Applications of the anonymous collapse

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Who is the anonymous collapse?

It is everyone,
and it is no one.

We will describe a forcing $A(\mu, \kappa)$ that turns a large cardinal κ into a successor cardinal μ^+, where μ is regular.

For a large class of "ordinary" collapsing posets $\mathbb{P}$, $A(\mu, \kappa)$ completely embeds into $\mathbb{P} * Add(\kappa)$ and absorbs all of the μ-sequences added by $\mathbb{P}$.

On the other hand, $A(\mu, \kappa)$ is not "ordinary," nor does anything "ordinary" embed into it. This makes the combinatorial structure of $A(\mu, \kappa)$ very difficult to understand.

Forcing preliminaries

Lemma

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are separative partial orders. $\mathcal{B}(\mathbb{P}) \cong \mathcal{B}(\mathbb{Q})$ iff the following holds. Letting $\dot{G}, \dot{H}$ be the canonical names for the generic filters for $\mathbb{P}, \mathbb{Q}$ respectively, there is a $\mathbb{P}$-name for a function $\dot{f}_0$ and a $\mathbb{Q}$-name for a function $\dot{f}_1$ such that:

(1) $\Vdash_{\mathbb{P}} \dot{f}_0(\dot{G})$ *is a $\mathbb{Q}$-generic filter,*

(2) $\Vdash_{\mathbb{Q}} \dot{f}_1(\dot{H})$ *is a $\mathbb{P}$-generic filter,*

(3) $\Vdash_{\mathbb{P}} \dot{G} = \dot{f}_1^{\dot{f}_0(\dot{G})}(\dot{f}_0(\dot{G}))$, *and* $\Vdash_{\mathbb{Q}} \dot{H} = \dot{f}_0^{\dot{f}_1(\dot{H})}(\dot{f}_1(\dot{H}))$.

An isomorphism is given by $p \mapsto ||\check{p} \in \dot{f}_1(\dot{H})||_{\mathcal{B}(\mathbb{Q})}$.

Forcing preliminaries

For any forcing $\mathbb{P}$ and any $\mathbb{P}$-name $\dot{X}$ for a set of ordinals, there is a canonically associated complete subalgebra $\mathcal{A}_{\dot{X}} \subseteq \mathcal{B}(\mathbb{P})$ that captures $\dot{X}$. $\mathcal{A}_{\dot{X}}$ has the property that whenever $G \subseteq \mathbb{P}$ is generic, $\dot{X}^G$ and $G \cap \mathcal{A}_{\dot{X}}$ are definable from each other using the parameters $\mathcal{B}(\mathbb{P})$ and its powerset, as computed in the ground model. In this case, we have
$V[\dot{X}^G] = V[G \cap \mathcal{A}_{\dot{X}}]$.

Layering and absorption

Definition

We will call a partial order $\mathbb{P}$ (μ, κ)-*nicely layered* when there is a collection $\mathcal{L}$ of regular suborders of $\mathbb{P}$ such that:

(1) for all $\mathbb{Q} \in \mathcal{L}$, $\mathbb{Q}$ is μ-closed and has size $< \kappa$,

(2) for all $\mathbb{Q}_0, \mathbb{Q}_1 \in \mathcal{L}$, if $\mathbb{Q}_0 \subseteq \mathbb{Q}_1$, then $\Vdash_{\mathbb{Q}_0} \mathbb{Q}_1 / \dot{G}$ is μ-closed, and

(3) for all $\mathbb{P}$-names $\dot{f}$ for a function from μ to the ordinals, and all $\mathbb{Q}_0 \in \mathcal{L}$, there is an $\mathbb{Q}_1 \in \mathcal{L}$ and an $\mathbb{Q}_1$-name $\dot{g}$ such that $\mathbb{Q}_0 \subseteq \mathbb{Q}_1$, and $\Vdash_{\mathbb{P}} \dot{f} = \dot{g}$.

We will say $\mathbb{P}$ is (μ, κ)-*nicely layered with collapses,* (μ, κ)-NLC, when additionally for all $\alpha < \kappa$ and all $\mathbb{Q}_0 \in \mathcal{L}$, there is $\mathbb{Q}_1 \in \mathcal{L}$ such that $\mathbb{Q}_0 \subseteq \mathbb{Q}_1$, $\Vdash_{\mathbb{Q}_0} |\mathbb{Q}_1 / \dot{G}| \geq |\alpha|$, and $\Vdash_{\mathbb{Q}_1} |\mathbb{Q}_1| = \mu$.

Proposition

If $\mathcal{L}$ witnesses that $\mathbb{P}$ is (μ, κ)-nicely layered, then $\mathbb{P}$ is κ-c.c. and $\bigcup \mathcal{L}$ is dense in $\mathbb{P}$.

Examples of (μ, κ)-NLC forcings

(1) The Levy collapse, $Col(\mu, < \kappa)$, when $\alpha^{<\mu} < \kappa$ for $\alpha < \kappa$. When $\kappa = \mu^+$, this is isomorphic to $Add(\mu, \kappa)$.

(2) The Silver collapse $\mathbb{S}(\mu, \kappa)$, when $\alpha^{\mu} < \kappa$ for $\alpha < \kappa$. This can be viewed as the collection of partial functions $p : \mu \times \kappa \to \kappa$, such that $|p| \leq \kappa$, $\forall\alpha\forall\beta p(\alpha, \beta) < \beta$, and there is some $\gamma < \mu$ such that $\text{dom}(p) \subseteq \gamma \times \kappa$.

(3) The Easton collapse $E(\mu, \kappa)$, when κ is Mahlo. This is the Easton-support product of $\langle Col(\mu, \alpha) : \alpha < \kappa \rangle$. Introduced by Shioya, 2011.

(4) If κ is weakly compact, $\mathbb{P}$ is κ-c.c., and $\Vdash_{\mathbb{P}} \kappa = \aleph_1$, $\mathbb{P}$ is (ω, κ)-NLC.

Layering and absorption

Lemma (Folklore)

If $\mathbb{P}$ is a μ-closed partial order such that $\Vdash_{\mathbb{P}} |\mathbb{P}| = \mu$, then $\mathcal{B}(\mathbb{P}) \cong \mathcal{B}(Col(\mu, |\mathbb{P}|))$.

Rearrangement Lemma

Suppose $\mu < \kappa$ are regular, and $\mathbb{P}$ is (μ, κ)-NLC. If G is $\mathbb{P}$-generic over V, then there is a forcing $\mathbb{R} \in V[G]$ such that $\mathbb{R}$ adds a filter $H \subseteq Col(\mu, < \kappa)$ which is generic over V and such that $(Ord^{\mu})^{V[G]} = (Ord^{\mu})^{V[H]}$.

In $V[G]$, let $\mathbb{R}$ be the collection of filters $h \subseteq Col(\mu, < \alpha)$ for $\alpha < \kappa$ which are generic over V, such that for some $\mathbb{Q} \in \mathcal{L}$, $V[h] = V[G \cap \mathbb{Q}]$. The ordering is end-extension.

The anonymous collapse

Let κ be a regular cardinal whose regularity is preserved by a forcing $\mathbb{P}$. Let $A(\mathbb{P})$ be the complete subalgebra of $\mathcal{B}(\mathbb{P} * Add(\kappa))$ generated by the canonical name for the $Add(\kappa)$-generic set. More precisely, if $e : \mathbb{P} * Add(\kappa) \to \mathcal{B}(\mathbb{P} * Add(\kappa))$ is the canonical dense embedding, $A(\mathbb{P})$ is completely generated by the elements of the form $e(\langle 1, \{\langle \dot{\alpha}, 1 \rangle\}\rangle)$.

In the case that $\alpha^{<\mu} < \kappa$ for all $\alpha < \kappa$ and $\mathbb{P} = Col(\mu, <\kappa)$, denote $A(\mathbb{P})$ by $A(\mu, \kappa)$, and write $B(\mu, \kappa)$ for $\mathcal{B}(Col(\mu, <\kappa) * Add(\kappa))$.

Lemma

If $\mathbb{P}$ is (μ, κ)-NLC, and $H \subseteq A(\mathbb{P})$ is generic over V, then $\mathcal{B}(\mathbb{P} * Add(\kappa))/H$ is κ-distributive in $V[H]$.

The anonymous collapse

Lemma

Let V be a countable transitive model of ZFC (or just assume generic extensions are always available), and assume $\Vdash^V_{\mathbb{P}} \kappa$ is regular. If $X \subseteq \kappa$, the following are equivalent:

(1) *X is $A(\mathbb{P})$-generic over V.*

(2) *There is $G \subseteq \mathbb{P}$ such that G is generic over V, and X is $Add(\kappa)$-generic over $V(P_0)$, where $P_0 = \mathcal{P}_\kappa(\kappa)^{V[G]}$.*

Theorem

For any $\mathbb{P}$ that is is (μ, κ)-NLC, $A(\mathbb{P})$ and $A(\mu, \kappa)$ are isomorphic.

The anonymous collapse

Proof: Let X be $A(\mathbb{P})$-generic over V. There is a κ-distributive forcing over $V[X]$ to get G such that $G * X$ is $\mathbb{P} * Add(\kappa)$-generic over V. By the "rearrangement lemma," we can do further forcing to obtain $H \subseteq Col(\mu, <\kappa)$ generic over V such that $(Ord^{\mu})^{V[H]} = (Ord^{\mu})^{V[G]}$. By the previous lemma, X is also $A(\mu, \kappa)$-generic over V.

We can also show that conversely, every $A(\mu, \kappa)$-generic X is $A(\mathbb{P})$-generic. This uses the previous lemmas, plus the "weak homogeneity" of the Levy collapse and Cohen forcing.

This implies that we have a canonical correspondence between $A(\mathbb{P})$- and $A(\mu, \kappa)$-generic filters, i.e. definable functions f, g such that for any generic H for $A(\mathbb{P})$, $f(H)$ is the generic for $A(\mu, \kappa)$ computed from X_H, and vice versa, and $g(f(H)) = H$. For $p \in A(\mathbb{P})$, put $\iota(p) = ||p \in g(\dot{H})||_{A(\mu,\kappa)}$. By the preliminary forcing fact, this is an isomorphism.

Non-absoluteness applications

It is easy to see for regular $\mu < \kappa$ such that $\alpha^{<\mu} < \kappa$ for all $\alpha < \kappa$, $Col(\mu, < \kappa) \times Add(\mu, \lambda)$ is (μ, κ)-NLC for every λ. Thus if X is $A(\mu, \kappa)$-generic, then for any λ, we may further force to obtain a model which is a $(Col(\mu, < \kappa) \times Add(\mu, \lambda)) * Add(\kappa)$-generic extension with the same Ord^{μ}.

Taking inner models given by such $Col(\mu, < \kappa) \times Add(\mu, \lambda)$-generic sets, we produce many models with the same cardinals and same $\mathcal{P}(\mu)$, each assigning a different cardinal value for 2^{μ}.

Loosely speaking, this means adding ω_1 many Cohen reals is the same as adding any number of Cohen reals.

Non-absoluteness applications

Theorem (Woodin-Shelah, Todorčević independently?)

If κ is a Woodin cardinal, then $\Vdash_{Col(\omega_1,<\kappa)}$ "There is an ω_2-saturated ideal on ω_1."

Theorem (Jech-Prikry)

If CH holds and there is an ω_2-saturated ideal on ω_1, then $2^{\omega_1} = \omega_2$.

Thus if κ is Woodin, forcing with $A(\omega_1, \kappa)$ (and going up and down) gets us two models with the same $\mathcal{P}(\omega_1)$, one of which has a saturated ideal on ω_1 and the other does not.

Non-absoluteness applications

If κ is weakly compact, then for every κ-c.c. partial order $\mathbb{P}$, and every $\mathbb{P}$-name for a function $f : \mu \to Ord$ with $\mu < \kappa$, there is a regular suborder $\mathbb{Q}$ of size $< \kappa$ that captures f. Furthermore, if $\mathbb{P}$ also forces $\kappa = \aleph_1$, then the set $\mathcal{L}$ of all regular suborders of size $< \kappa$ witnesses that $\mathbb{P}$ is (ω, κ)-NLC. Thus an extremely wide variety of forcing extensions with very different theories can be obtained, each sharing the same reals and same cardinals.

For example, we can start with a measurable κ and some weakly compact $\delta < \kappa$. If we force with $Col(\omega, < \delta) * \dot{\mathbb{Q}}$, where $\mathbb{Q}$ is *any* c.c.c. forcing, then this is (ω, δ)-NLC.

Non-absoluteness applications

We obtain different models with the same cardinals and reals satisfying:

(1) CH.
(2) MA $+ \mathfrak{c} = \omega_2$. (Solovay-Tennebaum)
(3) MA $+ \mathfrak{c}$ carries a c.c.c. ideal (thus is weakly Mahlo, Rowbottom, has the tree property, etc.)
(4) There is a nonmeager $A \subseteq \mathbb{R}$ of size ω_1, and $\mathfrak{c}$ carries a c.c.c. ideal, and there are $\mathfrak{c}$-many non-isomorphic Suslin trees.
(5) $\mathfrak{c}$ is real-valued measurable. (Solovay)
(6) There is a nonmeager set $A \subseteq \mathbb{R}$ such that $\mathcal{P}(A)/\textit{meager}$ is c.c.c. (Komjath)
(7) There is a set $A \subseteq \mathbb{R}$ of positive outer measure such that $\mathcal{P}(A)/\textit{null}$ is c.c.c. (Shelah)

Dense ideals

The main reason for exploring $A(\mu, \kappa)$ was to investigate ideals of minimal density on various spaces. To force such things, we start from almost-huge cardinals.

κ is called *almost-huge* if there is an elementary embedding $j : V \to M$ with critical point κ, such that $M^{<j(\kappa)} \subseteq M$. This is characterizable by a coherent system of normal measures of $\mathcal{P}_\kappa(\alpha)$ for $\kappa \leq \alpha < \delta = j(\kappa)$, with some special properties. This is called an "almost-huge tower of height δ."

The nice feature of these towers is they produce an embedding where the closure ends at δ. We will have $j(\delta) < \delta^+$ and $j[\delta]$ is cofinal in δ. This is useful for forcing constructions.

Dense ideals

We use a strengthening of "nicely layered."

Definition

$\mathbb{P}$ is (μ, κ)-*very nicely layered (with collapses)* when there is a sequence $\langle \mathbb{Q}_\alpha : \alpha < \kappa \rangle = \mathcal{L}$ such that:

(1) $\mathcal{L}$ witnesses that $\mathbb{P}$ is (μ, κ)-nicely layered (with collapses),

(2) $\mathcal{L}$ is $\subseteq$-increasing,

(3) every subset of $\mathbb{P}$ of size $< \mu$ with a lower bound has an infimum, and

(4) there is a system of continuous projection maps $\pi_\alpha : \mathbb{P} \to \mathbb{Q}_\alpha$ such that for each α, $\pi_\alpha \restriction \mathbb{Q}_\alpha = id$, and for $\beta < \alpha < \kappa$, $\pi_\beta = \pi_\beta \circ \pi_\alpha$.

Dense ideals

Theorem

Assume κ carries an almost-huge tower of height δ. Let μ, λ be regular such that $\mu < \kappa \leq \lambda < \delta$. Suppose $\Vdash_{A(\mu,\kappa)}$ "$\dot{\mathbb{P}}$ is (κ, δ)-very nicely layered and forces $\delta = \lambda^+$." If $X * H$ is $A(\mu, \kappa) * \dot{\mathbb{P}}$ is generic, then then in $V[X][H]$, there is a normal, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$.

This means that there is a normal ideal I on $Z = \mathcal{P}_\kappa(\lambda)$ such that the boolean algebra $\mathcal{P}(Z)/I$ has a dense subset of size λ. For a successor cardinal κ, this is the minimal possible value. It is also the minimum number of normal, κ-complete partial 2-valued measures needed to collectively measure all subsets of $\mathcal{P}_\kappa(\lambda)$.

Over $V[X][H]$, we force with $B(\mu, \kappa)/H_X \times Col(\mu, \lambda)$. There will be an ideal with a quotient algebra isomorphic to this forcing.

Dense ideals

The basic idea is the following. Let's describe it when $\kappa = \lambda$. We first add this $A(\mu, \kappa)$-generic $X \subseteq \kappa$. We get a generic H for something very nice like the Levy collapse to make $\delta = j(\kappa)$ equal to κ^+.

We then "complete" the $A(\mu, \kappa)$-generic with the quotient $B(\mu, \kappa)/H_X$. Then we force with $Col(\mu, \kappa)$.

The whole forcing $A(\mu, \kappa) * Col(\kappa, < \delta) * (B(\mu, \kappa)/H_X \times Col(\mu, \kappa)) \cong Col(\mu, < \kappa) * Col(\kappa, < \delta) * Col(\mu, \kappa)$ is (μ, δ)-NLC.

Invoking almost-hugeness and taking advantage of "very nicely layeredness," we can extend the embedding $j : V \to M$ to $\hat{j} : V[X][H] \to M[\hat{X}][\hat{H}]$ in the final model. The forcing to get to the final model from $V[X][H]$ had size κ, and we define a normal ideal on κ based on what this forces about $\hat{j}$.

Dense ideals

We can even get this for many spaces simultaneously. κ is called super-almost-huge when there are class-many $\delta > \kappa$ such that κ carries an almost-huge tower of height δ.

If κ is super-almost huge, we can do the following iteration:

Let $T = \{\alpha : \kappa$ carries an almost-huge tower of height $\alpha\}$. Let C be the closure of T, and let $\langle \alpha_\beta \rangle_{\beta \in Ord}$ be its continuous increasing enumeration. Over $V^{A(\mu,\kappa)}$, let $\mathbb{P}$ be the Easton support iteration of the following:

- Let $\mathbb{P}_0 = Col(\kappa, < \alpha_0)$.
- If β is zero or a successor ordinal, let $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * Col(\alpha_\beta, < \alpha_{\beta+1})$.
- If β is a limit ordinal such that α_β is singular, let $\mathbb{P}_{\beta+1} = Col(\alpha_\beta^+, < \alpha_{\beta+1})$.
- If β is a limit ordinal such that α_β is regular, let $\mathbb{P}_{\beta+1} = Col(\alpha_\beta, < \alpha_{\beta+1})$.

Dense ideals

One can show by induction that this iteration preserves the regularity of the members of T, the successors of the singular limit points of T, and the regular limit points of T. Further, the class of non-limit-points of T becomes the class of successors of regular cardinals above κ.

The general theorem applies to initial segments of this iteration to show dense ideals are constructed, and the tail ends preserve this. We get a model where for all regular $\lambda \geq \kappa$, there is a normal, κ-complete, λ-dense ideal on $\mathcal{P}_\kappa(\lambda)$.

Generic supercompactness and square

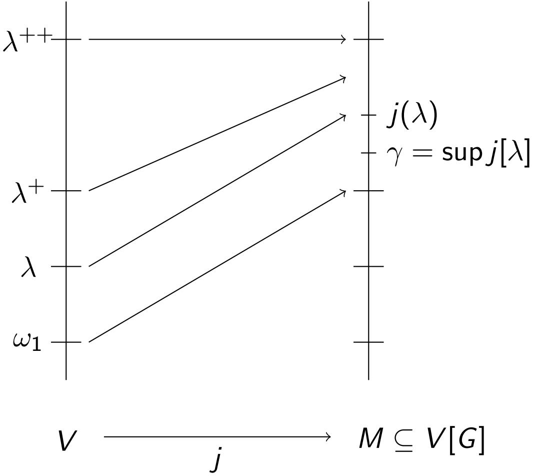
Taking $\mu = \omega$ we get a model in which for all regular $\lambda \geq \omega_1$, there is a normal and fine ideal I on $Z = \mathcal{P}_{\omega_1}(\lambda)$ such that $\mathcal{P}(Z)/I \cong \mathcal{B}(Col(\omega, \lambda))$. This is the strongest possible version of "ω_1 is generically supercompact."

Solovay proved that if κ is strongly compact, $\Box_\lambda$ fails for all $\lambda \geq \kappa$. The analogous statement does not hold for generic supercompactness.

The key to Solovay's theorem is that $\Box_\lambda$ contradicts stationary reflection at λ^+. To show stationary reflection at λ^+, we take a λ^+-supercompactness embedding $j : V \to M$, and note that $\sup j[\lambda^+] < j(\lambda^+)$, enabling a reflection argument. Ironically, this same feature lets us show squares are compatible with generic supercompactness.

If $j : V \to M \subseteq V[G]$ comes from a saturated ideal on $\mathcal{P}_{\omega_1}(\lambda)$, then $j(\omega_1) = \lambda^+$ and $M^\lambda \subseteq M$. We have GCH in V.

Generic supercompactness and square



Note: $M \models \mathrm{cf}(\gamma) = \omega$.
Also, $V[G] \models |j(\lambda)| = \omega_1$.

Generic supercompactness and square

The standard forcing $\mathbb{S}_\kappa$ to add a $\square_\kappa$ sequence has size κ^+ under GCH, so it is absorbed into $Col(\omega, \kappa^+)$. So if $\lambda \geq \kappa^+$, there is a regular embedding of $\mathbb{S}_\kappa$ into $\mathcal{P}(\mathcal{P}_{\omega_1}\lambda)/I$.

The conditions in $\mathbb{S}_\kappa$ are coherent sequences of clubs $\vec{C}_\alpha$ of length $\alpha < \lambda$. If H is generic for $\mathbb{S}_\kappa$, then H and $j \restriction H$ are in M.

If we pick any ω-sequence C_γ cofinal in γ, then $\bigcup\{j(\vec{C}) : \vec{C} \in H\}^\frown C_\gamma$ is a "master condition" in $j(\mathbb{S}_\kappa)$.

Since $\mathbb{S}_\kappa$ has size ω_1 in $V[G]$ and is countably closed, we can build an M-generic filter containing the master condition. Thus we may generically extend the embedding j to domain $V[H]$.

We get an ideal J in $V[H]$ such that $\mathcal{P}(\mathcal{P}_{\omega_1}\lambda)/I \cong \mathbb{S}_\kappa * \mathcal{P}(\mathcal{P}_{\omega_1}\lambda)/J$.

Generic supercompactness and square

This gets dense ideals on $\mathcal{P}_{\omega_1}(\lambda)$ for regular $\lambda > \omega_1$, plus $\square_\kappa$ with $\kappa < \lambda$. Note that we can add $\square_\kappa$ on cardinals $\geq \lambda$ and preserve the dense ideals, since $\mathbb{S}_\kappa$ adds no subsets of κ.

We can even iterate this to obtain dense ideals on $\mathcal{P}_{\omega_1}(\lambda)$ for every regular λ while having $\square_\kappa$ for all κ.

We can also get this with ω_1 replaced by a higher successor cardinal. This takes more care, and we have to absorb a "threading" forcing to get the master conditions.

This ability to absorb forcings into the ideal is what's missing from the ordinary supercompact picture.

Nonregular ultrafilters

These models also answer a question of Foreman related to an old conjecture in model theory.

Conjecture (all the people some of the time)

Suppose $\mathfrak{A}$ is an infinite structure, and U is a countably incomplete, uniform ultrafilter on a cardinal κ. Then $|\mathfrak{A}^\kappa/U| = |\mathfrak{A}|^\kappa$.

This was shown by Donder to be true in the core model below a measurable cardinal. But various people have forced various counterexamples from large cardinals. Done by Adler, Magidor, Woodin+Laver, Foreman-Magidor-Shelah, Foreman, Kunen+Huberich, and Jin-Shelah.

Nonregular ultrafilters

Question (Foreman)

Is it consistent that there is a uniform ultrafilter U on ω_3 such that ω^{ω_3}/U has cardinality ω_3? Is it consistent that there is a uniform ultrafilter U on $\aleph_{\omega+1}$ such that $\omega^{\aleph_{\omega+1}}/U$ has cardinality $\aleph_{\omega+1}$? Give a characterization of the possible cardinalities of ultrapowers.

Theorem

If a super-almost-huge cardinal is consistent, then it is consistent that for every regular uncountable cardinal κ, there is a uniform ultrafilter U on κ such that $|\omega^\kappa/U| = \kappa$. Further, ω can be replaced any small regular cardinal μ like ω_n or $\aleph_{\omega^2+1}$, considering then regular $\kappa > \mu$.

Burning questions

When we have a regular μ and an almost-huge $\kappa > \mu$ with a tower of height δ, we get a model where $\kappa = \mu^+$, and there is a κ-complete, κ-dense ideal on κ by forcing with $A(\mu, \kappa) * Col(\kappa, < \delta)$. This dense ideal would be preserved if we then added a Cohen subset to δ. So what happens if we force with $A(\mu, \kappa) * A(\kappa, \delta)$?

What if δ also almost-huge with a tower of height θ? Maybe we could force with $A(\omega, \kappa) * A(\kappa, \delta) * Col(\delta, < \theta)$ to get dense ideals on ω_1 and ω_2 simultaneously. Knowing whether this happens seems to depend on:

Question

How close is $A(\mu, \kappa)$ to being (μ, κ)-very nicely layered? In particular, does it have a μ-closed dense subset?

Also, we produced a normal ideal, for example on $\kappa = \omega_2$, with algebra isomorphic to the completion of $B(\omega_1, \kappa)/H_X \times Col(\omega_1, \omega_2)$. What the heck is the first part? Is it countably closed?

The bad news

Here's a reason why these questions are hard.

Theorem

If κ is inaccessible, no (μ, κ)-NLC forcing regularly embeds into $A(\mu, \kappa)$. In fact, no κ-c.c. forcing of size κ embeds.

Proof sketch: To show this, we first isolate two properties of a forcing extension that depend on two regular cardinals $\mu < \kappa$:

(1) *Levy(μ, κ):* $(\exists A \in [\kappa]^\kappa)(\forall y \in [\kappa]^\mu \cap V) y \not\subseteq A$.

(2) *Silver(μ, κ):* $(\exists A \in [\kappa]^\kappa)(\forall X \in [\kappa]^\kappa \cap V)(\exists y \in [X]^\mu \cap V) y \cap A = \emptyset$.

Note that these are both Σ_1 properties of the parameters $([\kappa]^\mu)^V$ and $([\kappa]^\kappa)^V$. For any partial order $\mathbb{P}$, and collection of dense subsets $\mathcal{D} \subseteq \mathcal{P}(\mathbb{P})$ the statement, "There is a filter $G \subseteq \mathbb{P}$ that is $\mathcal{D}$-generic," is also a Σ_1 property of $\mathbb{P}$ and $\mathcal{D}$.

The bad news

Now consider the following property:

$$(*)_{\mu,\kappa} : (\forall X \in [\mathbb{P}]^{\kappa})(\exists y \in [X]^{\mu}) y \text{ has a lower bound in } \mathbb{P}.$$

Lemma

If $\mathbb{P}$ is a partial order of size κ satisfying $(*)_{\mu,\kappa}$, then $\mathbb{P}$ forces Silver(μ, κ).

Lemma

If $\mathbb{P}$ is a κ-c.c. partial order of size κ satisfying $\neg(*)_{\mu,\kappa}$, then some $p \in \mathbb{P}$ forces Levy(μ, κ).

The bad news

Lemma

Suppose $\mu < \kappa$, μ is regular for all $\alpha < \kappa$, $\alpha^\mu < \kappa$. There are two (μ, κ)-NLC partial orders $\mathbb{P}_0$ and $\mathbb{P}_1$ such that $\mathbb{P}_0$ forces Levy $\wedge \neg$Silver, and $\mathbb{P}_1$ forces $\neg$Levy $\wedge$ Silver.

Corollary

Suppose μ, κ, $\mathbb{P}_0$, and $\mathbb{P}_1$ are as above. Let G be $\mathbb{P}_0$-generic and H be $\mathbb{P}_1$-generic over V. Let $\mathbb{Q} \in V$ be partial order. If $\mathbb{Q}$ forces Levy(μ, κ), then $V[H]$ has no $\mathbb{Q}$-generic, and if $\mathbb{Q}$ forces Silver(μ, κ), then $V[G]$ has no $\mathbb{Q}$-generic. If $\mathbb{Q}$ is κ-c.c. and of size κ, then no κ-closed forcing extension of $V[G]$ or $V[H]$ can introduce a generic for $\mathbb{Q}$.

The bad news

Proof of the theorem: Let $\mathbb{Q}$ be any κ-c.c. forcing of size κ. Whenever $X \subseteq \kappa$ is $A(\mu, \kappa)$-generic, there are two further forcings $\mathbb{R}_0, \mathbb{R}_1$ over $V[X]$ that respectively get filters G, H such that:

- $V[G][X]$ is $\mathbb{P}_0 * Add(\kappa)$-generic, and
- $V[H][X]$ is $\mathbb{P}_1 * Add(\kappa)$-generic.

Suppose to the contrary that $\mathbb{Q}$ embeds into $A(\mu, \kappa)$. If $(*)$ holds for $\mathbb{Q}$, then $V[G][X]$ has no $\mathbb{Q}$-generics, contradicting the supposition that there is one in $V[X]$. If $(*)$ fails for $\mathbb{Q}$, then we can take X such that $V[X]$ contains a $\mathbb{Q}$-generic K such that for some $q \in K$, $q \Vdash Levy$. But $V[H][X]$ has no such generic K.

Thanks for your attention.