On Partitioning Linearly Ordered Quadruples in Canonical Non-Choice-Contexts

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The partition relation

$$\rho \longrightarrow (\sigma \vee \tau, \varphi)^{\psi} \tag{1}$$

means that, given a set X of order-type ρ for any colouring $\chi : [X]^{\psi} \longrightarrow 2$ of the subsets of X having order-type ψ in two colours there is a subset of X having order-type σ or τ which is homogeneous in colour 0 or a subset of X having order-type φ which is homogeneous in colour 1.

The negation of (1) is written

$$\rho \not\longrightarrow (\sigma \vee \tau, \varphi)^{\psi}.$$

For any order-type ρ, ρ^* is the one attained by reversing the ordering. For any two order-types ρ and σ, $\rho+\sigma$ is the type of the orderings given by an ordering of type ρ left to an ordering of type σ. E.g., $\omega^* + \omega$ is the order-type of the integers.

We specify the axiom system used for proving a theorem, BP stands for the statement that all sets of real numbers have the property of Baire.

Previous Results

In [971E], Erdős, Milner and Rado proved, using the Axiom of Choice, the following three theorems.

Theorem 1 (ZFC). $\rho \not\longrightarrow (\omega^* + \omega, 4)^3$ *for any linear ordering* ρ.

Theorem 2 (ZFC)**.** $\rho \not\longrightarrow (\omega + \omega^*, 4)^3$ *for any linear ordering* ρ*.*

Theorem 3 (ZFC)**.** $\rho \not\longrightarrow (\omega^* + \omega \vee \omega + \omega^*, 5)^3$ *for any linear ordering* ρ*.*

New Results

Using a structural analysis from [981Bl] by Blass it is possible to prove analogous statements in a choiceless context for ${}^{\alpha}2$ lexicographically ordered for some ordinal number α:

Theorem 4 (ZF)**.** $\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (\omega^* + \omega, 5)^4$ *for any ordinal number* α*.*

Theorem 5 (ZF)**.** $\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (\omega + \omega^*, 5)^4$ *for any ordinal number* α*.*

Theorem 6 (ZF)**.** $\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (\omega^* + \omega \vee \omega + \omega^*, 7)^4$ *for any ordinal number* α*.*

One can, using a folklore variation of a theorem of Mycielski and Taylor, cf. [964My], [973GP], [978Ta] and [013D], prove a positive result for the Cantor space by assuming that every set of reals has the property of Baire.

Theorem 7 (ZF + BP)**.** $\langle {}^{\omega}2, <_{lex} \rangle \longrightarrow (1 + \omega^* + \omega + 1 \vee \omega + 1 + \omega^*, 5)^4$.

Furthermore, for countable ordinal numbers α one can strengthen theorem 6:

Theorem 8 (ZF)**.** $\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (\omega^* + \omega \vee \omega + \omega^*, 6)^4$ *for any* $\alpha < \omega_1$*.*

Similarly, one can prove the following two theorems:

Theorem 9 (ZF)**.** $\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (\omega^* + \omega \vee \omega + 2 + \omega^*, 5)^4$ *for any* $\alpha < \omega_1$*.*

Theorem 10 (ZF)**.** $\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (2 + \omega^* + \omega \vee \omega + \omega^*, 5)^4$ *for any* $\alpha < \omega_1$*.*

Outlook

We will close by speculating about the situation for $\langle {}^{\omega_1}2, <_{\mathrm{lex}} \rangle$ under the axiom of determinacy in light of the following theorems.

Theorem 11 (ZF + AD, cf. [964My])**.** BP*.*

Theorem 12 (ZF + AD, Martin, 1973, cf. [981K])**.** $\omega_1 \longrightarrow (\omega_1)^{\omega_1}_2$*.*

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