Simple definitions for complicated subsets of $\mathrm{H}(\omega_2)$

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Inner and Outer Model Theory
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The results to be presented in this talk are joint work with David Asperó and Peter Holy. They are contained in the following papers.

- David Asperó, Peter Holy, and Philipp Lücke.
  Forcing lightface definable well-orders without the GCH.
  Submitted.
- Peter Holy and Philipp Lücke.
  Locally Σ_1-definable well-orders of $\mathrm{H}(\kappa^+)$.
  To appear in *Fundamenta Mathematicae*.
- Peter Holy and Philipp Lücke.
  Simplest possible locally definable well-orders.
  Submitted.

Introduction

Given an infinite cardinal κ, we let $\mathrm{H}(\kappa^+)$ denote the collection of a sets of hereditary cardinality at most κ.

We are interested in the question whether *simple* formulas can define *complicated* subsets of $\mathrm{H}(\kappa^+)$ over the structure $\langle \mathrm{H}(\kappa^+), \in \rangle$.

We start by considering locally definable well-orders of $\mathrm{H}(\kappa^+)$.

A short argument shows that no well-ordering of $\mathrm{H}(\kappa^+)$ is definable over $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_0-formula with parameters.

Remember that a Σ_1-formula $\varphi(v_0, \ldots, v_{n-1})$ is $\Delta_1^{\mathrm{ZFC}^-}$ if there is a Π_1-formula $\psi(v_0, \ldots, v_{n-1})$ with

$$\mathrm{ZFC}^- \ \vdash \ \forall x_0, \ldots, x_{n-1} \ [\varphi(x_0, \ldots, x_{n-1}) \ \longleftrightarrow \ \psi(x_0, \ldots, x_{n-1})].$$

Note that every Σ_0-formula is $\Delta_1^{\mathrm{ZFC}^-}$.

Proposition

Given an infinite cardinal κ, there is no well-order of $\mathrm{H}(\kappa^+)$ that is definable over $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a $\Delta_1^{\mathrm{ZFC}^-}$-formula with parameters.

Proof.

Assume that there is a $\Delta_1^{\mathrm{ZFC}^-}$-formula φ and $z \in \mathrm{H}(\kappa^+)$ that define such a well-order. Let G be $\mathrm{Add}(\theta, (2^\theta)^+)$-generic over V for some regular cardinal $\theta > 2^\kappa$. In this situation, a folklore result says that there is no well-order of $\mathrm{H}(\theta^+)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\theta^+)^{\mathrm{V}[G]}, \in \rangle$.
By our assumptions, the Σ_1-*Reflection Principle* implies that φ and z do not define a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]} = \mathrm{H}(\kappa^+)^{\mathrm{V}}$ over this set, a contradiction. $\square$

Given $z \in \mathbb{R}$, a classical theorem of Mansfield says that the existence of a well-ordering of $\mathbb{R}$ that is a $\Sigma^1_2(z)$-subset of $\mathbb{R} \times \mathbb{R}$ is equivalent to the statement that every real is constructible from z.

Since a set of reals is $\Sigma^1_2(z)$-definable if and only if it is definable over the set $\mathrm{H}(\omega_1)$ by a Σ_1-formula with parameter z, we can state Mansfield's theorem in the following way.

Theorem (Mansfield)

The following statements are equivalent for every $z \in \mathbb{R}$.

- *Every real is constructible from z.*
- *There is a well-ordering of $\mathrm{H}(\omega_1)$ that is definable over $\langle \mathrm{H}(\omega_1), \in \rangle$ by a Σ_1-formula with parameter z.*

Corollary

If there is a well-ordering of $\mathrm{H}(\omega_1)$ that is definable over $\langle \mathrm{H}(\omega_1), \in \rangle$ by a Σ_1-formula with parameters, then CH holds and there are no measurable cardinals.

Obviously, the first implications of Mansfield's theorem still holds if we replace ω by an uncountable cardinal.

Proposition

Let κ be an infinite cardinal. If there is a subset z of κ such that every other subset of κ is constructible from z, then there is a well-ordering of $\mathrm{H}(\kappa^+)$ that is definable over $\mathrm{H}(\kappa^+)$ by a Σ_1-formula with parameter z.

Therefore it is natural to ask whether the other implication or its consequences also hold if we replace ω by an uncountable cardinal.

Question

What are the provable consequences of the existence of a well-ordering of $\mathrm{H}(\omega_2)$ that is definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a Σ_1-formula?

The next class of subsets of $\mathrm{H}(\kappa^+)$ that we consider are *Bernstein subsets*.

Let κ be a cardinal with $\kappa = \kappa^{<\kappa}$. Equip the set ${}^\kappa\kappa$ of all functions from κ to κ with the topology whose basic open subsets are of the form

$$N_s \ = \ \{x \in {}^\kappa\kappa \mid s \subseteq x\}$$

for some function $s : \alpha \longrightarrow \kappa$ with $\alpha < \kappa$. Note that the closed sets in this topology are the sets $[T]$ of all cofinal branches through subtrees T of ${}^{<\kappa}\kappa$.

- A closed subset of ${}^\kappa\kappa$ is *perfect* if it is homeomorphic to ${}^\kappa 2$ equipped with the subspace topology.
- A subset B of ${}^\kappa\kappa$ is a *Bernstein subset of* ${}^\kappa\kappa$ if B and its complement meet every perfect subset of ${}^\kappa\kappa$.

Proposition

If κ is an uncountable regular cardinal, then there is no Bernstein subset of ${}^{\kappa}\kappa$ that is definable over $\langle \mathrm{H}(\kappa^{+}), \in \rangle$ by a Σ_0-formula with parameters.

Proof.

Let $\varphi(u, v)$ be a Σ_0-formula and $z \in \mathrm{H}(\kappa^{+})$. Then there are subtrees T_0 and T_1 of ${}^{<\kappa}\kappa \times {}^{<\kappa}\kappa$ such that

$$({}^{\kappa}\kappa)^{\mathrm{V}[G]} \ = \ p[T_0]^{\mathrm{V}[G]} \ \dot{\cup} \ p[T_1]^{\mathrm{V}[G]}$$

and

$$p[T]^{\mathrm{V}[G]} \ = \ \{x \in ({}^{\kappa}\kappa)^{\mathrm{V}[G]} \mid \mathrm{V}[G] \models \varphi(x, z)\}$$

hold whenever $\mathrm{V}[G]$ is a forcing extension of V by a ${<}\kappa$-closed forcing.

Let $\dot{x}$ be the canonical $\mathrm{Add}(\kappa, 1)$-name for the generic function added by the forcing and let p be a condition in $\mathrm{Add}(\kappa, 1)$ that decides the statement "$\varphi(\dot{x}, \check{z})$". Then $p \Vdash$ "$\dot{x} \in p[\check{T}_i]$" for some $i < 2$ and this allows us to construct a perfect subset of $p[T_i]$ in V. $\square$

The following theorem of Jörg Brendle and Benedikt Löwe establishes a criterion for the existence of $\mathbf{\Delta}^1_2$-Bernstein subsets of ${}^\omega\omega$ that closely resembles Mansfield's result.

Theorem (Brendle–Löwe)

The following statements are equivalent for every $z \in {}^\omega\omega$.

- *Every real is constructible from z.*
- *There is a Bernstein subset of ${}^\omega\omega$ that is definable over $\langle \mathrm{H}(\omega_1), \in \rangle$ by a Σ_1-formula with parameter z.*
- *There is a Bernstein subset of ${}^\omega\omega$ that is Δ_1-definable over $\langle \mathrm{H}(\omega_1), \in \rangle$ with parameter z.*

Corollary

If there is a Bernstein subset of ${}^\omega\omega$ that is Δ_1-definable over $\langle \mathrm{H}(\omega_1), \in \rangle$ with parameters, then CH holds and there are no measurable cardinals.

As above, the certain implications still holds if we replace ω by an arbitrary infinite cardinal.

Proposition

Let κ be an infinite cardinal. If there is a subset z of κ such that every other subset of κ is constructible from z, then there is a Bernstein subset of ${}^{\kappa}\kappa$ that is Δ_1-definable over $\langle \mathrm{H}(\omega_1), \in \rangle$ with parameter z.

As above, this raises the following question.

Question

What are the provable consequences of the existence of a Bernstein subset of ${}^{\omega_1}\omega_1$ that is Δ_1-definable over $\langle \mathrm{H}(\omega_2), \in \rangle$?

Boldface Σ_1-definitions

The following results answers the above questions in the negative for Σ_1-definitions using arbitrary parameters.

Theorem (Holy–L.)

Let κ be an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$ and 2^κ is regular. Then there is a partial order $\mathbb{P}$ with the following properties.

- *$\mathbb{P}$ is $<\kappa$-closed and forcing with $\mathbb{P}$ preserves cofinalities $\leq 2^\kappa$ and the value of 2^κ.*
- *If G is $\mathbb{P}$-generic over the ground model V, then there is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameters.*
- *If G is $\mathbb{P}$-generic over the ground model V, then there is a Bernstein subset of ${}^\kappa\kappa$ that is Δ_1-definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ with parameters.*

Corollary

The existence of a well-ordering of $\mathrm{H}(\omega_2)$ *that is definable over* $\mathrm{H}(\omega_2)$ *by a* Σ_1*-formula with parameters is consistent with a failure of the* GCH *at* ω_1 *and all consistent large cardinal notions.*

Corollary

The existence of a Bernstein subset of ${}^{\omega_1}\omega_1$ *that is* Δ_1*-definable with parameters over* $\langle \mathrm{H}(\omega_2), \in \rangle$ *is consistent with a failure of the* GCH *at* ω_1 *and all consistent large cardinal notions.*

The forcing that witnesses the above theorem is constructed inductively using techniques developed by David Asperó and Sy Friedman in their paper "*Definable well-orders of* $\mathrm{H}(\omega_2)$ *and* GCH".

In the following, we sketch the construction of this forcing.

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and 2^κ regular.

We will later show that there is a family $\langle \mathbb{P}(A) \mid A \subseteq {}^\kappa\kappa \rangle$ of partial orders such that the following properties hold for all $A \subseteq {}^\kappa\kappa$.

- $\mathbb{P}(A)$ is $<\kappa$-closed and satisfies the κ^+-chain condition.
- If G is $\mathbb{P}(A)$-generic over V, then A is definable in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameters.
- If $B \subseteq A$, then $\mathbb{P}(B)$ is a complete subforcing of $\mathbb{P}(A)$.

We use this family of coding forcings to construct a sequence $\langle \mathbb{P}_\gamma \mid \gamma \leq 2^\kappa \rangle$ of partial orders such that the following statements hold for all $\gamma \leq 2^\kappa$.

- Forcing with $\mathbb{P}_\gamma$ preserves cofinalities $\leq 2^\kappa$.
- If $\bar{\gamma} < \gamma$, then $\mathbb{P}_{\bar{\gamma}}$ is a complete subforcing of $\mathbb{P}_\gamma$ and there is a projection map $\pi_{\bar{\gamma},\gamma} : \mathbb{P}_\gamma \longrightarrow \mathbb{P}_{\bar{\gamma}}$.
- A condition in $\mathbb{P}_\gamma$ is a pair $p = \langle \vec{A}_p, \bar{p} \rangle$ such that
  - $\vec{A}_p = \langle \dot{A}_{p,\delta} \mid \delta < \gamma_p \rangle$ is a sequence of length $\gamma_p < \min\{\gamma + 1, \ 2^\kappa\}$ such that each $\dot{A}_{p,\delta}$ is a $\mathbb{P}_\delta$-name for an element of $\mathrm{H}(\kappa^+)$ and every condition in $\mathbb{P}_\delta$ forces the sequence $\vec{A}_p \restriction (\delta + 1)$ to be injective.
  - $\bar{p}$ is a condition in $\mathbb{P}(A_p)$, where $A_p \subseteq {}^\kappa 2$ consists of functions that encode the information about the $\dot{A}_{p,\delta}$ decided by $\pi_{\delta,\gamma}(p)$.
- We define $p \leq_{\mathbb{P}_\gamma} q$ to hold if $\vec{A}_q \subseteq \vec{A}_p$, $A_q \subseteq A_p$ and $\bar{p} \leq_{\mathbb{P}(A_p)} \bar{q}$.

Let G be $\mathbb{P}_{2^\kappa}$-generic over V. Given $x, y \in \mathrm{H}(\kappa^+)$, we define

$$x \prec y \iff \exists p \in G \ \exists \delta_0 < \delta_1 < \gamma_p \ [x = \dot{A}^G_{p,\delta_0} \ \wedge \ y = \dot{A}^G_{p,\delta_1}].$$

With the help of a preparatory forcing, we can ensure that this relation is a locally Σ_1-definable well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$.

We show how the above family of coding forcings can be constructed.

The starting point is the generalization of Solovay's *almost disjoint coding forcing* to uncountable cardinals.

Definition

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and A be a non-empty subset of ${}^{\kappa}\kappa$. We define a partial order $\mathbb{C}(A)$ by the following clauses.

- A condition in $\mathbb{C}(A)$ is a triple $p = \langle a_p, s_p, t_p \rangle$ such that $a_p \in [A]^{<\kappa}$ and there is an $\alpha_p < \kappa$ with $s_p : \alpha_p \longrightarrow {}^{<\kappa}\kappa$ and $t_p : \alpha_p \longrightarrow 2$.
- We have $q \leq_{\mathbb{C}(A)} p$ if and only if $a_p \subseteq a_q$, $s_p \subseteq s_q$, $t_p \subseteq t_q$ and

$$s_q(\beta) \subseteq x \;\longrightarrow\; t_q(\beta) = 1$$

for every $x \in a_p$ and $\alpha_p \leq \beta < \alpha_q$.

Proposition

The partial order $\mathbb{C}(A)$ is $<\kappa$-closed and satisfies the κ^+-chain condition.

Let G be $\mathbb{C}(A)$-generic over V. Define

$$s_G = \bigcup_{p \in G} s_p : \kappa \longrightarrow {}^{<\kappa}\kappa \quad \textit{and} \quad t_G = \bigcup_{p \in G} t_p : \kappa \longrightarrow 2.$$

Given $\alpha < \kappa$, define T^G_α to be the subtree

$$\{u \in {}^{<\kappa}\kappa \mid \forall \alpha < \beta < \kappa \; [s_G(\beta) \subseteq u \longrightarrow t_G(\beta) = 1]\}$$

of ${}^{<\kappa}\kappa$.

Theorem

Let G be $\mathbb{C}(A)$-generic over V. Then the sequence $\langle T^G_\alpha \mid \alpha < \kappa \rangle$ witnesses that A is a $\mathbf{\Sigma}^0_2$-subset of ${}^\kappa\kappa$ in $\mathrm{V}[G]$, i.e. we have $A = \bigcup_{\alpha<\kappa} [T^G_\alpha]^{\mathrm{V}[G]}$.

This shows that the family $\langle \mathbb{C}(A) \mid A \subseteq {}^{\kappa}\kappa \rangle$ of partial orders satisfies the first and the second statement listed above.

Unfortunately, it does not satisfy the "*complete subforcing*" property.

Proposition

If $\emptyset \neq B \subsetneq A \subset {}^{\kappa}\kappa$, then $\mathbb{C}(B)$ is not a complete subforcing of $\mathbb{C}(A)$.

Proof.

Assume, towards a contradiction, that there are $\emptyset \neq B \subsetneq A \subset {}^{\kappa}\kappa$ such that $\mathbb{C}(B)$ is a complete subforcing of $\mathbb{C}(A)$. Let G be $\mathbb{C}(A)$-generic over V and define $\bar{G} = \{p \in G \mid a_p \subseteq B\}$. By our assumption, $\bar{G}$ is $\mathbb{C}(B)$-generic over V with $s_G = s_{\bar{G}}$, $t_G = t_{\bar{G}}$ and $T^G_\alpha = T^{\bar{G}}_\alpha$ for every $\alpha < \kappa$.
Pick $x \in A \setminus B$. By the above theorem, there is an $\alpha < \kappa$ with $x \in [T^G_\alpha]^{\mathrm{V}[G]} = [T^{\bar{G}}_\alpha]^{\mathrm{V}[G]}$. Since x and T^G_α are both contained in $\mathrm{V}[\bar{G}]$, we have $x \in [T^{\bar{G}}_\alpha]^{\mathrm{V}[\bar{G}]} \subseteq B$, a contradiction. □

Fortunately, it is possible to modify the above definition to obtain a family of coding forcings that satisfies all three properties.

The following construction and results are contained in joint work with David Asperó and Peter Holy.

Definition

Let κ be an uncountable cardinal with $\kappa = \kappa^{<\kappa}$ and A be a non-empty subset of ${}^{\kappa}\kappa$. We define a partial order $\mathbb{C}^*(A)$ by the following clauses.

- A condition in $\mathbb{C}^*(A)$ is a triple $p = \langle s_p, t_p, \langle c_{p,x} \mid x \in a_p \rangle \rangle$ with
  - $s_p : \alpha_p \longrightarrow {}^{<\kappa}\kappa$ and $t_p : \alpha_p \longrightarrow 2$ for some successor ordinal α_p less than κ.
  - $a_p \in [A]^{<\kappa}$ and $c_{p,x}$ is a closed subset of α_p with

$$\forall \beta \in c_{p,x} \; [s_p(\beta) \subseteq x \;\longrightarrow\; t_p(\beta) = 1]$$

    for every $x \in a_p$.
- We have $q \leq_{\mathbb{C}^*(A)} p$ if and only if $a_p \subseteq a_q$, $s_p \subseteq s_q$, $t_p \subseteq t_q$ and $c_{p,x} = c_{q,x} \cap \alpha_p$ for every $x \in a_p$.

Proposition

The partial order $\mathbb{C}^*(A)$ is $<\kappa$-closed and satisfies the κ^+-chain condition.

Proposition

If $\emptyset \neq B \subsetneq A \subset {}^\kappa\kappa$, then $\mathbb{C}^*(B)$ is a complete subforcing of $\mathbb{C}^*(A)$.

Theorem (Asperó–Holy–L.)

Let G be $\mathbb{C}^*(A)$-generic over V. Define s_G and t_G as above. In $\mathrm{V}[G]$, the set A is equal to the set of all $x \in {}^\kappa\kappa$ such that

$$\forall \beta \in C \; [s_G(\beta) \subseteq \beta \longrightarrow t_G(\beta) = 1]$$

holds for some club C in κ. In particular, A is definable over $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameters.

Lightface Σ_1-definitions

The parameter used in the above Σ_1-definitions is a subset of κ added by the forcing and therefore is a very complicated object.

We want to construct models of set theory with local Σ_1-definitions for well-orders and Bernstein sets that only use *simple parameters*.

The first step towards this goal is the next result that produces Σ_1-definitions in the generic extension whose parameters are elements of the ground model.

The construction of the corresponding forcing combines the techniques of the above proof with iterated club shooting and variations of Shelah's notion of S*-complete forcings*.

Theorem (Holy–L.)

Let κ be the successor of a regular cardinal η such that $\eta = \eta^{<\eta}$, $\kappa = 2^\eta$ and 2^κ is regular. If $\vec{S}$ is a κ-sequence of pairwise disjoint stationary subsets of S^κ_η, then there is a partial order $\mathbb{P}_{\vec{S}}$ with the following properties.

- *$\mathbb{P}_{\vec{S}}$ is $<\kappa$-distributive and forcing with $\mathbb{P}$ preserves cofinalities $\leq 2^\kappa$ and the value of 2^κ.*
- *If G is $\mathbb{P}_{\vec{S}}$-generic over V, then there is a well-order of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameter $\vec{S}$.*
- *If G is $\mathbb{P}_{\vec{S}}$-generic over V, then there is a Bernstein subset of ${}^\kappa\kappa$ that is Δ_1-definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ with parameter $\vec{S}$.*

By forcing over certain canonical inner models, the above theorem can be used to produce Σ_1-definitions that only use the cardinal κ as a parameter.

Definition

Given an uncountable cardinal κ, we say that an inner model M is *κ-suitable* if there is a regular cardinal $\theta > \kappa$ in M and a well-order $\prec$ of $\mathrm{H}(\theta)^M$ such that the following statements hold.

- The sets $\mathrm{H}(\kappa)^M$ and $\prec \cap \mathrm{H}(\kappa)^M$ are definable in $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameter κ whenever $\mathrm{V}[G]$ is a $<\kappa$-distributive forcing extension of V.
- In M, there is a continuous ascending sequence $\langle M_\alpha \mid \alpha < \kappa \rangle$ of elementary submodels of $\mathrm{H}(\theta)$ of cardinality less than κ and a strictly increasing continuous sequence $\langle \kappa_\alpha \mid \alpha < \kappa \rangle$ of ordinals such that the following statements hold for all $\alpha < \kappa$.
  - $\kappa \in M_0$ and $\kappa_\alpha = M_\alpha \cap \kappa$.
  - If $\pi_\alpha : \langle M_\alpha, \in, \prec \rangle \longrightarrow \langle N_\alpha, \in, \prec_\alpha \rangle$ is the corresponding transitive collapse and $P_\alpha = \pi_\alpha(\mathrm{H}(\kappa^+))$, then $\prec \cap P_\alpha = \prec_\alpha \cap P_\alpha$ and the set N_α is $\prec$-downwards-closed below every element of P_α.

Theorem (Holy–L.)

In the situation of the above theorem, assume that M is κ-suitable inner model such that every stationary subset of κ in M is stationary in V. Then there is a κ-sequence $\vec{S}$ of pairwise disjoint stationary subsets of S^{κ}_{η} such that the following statements hold in every $\mathbb{P}_{\vec{S}}$-generic extension $\mathrm{V}[G]$ of the ground model V.

- *There is a well-ordering of $\mathrm{H}(\kappa^+)^{\mathrm{V}[G]}$ that is definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ by a Σ_1-formula with parameter κ.*
- *There is a Bernstein subset of ${}^{\kappa}\kappa$ that is Δ_1-definable over $\langle \mathrm{H}(\kappa^+)^{\mathrm{V}[G]}, \in \rangle$ with parameter κ.*

Lemma

If κ is an uncountable regular cardinal, then L is κ-suitable.

Corollary

The existence of a well-ordering of $\mathrm{H}(\omega_2)$ that is definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a Σ_1-formula with parameter ω_1 is consistent with a failure of the GCH at ω_1.

Corollary

The existence of a Bernstein subset of ${}^{\omega_1}\omega_1$ that is Δ_1-definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ with parameter ω_1 is consistent with a failure of the GCH at ω_1.

Lemma

Let κ be an uncountable regular cardinal and U be a normal ultrafilter over $\delta > \kappa$, then $\mathrm{L}[U]$ is κ-suitable.

Corollary

If the existence of a measurable cardinal is consistent, then the existence of a measurable cardinal is consistent with the existence of a well-ordering of $\mathrm{H}(\omega_2)$ that is definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a Σ_1-formula with parameter ω_1.

Corollary

If the existence of a measurable cardinal is consistent, then the existence of a measurable cardinal is consistent with the existence of a Bernstein subset of ${}^{\omega_1}\omega_1$ that is Δ_1-definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ with parameter ω_1.

In contrast, it is possible to use results of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$ to show that the existence of larger large cardinals is incompatible with the existence of such objects.

Proposition

Assume that there are infinitely many Woodin cardinals with a measurable cardinal above them all. If there is a well-order of $\mathrm{H}(\omega_2)$ that is definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ by a Σ_1-formula with parameter $z \subseteq \omega_1$, then $z \notin \mathrm{L}(\mathbb{R})$.

Proof.

Assume that there is a Σ_1-formula ϕ and $z \in \mathcal{P}(\omega_1)^{\mathrm{L}(\mathbb{R})}$ that define a well-order of $\mathrm{H}(\omega_2)$ over $\langle \mathrm{H}(\omega_2), \in \rangle$. Let G be $\mathbb{P}_{\mathrm{max}}$-generic over $\mathrm{L}(\mathbb{R})$. By the Π_2-maximality of the $\mathbb{P}_{max}$-extension, we know that φ and z define a well-order of $\mathrm{H}(\omega_2)^{\mathrm{L}(\mathbb{R})[G]}$ over $\langle \mathrm{H}(\omega_2)^{\mathrm{L}(\mathbb{R})[G]}, \in \rangle$. This shows that there is well-ordering of $\mathbb{R}$ that is definable in $\mathrm{L}(\mathbb{R})[G]$ by a formula with parameter $z \in \mathrm{L}(\mathbb{R})$.

Since the partial order $\mathbb{P}_{\mathrm{max}}$ is homogeneous in $\mathrm{L}(\mathbb{R})$, this implies that there is a well-ordering of $\mathbb{R}$ in $\mathrm{L}(\mathbb{R})$. But results of Woodin show that our assumptions imply that AD holds in $\mathrm{L}(\mathbb{R})$, a contradiction. □

It is possible to run a similar argument to prove an analog statement for Bernstern subsets of ${}^{\omega_1}\omega_1$.

Proposition

Assume that there are infinitely many Woodin cardinals with a measurable cardinal above them all. If there is a Bernstein subset of ${}^{\omega_1}\omega_1$ that is Δ_1-definable over $\langle \mathrm{H}(\omega_2), \in \rangle$ with parameter $z \subseteq \omega_1$, then $z \notin \mathrm{L}(\mathbb{R})$.

Questions

Question

Is it possible to develop a *descriptive set theory of lightface Δ_1-subsets of* ${}^{\omega_1}\omega_1$ in the presence of class-many Woodin cardinals?

Question

Given a regular cardinal $\kappa > \omega_1$, does the existence of a well-order of $\mathrm{H}(\kappa^+)$ that is definable over $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula with parameter κ imply that there are no supercompact cardinals above κ?

Question

What is the weakest large cardinal whose existence implies that no well-order of $\mathrm{H}(\omega_2)$ is locally definable by a Σ_1-formula with parameter ω_1?

Question

Does the existence of a well-order of $\mathrm{H}(\kappa^+)$ that is definable over $\langle \mathrm{H}(\kappa^+), \in \rangle$ by a Σ_1-formula without parameters imply that the GCH holds at κ?

Thank you for listening!