Generic absoluteness, determinacy, and the form of $\mathrm{Ultimate}\text{-}L$

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The building blocks for inner models: Extenders

Suppose that

$$j : V \to M$$

is an elementary embedding with critical point κ, $\kappa < \eta$, and that

$$\mathcal{P}(\eta) \subset M.$$

The (strong) extender E of length η derived from j

The **extender** *E of* **length** *η defined from j is the function:*

$$E : \mathcal{P}(\eta) \to \mathcal{P}(\eta)$$

where $E(A) = j(A) \cap \eta$.

Two ordinals associated to the extender E:

- $\mathrm{CRT}(E) = \min\{\alpha \mid E(\alpha) \neq \alpha\} = \kappa$.
- $\mathrm{LTH}(E) = \eta$ where $\mathrm{dom}(E) = \mathcal{P}(\eta)$.

Large cardinal axioms in terms of extenders

δ is a **strong cardinal** *if for each $\gamma > \delta$ there exists an extender E such that $\mathrm{CRT}(E) = \delta$ and $\mathrm{LTH}(E) \geq \gamma$.*

δ is an **extendible cardinal** *if for each $\gamma > \delta$ there exists an extender E such that $\mathrm{CRT}(E) = \delta$, $E(\delta) > \gamma$, and $\mathrm{LTH}(E) > E(\gamma)$.*

Using Magidor's Lemma:

δ is a **supercompact cardinal** *if for each $\gamma > \delta$ there exists an extender E such that $E(\mathrm{CRT}(E)) = \delta$ and $\mathrm{LTH}(E) \geq \gamma$.*

Gödel's transitive class HOD

- For each set X, $\mathrm{TC(X)}$ is the smallest transitive set M with $X \in M$.

Definition

For each ordinal α, $\mathrm{HOD}_{\alpha+1}$ is the set of all sets $X \subseteq V_\alpha$ such that:

1. X is definable in V_α from ordinal parameters.
2. If $Y \in \mathrm{TC(X)}$ then Y is definable in V_α from ordinal parameters.

- The definition of $\mathrm{HOD}_{\alpha+1}$ is a mixture of the definition of $L_{\alpha+1}$ and $V_{\alpha+1}$.

Definition (Gödel)

HOD is the class of all sets X such that $X \in \mathrm{HOD}_{\alpha+1}$ for some α.

Weak extender models

For a large cardinal axiom Φ:

Definition

A transitive class N is a **weak extender model for** Φ if Φ is witnessed to hold in N by extenders E of N such that

$$E = F|N$$

for some extender F.

- If Φ holds in V then V is a weak extender model for Φ.

For a large cardinal axiom Φ and weak extender models, the simplest goal of the Inner Model Program is to answer the question:

Question

Assume that Φ holds. **Must** *there exist a weak extender model N for Φ such that $N \subseteq \mathrm{HOD}$?*

The Universality Theorem

Theorem (Universality Theorem)

Suppose that N is a weak extender model for δ is supercompact. Suppose that F is an extender such that:

- $\mathrm{CRT}(F) \geq \delta$ *and N is closed under F.*

Then $F|N \in N$.

- For any extender F, L is closed under F but $F|L \notin L$.
- **Any** weak extender model for δ is supercompact inherits **all** large cardinals from V which occur above δ.
  - (in a strong enough form)

Conclusion

The extension of the Inner Model Program to the level of one supercompact cardinal must yield the ultimate inner model

- *it must yield an ultimate version of L.*

Weak extender models for supercompact cardinals

Theorem

Suppose that δ is an extendible cardinal. Then the following are equivalent.

(1) *There exists a regular cardinal $\kappa > \delta$ which is not a measurable cardinal in HOD.*

(2) *There is a weak extender model N for δ is a supercompact cardinal such that $N \subseteq \mathrm{HOD}$.*

(3) HOD *is a weak extender model for δ is a supercompact cardinal.*

- Perhaps we need a different test question for the Inner Model Program.
  - One option is to refine the notion of a weak extender model.

Partial-extenders and partial-extender models

A **partial-extender** *E of length η is obtained from an elementary embedding*

$$j : N \to M$$

where $N \cap \mathcal{P}(\eta) = M \cap \mathcal{P}(\eta)$:

1. *E has domain $N \cap \mathcal{P}(\eta)$;*
2. *$E(A) = j(A) \cap \eta$.*

Definition

A transitive class N is a **partial-extender model for** Φ if for some sequence $\mathbb{E}$ of partial-extenders:

1. $N = L[\mathbb{E}]$,
2. N is a weak extender model for Φ and this is witnessed by the partial extenders on the sequence $\mathbb{E}$.

Good partial-extender models

Definition (Coherence)

$\mathbb{E}$ is **coherent** if for all $\alpha \in \mathrm{dom}(\mathbb{E})$, $E(\mathbb{E}|\eta) = \mathbb{E}|\eta$ where E is the partial extender given by $\mathbb{E}_\alpha$ and $\mathrm{dom}(E) = \mathcal{P}(\eta) \cap N$.

Definition

Suppose $L[\mathbb{E}]$ is a partial-extender model. Then $L[\mathbb{E}]$ is a **good partial-extender model** if $\mathbb{E}$ is coherent and for all $\gamma < \alpha$, if

$$X \prec (L_\alpha[\mathbb{E}], \mathbb{E} \cap L_\alpha[\mathbb{E}])$$

is the elementary substructure given by the elements which are definable with parameters from γ and if γ is an $L[\mathbb{E}]$-cardinal, then

$$X \cong (L_\beta[\mathbb{E}], \mathbb{E} \cap L_\beta[\mathbb{E}])$$

for some β.

- If $L[\mathbb{E}]$ is a good partial-extender model then the Generalized Continuum Hypothesis holds in $L[\mathbb{E}]$.

Mitchell-Steel models

- The basic framework for good partial-extenders models for large cardinals up to the level of superstrong cardinals originates in the constructions of Mitchell and Steel.
  - There is an important variation due to Jensen which is equivalent but yields models with stronger condensation properties.

Theorem (Mitchell-Steel et al)

Assume the Iteration Hypothesis and that there is a proper class of superstrong cardinals. Then there is a partial-extender model $L[\mathbb{E}]$ for a proper class of superstrong cardinals such that

(1) $L[\mathbb{E}] \subseteq \mathrm{HOD}$,

(2) $L[\mathbb{E}]$ *is a good partial-extender model.*

Generalized Mitchell-Steel models

Definition

A cardinal κ is an α-**extendible cardinal** where $\alpha < \kappa$ if there exists an elementary embedding

$$j : V_{\kappa+\alpha} \to V_{j(\kappa)+\alpha}$$

with critical point κ.

Theorem

Assume the Iteration Hypothesis and that there is a proper class of $(\omega+1)$-extendible cardinals. Then there is a partial-extender model $L[\mathbb{E}]$ for a proper class of ω-extendible cardinals such that

(1) $L[\mathbb{E}] \subseteq \mathrm{HOD}$,

(2) $L[\mathbb{E}]$ *is a good partial-extender model.*

(3) $L[\mathbb{E}] \models$ *"There is a Σ^2_2-wellordering of $\mathbb{R}$".*

Good partial extender models and factoring

Definition: cut-point

Suppose $L[\mathbb{E}]$ is a partial extender model. Then κ is a **cut-point** *of $\mathbb{E}$ if*

$$\mathrm{LTH}(\mathbb{E}_\alpha) < \kappa$$

for all $\alpha \in \mathrm{dom}(\mathbb{E})$ such that $\mathrm{CRT}(\mathbb{E}) < \kappa$.

Definition: factoring

A good partial extender model $L[\mathbb{E}]$ satisfies **factoring** *if for all $L[\mathbb{E}]$-cardinals κ which are strongly inaccessible in $L[\mathbb{E}]$, if κ is a cut-point of $L[\mathbb{E}]$ and*

$$L_\kappa[\mathbb{E}][g] = M[h],$$

then there exists M^* such that

$$L[\mathbb{E}][g] = M^*[h]$$

and such that $M \cap V_\kappa = M^* \cap V_\kappa$.

Good partial extender models and strong factoring

Definition: strong factoring

Suppose that $L[\mathbb{E}$ is a good partial extender model which satisfies factoring.

Then $L[\mathbb{E}]$ satisfies **strong factoring**, *if for all $L[\mathbb{E}]$-cardinals κ which are strongly inaccessible in $L[\mathbb{E}]$:*

- *If κ is a cut-point of $L[\mathbb{E}]$ and there is a Woodin cardinal of $L[\mathbb{E}]$ below κ, then*

  $$L_\kappa[\mathbb{E}]$$

  is a nontrivial generic extension.

Theorem

Suppose $L[\mathbb{E}]$ is an iterable generalized Mitchell-Steel model. Then $L[\mathbb{E}]$ satisfies strong factoring.

Is $\mathrm{Ultimate}\text{-}L$ a generalized Mitchell-Steel model?

Assume the Iteration Hypothesis holds in V and that there is a proper class of measurable Woodin cardinals.

- It is not known if there exists a Mitchell-Steel model $L[\mathbb{E}]$ for a proper class of measurable Woodin cardinals within which $\mathbb{E}$ is definable (even from parameters).
- Suppose $L[\mathbb{E}]$ is a Mitchell-Steel model within which there exists a Woodin cardinal. The inductive first order requirements on $L_\alpha[\mathbb{E}]$ are very complicated:
  - things only get worse for the generalized Mitchell-Steel models.

Two questions

1. *Is there a simple candidate for the axiom "$V = \mathrm{Ultimate}\text{-}L$"?*
2. *Is $\mathrm{Ultimate}\text{-}L$ even a good partial-extender model which satisfies strong factoring?*

Universally Baire sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}$ is **universally Baire** if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}$, the preimage of A by π has the property of Baire in the space Ω.

- Universally Baire sets are an abstract generalization of the borel sets.

Theorem

Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire.

(1) *Every set $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is universally Baire.*

(2) $L(A, \mathbb{R}) \models \mathrm{AD}^+$.

The Inner Model Program and the universally Baire sets

- $\mathrm{HOD}^{L(A,\mathbb{R})}$ denotes HOD as defined in $L(A,\mathbb{R})$.

For a large cardinal axiom Φ, the non-trivial test question for the Inner Model Program is:

Question

Assume there is a proper class of Woodin cardinals.

- **Can** *there exist a weak extender model N for Φ such that for all $x \in \mathbb{R} \cap N$ there exists a universally Baire set $A \subseteq \mathbb{R}$ such that*

$$x \in \mathrm{HOD}^{L(A,\mathbb{R})}?$$

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and large cardinal axioms

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

- $\Theta^{L(A,\mathbb{R})}$ is a measure of the complexity of A.

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and the Inner Model Program

Theorem (Steel)

Suppose that there is a proper class of Woodin cardinals and let $\delta = \Theta^{L(\mathbb{R})}$.

Then $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$ is a Mitchell-Steel model.

Theorem

Suppose that there is a proper class of Woodin cardinals.

Then $\mathrm{HOD}^{L(\mathbb{R})}$ is **not** *a Mitchell-Steel model.*

- *It is a strategic partial-extender model.*

There is another class of solutions to the Inner Model Program.

The axiom for $V = \mathrm{Ultimate}\text{-}L$

A sentence φ is a Σ_3-sentence if it is of the form:

- There exists α such that $V_\alpha \models \psi$ and such that $V_\alpha \prec_{\Sigma_2} V$;

for some sentence ψ.

Conjecture: The axiom for $V = \mathrm{Ultimate}\text{-}L$

- *There is a strong cardinal which is a limit of Woodin cardinals.*
- *For each Σ_3-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that*

  $$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \models \varphi$$

  where $\Theta = \Theta^{L(A,\mathbb{R})}$.

- This axiom settles (modulo axioms of infinity) *all* sentences about $\mathcal{P}(\mathbb{R})$ (and much more) which have been shown to be independent by Cohen's method.

Consequences of $V = \mathrm{Ultimate}\text{-}L$

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

The Continuum Hypothesis holds.

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

$V = \mathrm{HOD}$.

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

For every real x there exists a universally Baire set $A \subseteq \mathbb{R}$ such that $x \in \mathrm{HOD}^{L(A,\mathbb{R})}$.

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

V does not satisfy strong factoring.

- *V is not a nontrivial generic extension of any inner model,*
  - *V is the minimum universe of the Generic-Multiverse.*

The Ultimate-L Conjecture

Ultimate-L Conjecture

(ZFC) *Suppose that δ is an extendible cardinal. Then there is a transitive class N such that:*

1. *N is a weak extender model for δ is supercompact.*
2. *$N \subseteq \mathrm{HOD}$.*
3. *$N \models$ "$V = \mathrm{Ultimate}\text{-}L$".*

Two questions

1. *Is there a version of the axiom, $V = \mathrm{Ultimate}\text{-}L$, for good partial extender models which satisfy strong factoring?*
2. *If both versions exist, how do we choose between them?*

Generic absoluteness

Theorem

Suppose there is a proper class of measurable Woodin cardinals.

- *Then Σ^2_1-sentences are absolute between all generic extensions of V in which CH holds.*

Theorem

Suppose that there is a proper class of measurable Woodin cardinals and the Iteration Hypothesis holds.

- *Then for each $A \in \Gamma^\infty$, for each Σ^2_1-formula $\varphi(x)$, either*

  $$\mathrm{ZFC} + \mathrm{CH} \vdash_\Omega \varphi[A]$$

  or $\mathrm{ZFC} + \mathrm{CH} \vdash_\Omega (\neg\varphi)[A]$.

Beyond Σ^2_1-generic absoluteness

Question

Does some large cardinal hypothesis imply that Σ^2_2-sentences are absolute between all generic extensions of V in which $\diamond$ holds?

Lemma ($\diamond$)

Suppose that Σ^2_2-sentences are absolute between all generic extensions of V in which $\diamond$ holds.

- *Then there is no Σ^2_2-wellordering of the reals.*

Claim

Any plausible extension of the theory of Mitchell-Steel extender models to the level of a weak extender model of supercompactness

- *must yield a weak extender model of supercompactness in which $\diamond$ holds and there is a Σ^2_2-wellordering of the reals.*

Games of length ω_1

The following theorem is a corollary of a key theorem of Neeman:

Theorem

Suppose that ZFC *is consistent with a proper class of Woodin cardinals which are limits of Woodin cardinals.*

- *Then* ZFC *is consistent with a proper class of Woodin cardinals and every set*

  $$X \subset \{0,1\}^{\omega_1}$$

  which is ordinal definable from a universally Baire set, is determined.

Question

Which games of length ω_1 *can be* **provably** *determined from large cardinal hypotheses?*

Universally Baire closed games of length ω_1

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and $\varphi(x)$ is a formula.

- The closed set
  $$\mathcal{C}^A_\varphi = \{f \in \{0,1\}^{\omega_1} \mid (H(\omega_1), A, \in) \models \varphi[f|\alpha] \text{ for all } \alpha < \omega_1\}$$
  is the closed set given by (A, φ).

Strategies

1. *Strategies are functions $\tau : \{0,1\}^{<\omega_1} \to \{0,1\}$.*
2. *To define universally Baire strategies, one must use a more refined notion.*

Suppose

$$\tau_0 : \mathbb{R} \to \{0,1\},\ \tau_1 : \mathbb{R} \to \mathbb{R}$$

are functions

(τ_0, τ_1) as a winning quasi-strategy

Definition

Suppose $X \subseteq \{0,1\}^{\omega_1}$.

- (τ_0, τ_1) is a **winning quasi-strategy for Player I** in the X-game if $g \in X$ for all
$$(f, g) \in \mathbb{R}^{\omega_1} \times \{0,1\}^{\omega_1}$$
such that for all even $\alpha < \omega_1$,
  1. $f(\alpha)$ codes $(f|\alpha, g|\alpha, \tau_1 \circ f|\alpha)$,
  2. $\tau_0(f(\alpha)) = g(\alpha)$.
- (τ_0, τ_1) is a **winning quasi-strategy for Player II** in the X-game if $g \notin X$ for all
$$(f, g) \in \mathbb{R}^{\omega_1} \times \{0,1\}^{\omega_1}$$
such that for all odd $\alpha < \omega_1$,
  1. $f(\alpha)$ codes $(f|\alpha, g|\alpha, \tau_1 \circ f|\alpha)$,
  2. $\tau_0(f(\alpha)) = g(\alpha)$.

Determinacy and winning quasi-strategies

Theorem

Suppose that there exists a proper class of Woodin cardinals which are limits of Woodin cardinals. Let Γ^∞ be the set of all $A \subseteq \mathbb{R}$ such that A is universally Baire. Then the following are equivalent.

(1) *For each $A \in \Gamma^\infty$ and for each formula $\varphi(x)$,*

$$\mathrm{ZFC} \vdash_\Omega \text{“}\mathcal{C}^A_\varphi \text{ is determined.”}$$

(2) *For each $A \in \Gamma^\infty$, for each formula $\varphi(x)$, there is a universally Baire quasi-strategy (τ_0, τ_1) which witnesses that the game $\mathcal{C}^A_\varphi$ is determined.*

Σ^2_1-generic absoluteness and determinacy

Theorem

Suppose that there exists a proper class of Woodin cardinals which are limits of Woodin cardinals. Let Γ^∞ be the set of all $A \subseteq \mathbb{R}$ such that A is universally Baire. Then the following are equivalent.

(1) *For each $A \in \Gamma^\infty$, for each Σ^2_1-formula $\varphi(x)$, one of the following hold.*
   - $\mathrm{ZFC} + \mathrm{CH} \vdash_\Omega \varphi[A]$.
   - $\mathrm{ZFC} + \mathrm{CH} \vdash_\Omega (\neg\varphi)[A]$.

(2) *For each $A \in \Gamma^\infty$ and for each formula $\varphi(x)$,*

$$\mathrm{ZFC} \vdash_\Omega \text{“}\mathcal{C}^A_\varphi \text{ is determined.”}$$

Question

Is there a version of this theorem for Σ^2_2-generic absoluteness conditioned on $\diamond$?

The term relation of A and condensation

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

- τ_A^∞ *is the class of all* $(\mathbb{P}, \sigma, p)$ *such that*

$$p \Vdash_{\mathbb{P}} \text{“}\sigma \in A_G\text{”}.$$

- τ_A^∞ *satisfies* **condensation** *if for all strongly inaccessible* κ *and for all*

$$\mathcal{X} \prec (V_\kappa, \tau_A^\infty \cap V_\kappa, \in)$$

with transitive collapse (M, τ),

$$\tau = \tau_A^\infty \cap M.$$

Lemma (Proper class of Woodin cardinals)

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

- *Then there exists a universally Baire set B such that τ_B^∞ satisfies condensation and such that $A \leq B$.*

The sets $Y_\varphi^A \subseteq \{0,1\}^{\omega_1}$

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and $\varphi(x)$ is a Σ_1-formula.

- Y_φ^A denotes the set of all $f \in \{0,1\}^{\omega_1}$ such that

$$(H(\omega_2), \tau_A^\infty \cap H(\omega_2), \in) \models \varphi[f].$$

Lemma (Proper class of strongly inaccessible cardinals)

Suppose that $A \subseteq \mathbb{R}$ is universally Baire, τ_A^∞ satisfies condensation, and $X \subset \{0,1\}^{\omega_1}$. Then the following are equivalent.

1. *X is Σ_1-definable from in $(V, \tau_A^\infty, \in)$ from ω_1.*
2. *There exists a Σ_1-formula $\varphi(x)$ such that $X = Y_\varphi^A$.*

The sets Y^A_φ and CH

Assume CH, $A \subseteq \mathbb{R}$ is universally Baire, and there is a proper class of Woodin cardinals:

- The set $Y = \{f \in \{0,1\}^{\omega_1} \mid \mathbb{R} \subset L[f]\}$ is not of the form Y^A_φ for any choice of φ,
  - but it is Σ_1-definable from $\mathbb{R}$.

Lemma (CH)

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and τ^∞_A satisfies condensation. Then the following are equivalent.

(1) *Every $X \subset \{0,1\}^{\omega_1}$ which Σ_1-definable from $(A, \mathbb{R})$, is determined.*

(2) *Every $X \subset \{0,1\}^{\omega_1}$ which is of the form Y^A_φ, is determined.*

A generic form of $\diamond$

Definition: $\diamond_G^\infty$

For each Σ_2-sentence, φ, and for each universally Baire set A,

$$(H(\omega_2), \tau_A^\infty \cap H(\omega_2), \in) \models \varphi$$

if and only if

$$(H(\omega_2), \tau_A^\infty \cap H(\omega_2), \in)^{V[G]} \models \varphi$$

where $G \subset \mathrm{Coll}(\omega_1, \mathbb{R})$ is V-generic.

The main theorem

Theorem (IH + Proper class of measurable Woodin cardinals)

The following are equivalent.

(1) *For each $A \in \Gamma^\infty$ and for each Σ^2_2-formula $\varphi(x)$ one of the following hold.*

- $\mathrm{ZFC} + \diamond^\infty_G \vdash_\Omega \varphi[A]$.
- $\mathrm{ZFC} + \diamond^\infty_G \vdash_\Omega (\neg\varphi)[A]$.

(2) *For each $A \in \Gamma^\infty$,*

$$\mathrm{ZFC} + \diamond^\infty_G \vdash_\Omega \text{“}\Sigma_1(A)\text{-determinacy holds for } \{0,1\}^{\omega_1}\text{”}.$$

Why $\diamond_G^\infty$ and not just $\diamond$?

Theorem (Paul Larson)

Suppose that there is a proper class of Woodin cardinals.

- *Then there is a Σ_1-formula $\varphi(x)$ and a partial order $\mathbb{P}$ such that in $V[G]$, the game $Y_\varphi^{\mathbb{R}}$ is not determined.*

Claim

Larson's theorem shows one cannot hope to prove from any (consistent) large cardinal hypothesis that that all the games Y_φ^A are determined.

- Recall that assuming the Iteration Hypothesis and that there is a proper class of measurable Woodin cardinals then all the games $\mathcal{C}_\varphi^A$ are $\vdash_\Omega$-provably determined.
  - This then gives Σ_1^2-generic absoluteness conditioned on CH.

Perhaps this is why

- There is a fundamental asymmetry between the winning strategies for Player I versus Player II in the $\Sigma_1(A)$-games.

Theorem (IH + Proper class of measurable Woodin cardinals)

The following are equivalent.

(1) *For each $A \in \Gamma^\infty$,*

$$\mathrm{ZFC} + \diamond_G^\infty \vdash_\Omega \text{“}\Sigma_1(A)\text{-determinacy holds for } \{0,1\}^{\omega_1}\text{”}.$$

(2) *For each $A \in \Gamma^\infty$, if $X \subset \{0,1\}^{\omega_1}$ is Σ_1-definable in $(V, \tau_A^\infty, \in)$ from ω_1 then one of the following hold.*
   - *There is a universally Baire quasi-strategy (τ_0, τ_1) for Player I such that*

     $$\mathrm{ZFC} \vdash_\Omega \text{“}(\tau_0, \tau_1) \text{ is a winning quasi-strategy in } X\text{”}.$$
   - *There is a universally Baire term $\tau \in V^{\mathrm{Coll}(\omega_1, \mathbb{R})}$ for a strategy for Player II such that*

     $$\mathrm{ZFC} \vdash_\Omega \text{“}\tau \text{ is a term for a winning strategy in } X\text{”}.$$

Summary

The following now seems a very plausible conjecture.

Conjecture

1. *From some large cardinal hypothesis, Σ_2^2-generic absoluteness holds conditioned on $\diamond_G^\infty$.*
2. *There is no non-strategic version of* $\mathrm{Ultimate}$-L.