement

$F_\omega$ acts on $(2^\omega)^{F_\omega}$ via the shift action. Let $F(F_\omega, 2^\omega)$ be the orbit equivalence relation of the free part of this action. The following theorem gains us the possibility of using algebraic tools to understand Martin’s conjecture.
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**Theorem (M.)**

Martin’s conjecture is true iff every Borel Turing invariant function is equivalent to a Borel uniformly invariant function from $F(\mathbb{F}_\omega, 2^\omega)$ to $\equiv_T$ on a Turing cone.

This theorem is proved by copying Slaman and Steel’s argument, but replacing their recursion theoretic arguments which use pointed perfect sets, the recursion theorem, Martin’s ultrafilter on Turing invariant sets, etc. with group theoretic ideas (equivariant embeddings, tilings of the free group, an ultrafilter on $\mathbb{F}_\omega$-invariant sets) to refine Slaman and Steel’s result.
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A rigidity conjecture for Borel cocycles of $F(\mathbb{F}_\omega, 2^\omega)$

So Martin’s conjecture now reduces to proving Borel rigidity results for cocycles of the shift action of $\mathbb{F}_\omega$. 

Recall that a cocyle of an equivalence relation $F$ into a group $\Gamma$ is a map $\rho : F \rightarrow \Gamma$ such that $\rho(x,y) \rho(y,z) = \rho(x,z)$. The following rigidity for Borel cocycles of $F(\mathbb{F}_\omega, 2^\omega)$ implies Martin’s conjecture:

**Conjecture (M.)**

Let $\rho : F(\mathbb{F}_\omega, 2^\omega) \rightarrow \Gamma$ be a Borel cocycle of $F(\mathbb{F}_\omega, 2^\omega)$ to a countable group $\Gamma$. Then there exists a continuous injective homomorphism $f$ from $F(\mathbb{F}_\omega, 2^\omega)$ to itself so that for every $x \in \text{Free}((2^\omega \mathbb{F}_\omega))$, the map $\gamma \mapsto \rho(f(\gamma \cdot x), f(x))$ is a group homomorphism whose value does not depend on $x$.

If this conjecture fails in a particularly strong way, then one can construct counterexamples to slight generalizations of Martin’s conjecture (for instance, a Borel homomorphism from $\equiv_T$ to $\equiv_1$ which is not uniform on any pointed perfect set). This is shown by use recursion-theoretic methods to construct Borel homomorphisms realizing a Borel cocycle exhibiting a strong failure of rigidity.
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Uniformly universal countable Borel equivalence relations

Having fixed some family of functions generating an equivalence relation $E\{\psi_i\}$, we say that $E\{\psi_i\}$ is uniformly universal if for every other countable Borel equivalence relation $E\{\phi_i\}$, there is a uniform reduction from $E\{\phi_i\}$ to $E\{\psi_i\}$. This notion was introduced by Montalban, Reimann, and Slaman.
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One may safely think of the uniformly universal countable Borel equivalence relations as those which we have any hope of currently proving universal without dramatically new techniques; all current universality proofs are uniform in this way. Further, if significant rigidity occurs for “near-universal” Borel equivalence relations, as I’ve suggested above, then its possible that a countable Borel equivalence relation may be universal if and only if it is universal for all possible ways it may be generated.
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Of the numerous open problems concerning universal countable Borel equivalence relations, what can we say if we ask instead about uniform universality?
Properties of uniformly universal equivalence relations

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## Properties of uniformly universal equivalence relations

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3. **Both the uniformly universal and non-uniformly universal countable Borel equivalence relations are cofinal under** $\subseteq$. 

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**Note:** The passage includes references to mathematical concepts and terminology, such as Borel actions and uniformly universal equivalence relations, which are fundamental in the study of equivalence relations in descriptive set theory. The theorem highlights the existence of specific actions and relations that are universal within certain classes, and it also discusses the properties of these relations under various conditions.
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</tbody>
</table>
Which equivalence relations are uniformly universal?

Theorem (M.)

1. If $\Gamma$ is a countable group, then the shift action of $\Gamma$ on $2^\Gamma$ is uniformly universal relation if and only if $\Gamma$ contains $\mathbb{F}_2$.
Which equivalence relations are uniformly universal?

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1. If $\Gamma$ is a countable group, then the shift action of $\Gamma$ on $2^{\Gamma}$ is uniformly universal relation if and only if $\Gamma$ contains $\mathbb{F}_2$.

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1. If \( \Gamma \) is a countable group, then the shift action of \( \Gamma \) on \( 2^\Gamma \) is uniformly universal relation if and only if \( \Gamma \) contains \( \mathbb{F}_2 \).

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3. (Joint with Jay Williams) If \( \Gamma \) is a countable subgroup of \( S_\infty \) and \( X \) is a standard Borel space of cardinality \( \geq 3 \), then the permutation action of \( \Gamma \) on \( X^\omega \) is uniformly universal if and only if there exists some \( n \in \omega \) and a subgroup \( \Delta \leq \Gamma \) isomorphic to \( \mathbb{F}_2 \) such that the map \( \Delta \to \omega \) given by \( \delta \mapsto \delta(n) \) is injective.

4. For every additively indecomposable \( \alpha < \omega_1 \), let \( \equiv_\alpha \) be the equivalence relation on \( 2^\omega \) defined by \( x \equiv_\alpha y \) if and only if \( x(\beta) \geq T y \) and \( y(\beta) \geq T x \) for some \( \beta < \alpha \). \( \equiv_\alpha \) is uniformly universal if and only if there is a \( \beta < \alpha \) such that \( \beta \cdot \omega = \alpha \).
Which equivalence relations are uniformly universal?

**Theorem (M.)**

1. If \( \Gamma \) is a countable group, then the shift action of \( \Gamma \) on \( 2^\Gamma \) is uniformly universal relation if and only if \( \Gamma \) contains \( F_2 \).

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Which equivalence relations are uniformly universal?

**Theorem (Continued)**

5. If $E_{\{\varphi_i\}}$ is a countable Borel equivalence relation on $2^\omega$ coarser than recursive isomorphism and closed under countable recursive joins, then $E_{\{\varphi_i\}}$ is not uniformly universal. (This includes many-one equivalence, tt equivalence, wtt equivalence, Turing equivalence, enumeration equivalence, etc.)