Perfect subsets of generalized Baire spaces and Banach-Mazur games

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1. Perfect subsets
   - Setting
   - Factoring generics
   - Perfect set property
   - Global version

2. Banach-Mazur games
   - Baire property
   - Banach-Mazur games

3. Applications
   - Ramsey properties
   - Graphs and equivalence relations
   - Generalized Choquet spaces

Setting

Let κ always denote an uncountable regular cardinal with $\kappa^{<\kappa} = \kappa$.

Definition

We consider the generalized Baire space ${}^{\kappa}\kappa$ of functions $f : \kappa \to \kappa$ with basic open sets $U_s = \{f \in {}^{\kappa}\kappa \mid s \subseteq f\}$ for $s \in {}^{<\kappa}\kappa$.

Definition

A $\Sigma^1_n({}^{\kappa}\kappa)$-formula is of the form $\exists x_1 \forall x_2 \ldots \exists x_n \varphi$, where $\exists x_i$ and $\forall x_i$ range over ${}^{\kappa}\kappa$ and quantifiers in φ range over κ.
A $\Sigma^1_n({}^{\kappa}\kappa)$ subset of ${}^{\kappa}\kappa$ is defined by a $\Sigma^1_n({}^{\kappa}\kappa)$-formula with parameters in ${}^{\kappa}\kappa$.

The study of the descriptive set theory of these spaces was initiated by Mekler and Väänänen.

Perfect trees and sets

Definition

A subtree $T \subseteq {}^{<\kappa}\kappa$ is (κ-)perfect if it is $< \kappa$-closed and its splitting nodes are cofinal in T.
A subset $A \subseteq {}^{\kappa}\kappa$ is (κ-)perfect if $A = [T]$ for some (κ-)perfect tree T.

Definition

Let PSP^{κ}_{od} denote the perfect set property for subsets of ${}^{\kappa}\kappa$ definable from ordinals and subsets of κ, i.e. the condition that every such set either has size $\leqslant \kappa$ or has a perfect subset.

Lemma

Trees of height $\kappa = \lambda^+$ with levels of size $< \kappa$ don't have perfect subtrees. In particular if T is a κ-Kurepa tree, then $[T]$ is a closed counterexample to PSP^{κ}_{od}.

Lemma (Silver)

If $\lambda > \kappa$ is inaccessible, then $Col(\kappa, < \lambda)$ forces that there are no κ-Kurepa trees.

Solovay's Theorem

Theorem (Solovay)

Suppose V is a $Col(\omega, <\lambda)$-extension where λ is inaccessible in the ground model. Then every subset of ${}^{\omega}\omega$ definable from ordinals and reals has the perfect set property, the property of Baire, and is Lebesgue measurable.

Definition

Suppose $V[G]$ is a generic extension of V by a $<\kappa$-closed forcing and $x \in V[G]$. Then x *factors* in $(V, V[G])$ if $V[G]$ is a generic extension of $V[x]$ by a $<\kappa$-closed forcing.

Lemma (Solovay)

If G is $Add(\omega, 1)$-generic over V, then $V[G]$ is an $Add(\omega, 1)$-generic extension of $V[x]$ for every $x \in V[G]$ with $V[x] \neq V[G]$. So x factors in $(V, V[G])$.

This is used in the proof of Solovay's Theorem but fails for $\kappa \geqslant \omega_1$.

Factoring

Example

Suppose $\kappa^{<\kappa} = \kappa \geqslant \omega_1$. Let $\mathbb{P}$ consist of pairs $(p, q) \in ({}^{\alpha}2)^2$ with $\alpha < \kappa$ such that s_q is closed and $s_q \cap s_p = \varnothing$ for the sets $s_p, s_q \subseteq \kappa$ with characteristic functions p, q, ordered by coordinatewise end extension.

Remark

The forcing $\mathbb{P}$ is equivalent to $Add(\kappa, 1)$. It factors as $\mathbb{Q} * \dot{\mathbb{Q}}$, where $\mathbb{Q}$ adds a Cohen subset of κ, which is stationary, and $\dot{\mathbb{Q}}$ shoots a club through its complement.
Since $Add(\kappa, 1)$ is $< \kappa$-closed and hence $\Sigma^1_1({}^{\kappa}\kappa)$-absolute, it does not destroy the stationarity of subsets of κ. So $\dot{\mathbb{Q}}$ is not $< \kappa$-closed and not equivalent to $Add(\kappa, 1)$.

Perfect subsets

Question (Väänänen)

Is the perfect set property for definable subsets of ${}^{\kappa}\kappa$ consistent?

Theorem

Suppose $\kappa \geqslant \omega_1$ is regular and $\lambda > \kappa$ is inaccessible. Then Col$(\kappa, < \lambda)$ forces PSP$^{\kappa}_{od}$.

Proof sketch

We add a perfect tree all of whose branches are factoring Cohen subsets of κ.

Let $|s| = dom(s)$ for $s \in {}^{<\kappa}2$ and $|t| = \sup_{s \in t} |s|$ for $t \subseteq {}^{<\kappa}2$.

Let $\mathbb{P}$ denote the forcing consisting of conditions (t, s) where

- $t \subseteq {}^{<\kappa}2$ is a $< \kappa$-closed tree,
- $s \subseteq {}^{<\kappa}2$ with
  - $s \cap t = \varnothing$,
  - every node in s is of the form $r \frown i$ with $r \frown (1 - i) \in t$,
- t is the closure of a tree of size $< \kappa$.
- $|s| < \kappa$.

The conditions are ordered by coordinatewise reverse inclusion.

$\mathbb{P}$ is equivalent to $Add(\kappa, 1)$, since it is $< \kappa$-closed and $|\mathbb{P}| = \kappa^{<\kappa} = \kappa$.

$\mathbb{P}$ adds a perfect set of mutually generic factoring Cohen subsets of κ.

Suppose G is $Col(\kappa, < \lambda)$-generic over V. Suppose $A \subseteq {}^{\kappa}\kappa$ is defined in $V[G]$ from parameters in V.

Since $|A| \geqslant \kappa^+$ in $V[G]$, there is $\delta < \lambda$ so that $G_{\delta+1} = G \upharpoonright (\delta + 1)$ adds a new element to A.

We write $V[G] = V[G_{\delta+1}][g][h][i]$, where g, h, i are mutually generic for $Col(\kappa, < \delta + 1)$, $Col(\kappa, < \lambda)$, $Add(\kappa, 1)$ over $V[G_{\delta+1}]$.

Then g adds a new element x to A over V and $x \notin V[G_{\delta+1}][h]$.

Note that $Col(\kappa, < \delta + 1)$ is equivalent to $Add(\kappa, 1)$ in $V[G_{\delta+1}][h]$. Let $\dot{x}$ be a name in $V[G_{\delta+1}][h]$ for a new element of A.

We add a perfect set of mutually generic factoring subsets of κ over $V[G_{\delta+1}][h]$. The image under $\dot{x}$ is a perfect subset of A.

Perfect set games

Definition

In the perfect set game of length κ for a subset $A \subseteq {}^{\kappa}\kappa$, two players play a strictly increasing sequence $(s_\alpha)_{\alpha<\kappa}$ in ${}^{<\kappa}\kappa$. Player I plays at all even stages. Player II extends the sequence by a single ordinal in each move. Player I wins if $\bigcup_{\alpha<\kappa} s_\alpha \in A$.

Lemma (Kovachev)

A subset A of ${}^{\kappa}\kappa$ has a perfect subset if and only if player I has a winning strategy in the perfect set game for A. Also $|A| \leqslant \kappa$ if and only if player II has a winning strategy.

Corollary

In $V[G]$, the perfect set game of length κ is determined for all sets definable from ordinals and subsets of κ.

Global version

Theorem (Global PSP)

Suppose there is a proper class of inaccessible cardinals. Then there is a class forcing extension in which PSP^{κ}_{od} *holds for all infinite regular cardinals.*

Corollary

The consistency strength of PSP^{κ}_{od} *at some regular cardinal* κ *is that of an inaccessible cardinal. The consistency strength of* PSP^{κ}_{od} *at all regular cardinals is that of a proper class of inaccessible cardinals.*

Remark (Ikegami-S. - Failure of effective PSP)

In $L^{Col(\omega_1,<\lambda)}$ *for* λ *inaccessible, the set of non-stationary subsets of* ω_1 *which are stationary in an intermediate extension is definable without parameters, but does not have a perfect subset coded in* L*.*

Perfect subsets from maximality principles

Definition

Suppose $A \subseteq {}^{\kappa}\kappa$ is defined from ordinals and subsets of κ. A is *necessarily* $\geqslant \kappa^+$ if $|A| \geqslant \kappa^+$ in every $< \kappa$-closed forcing extension.

Lemma

If A is Σ^1_1 and necessarily $\geqslant \kappa^+$, then A has a perfect subset.

This fails for Π^1_1 subsets of ${}^{\kappa}\kappa$ in L.

Definition (Maximality principles)

Suppose that Γ is a class of forcings. The *maximality principle* $MP_\Gamma(H_\mu)$ is the assertion the for every statement φ with parameters in H_μ, if there is an extension $V[G]$ by a forcing in Γ so that φ holds in $V[G]$ and in all further extensions by forcings in Γ, then φ holds in V.

Remark

Let Γ^κ_{col} denote the class of forcings $Col(\kappa, <\nu)$ for regular ν closed under the function $f(\alpha) = \alpha^{<\kappa}$. It follows from a proof of Fuchs that if $\kappa < \delta$ are regular and $V_\delta \prec V$, then $Col(\kappa, <\delta)$ forces $MP_{\Gamma^\kappa_{col}}(H_{\kappa^+})$.

Lemma

Suppose $\kappa > \omega$ is regular. Then $MP_{\Gamma^\kappa_{col}}$ implies the perfect set property for all subsets of ${}^\kappa\kappa$ definable from subsets of κ.

Baire property

Definition

Suppose X is a topological space. $A \subseteq X$ is (κ-)meager if it is a union of κ nowhere dense sets. A set has the (κ-)Baire property if there is an open set $U \subseteq X$ such that $A \triangle U$ is κ-meager.

Example (Halko-Shelah)

The club filter on κ does not have the property of Baire and is Σ^1_1.

Banach-Mazur games

Definition

In the Banach-Mazur game of length κ for a set $A \subseteq {}^{\kappa}\kappa$, two players play a strictly increasing sequence $(s_\alpha)_{\alpha<\kappa}$ in ${}^{<\kappa}\kappa$. Player I plays at all even stages. Player II wins if $\bigcup_{\alpha<\kappa} s_\alpha \in A$.

Lemma (Kovachev)

A subset A of ${}^{\kappa}\kappa$ is co-meager if and only if the second player has a winning strategy for A in the Banach-Mazur game.

Theorem

Suppose κ is regular and $\lambda > \kappa$ is inaccessible. Let G be $Col(\kappa, < \lambda)$-generic over V. Suppose $A \subseteq {}^{\kappa}\kappa$ is definable from ordinals and subsets of κ in $V[G]$. Then in $V[G]$, the Banach-Mazur game for A is determined.

Proof sketch

Conditions $p \in \mathbb{P}$ consist of strictly increasing sequences in ${}^{<\kappa}\kappa$ such that

- p is closed under initial segments,
- $|p| < \kappa$,
- if $s, t \in p$, $dom(s) = dom(t) = \gamma + 1$, $\gamma \in Lim$, and $s \upharpoonright \gamma = t \upharpoonright \gamma$, then $s(\gamma) = t(\gamma)$.

The conditions are ordered by reverse inclusion.

If G is $\mathbb{P}$-generic over V, let $S = \{b \in {}^{\kappa}\kappa \mid \exists f \in {}^{\kappa}({}^{<\kappa}\kappa))\ \forall \alpha < \kappa \bigcup(f \upharpoonright \alpha) \in \bigcup G$ and $b = \bigcup_{\alpha<\kappa} f(\alpha)\}$.

Then all $b \in S$ factor in $(V, V[G])$.

Player I has a winning strategy for S in the Banach-Mazur game.

If A is meager, then the player II has a strategy to play outside of A in the Banach-Mazur game.

Otherwise we write $V[G] = V[g][h][j]$, where g, h, j are mutually generic for $Col(\kappa, <\lambda)$, $Add(\kappa, 1)$, $\mathbb{Q}^h$, and the parameters are in $V[g]$. We can choose h so that $h \in A$.

Then $i \in A$ for every $Add(\kappa, 1)$-generic i over $V[g]$ whose quotient is equivalent to $\dot{\mathbb{Q}}^i$. If $\mathbb{Q}^{g \times i}$ is $Add(\kappa, 1)$, then the player I has a winning strategy for $S \subseteq A$. Otherwise use a variant of the forcing.

Failure of Banach-Mazur determinacy

Remark

Suppose there is a wellorder of ${}^\kappa\kappa$ definable from ordinals and subsets of κ. Then there is a set $A \subseteq {}^\kappa\kappa$ definable from ordinals and subsets of κ for which the Bernstein property fails, i.e. neither A nor ${}^\kappa\kappa \setminus A$ have a perfect subset.

This works by the usual construction along a well-order of the perfect trees. In L we obtain a $\Sigma^1_1({}^\kappa\kappa)$ counterexample.

Then the first player does not have a winning strategy for A or for ${}^\kappa\kappa \setminus A$ in the Banach-Mazur game.

Problem

Is there a family of κ^+ disjoint subsets of ${}^\kappa\kappa$ so that the first player has a winning strategy for each set in the Banach-Mazur game?

Ramsey properties

Lemma

In $V[G]$ every function $f : {}^{\kappa}\kappa \to {}^{\kappa}\kappa$ definable from ordinals and subsets of κ is continuous on a set $A \subseteq {}^{\kappa}\kappa$ such that the first player has a winning strategy in the Banach-Mazur game for A.

Problem

Are there other (consistent) canonization theorems for definable subsets of ${}^{\kappa}\kappa$?

Graphs and equivalence relations

Suppose S is a dense subset of ${}^{<\kappa}2$. Let G_0^S denote the set of pairs $(s \frown 0 \frown x, s \frown 1 \frown x)$ with $s \in S$ and $x \in {}^{\kappa}2$.

Lemma

There is no coloring of G_0^S in $\leqslant \kappa$ colors definable from ordinals and subsets of κ in $V[G]$.

Let E_0 denote equality up to bounded error on ${}^{\kappa}2$.

Lemma

There is no reduction f of E_0 to equality on ${}^{\kappa}2$ definable from ordinals and subsets of κ in $V[G]$.

Problem

Do these results follow from Banach-Mazur determinacy?

Generalized Choquet spaces

Choquet spaces were introduced to prove completeness of subsets of function spaces.

Definition

Suppose that X is a normal space without isolated points and that X has a basis of size $\leqslant \kappa$. The *strong κ-Choquet game* in X is played by two players I (empty) and II (nonempty).

I	U_0, x_0		U_1, x_1		$\dots$	U_λ, x_λ		$\dots$
II		V_0		V_1			V_λ	

In the top half of each inning, I plays U_α, x_α such that $x_\alpha \in U_\alpha$ and U_α is relatively open in $\bigcap_{\beta<\alpha} U_\beta$. In the bottom half of each inning, II responds with V_α such that $x_\alpha \in V_\alpha$ and V_α is relatively open in U_α.
We say that II *wins the play* if (i) player I has a valid play at each limit stage, and (ii) $\bigcap_{\alpha<\kappa} U_\alpha \neq \varnothing$.

Definition

A space is called (strong) κ-Choquet if player II has a winning strategy in the (strong) κ-Choquet game.

Theorem (Coskey-S.)

If X is a normal strong κ-Choquet space with a dense subset of size $\leqslant \kappa$, then there is a $< \kappa$-closed tree $T \subseteq {}^{<\kappa}\kappa$ without end nodes and a continuous bijection $f : [T] \to X$.

Corollary

Suppose an inaccessible cardinal $\lambda > \kappa$ is Levy-collapsed to κ^+. Suppose A is a subset of a strong κ-Choquet space X as above and X, A are definable from ordinals and subsets of κ. Then either $|A| \leqslant \kappa$ or there is a continuous injection $f : {}^{\kappa}2 \to A$.