Condensation does not imply Square

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Condensation

Lemma (Gödel)

If $M \prec (L_\alpha, \in)$, then for some $\bar{\alpha} \leq \alpha$, $M \cong (L_{\bar{\alpha}}, \in)$.

We want to consider generalizations of this principle that apply to models other than L:

Condensation in models of the form L[A]

Assume $A \subseteq \mathrm{Ord}$. If M is a substructure of $(L_\alpha[A], \in, A)$, we say that M *condenses* if for some $\bar{\alpha} \leq \alpha$, $M \cong (L_{\bar{\alpha}}[A], \in, A)$.

Local Club Condensation at κ

If $\kappa = \lambda^+$, LCC at κ is the statement that there is $A \subseteq \kappa$ s.t. $H_\kappa = L_\kappa[A]$ and if $\alpha \in [\lambda, \kappa)$ and $\mathcal{A} = (L_\alpha[A], \in, A, \ldots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \mid \gamma < \lambda \rangle$ of condensing substructures of $\mathcal{A}$ whose domains have union $L_\alpha[A]$, where each $B_\gamma = \mathrm{dom}(\mathcal{B}_\gamma)$ is s.t. $|B_\gamma| < \lambda$ and $\gamma \subseteq B_\gamma$.

Lemma (Friedman, Holy, Wu)

If $\kappa = \lambda^+$, $\mathrm{cof}(\lambda) \geq \omega_1$ *and* LCC *at* κ *holds, then there is a structure* $\mathcal{M}$ *for a countable language with domain* H_κ *such that* X *condenses whenever* X *is a substructure of* $\mathcal{M}$ *and is transitive below* λ*.*

If $\kappa = \omega_2$, every substructure of such $\mathcal{M}$ will be transitive below ω_1, hence we obtain the following.

Corollary

If LCC *at* ω_2 *holds, then there is a structure* $\mathcal{M}$ *for a countable language with domain* H_{ω_2} *such that every substructure* X *of* $\mathcal{M}$ *condenses.*

This is what Woodin introduced as Strong Condensation for ω_2.

Theorem (Wu)

Assuming the consistency of a stationary limit of measurable cardinals, Strong Condensation for ω_2 *is consistent with the failure of* $\square_{\omega_1}$*.*

We will improve and generalize this result.

Square

Definition

If $\lambda \geq \omega_1$, $\square_\lambda$ is the statement that there exists a sequence $\langle C_\alpha \mid \alpha < \lambda^+ \rangle$ such that

1. Whenever α is a limit ordinal, C_α is a closed unbounded subset of α.
2. If β is a limit point of C_α then $C_\beta = C_\alpha \cap \beta$.
3. For every α, $\mathrm{ot}(C_\alpha) \leq \lambda$.

$\square_\lambda$ holds in L for every uncountable cardinal λ. All known proofs of this fact use some sort of fine structural machinery. It is generally believed that this is in fact necessary; we support this belief (as did Liuzhen Wu for $\square_{\omega_1}$) by showing that (Local Club) Condensation is not sufficient to imply $\square_\lambda$.

Square and Forcing

Theorem (Solovay)

If λ is regular and uncountable and $\kappa > \lambda$ is a Mahlo cardinal, then after performing a Lévy collapse so that κ becomes λ^+, $\Box_\lambda$ fails.

We want to collapse some large cardinal κ to become λ^+ while forcing LCC at λ^+ and then show that we can still verify the failure of $\Box_\lambda$ in the resulting model.

Theorem (H)

Assume GCH *holds and $\lambda < \kappa$ are regular. There is a $<\lambda$-directed closed, κ-cc notion of forcing which ensures that $\kappa = \lambda^+$ and Local Club Condensation at κ hold in any generic extension.*

Rough idea of proof: BLACKBOARD.

Proof of Solovay's theorem

Theorem (Solovay)

If λ is regular and uncountable and $\kappa > \lambda$ is a Mahlo cardinal, then after performing a Lévy collapse so that κ becomes λ^+, $\square_\lambda$ fails.

Proof: Let $P = P(\lambda, \kappa)$ denote the above Lévy Collapse. Assume $\dot{C} = \langle \dot{C}_\eta \mid \eta < \kappa \rangle$ is a P-name for a $\square_\lambda$-sequence in a P-generic extension. Using that the forcing is κ-cc and conditions have *bounded support*, there is a club of $\eta < \kappa$ such that $\dot{C} \restriction \eta$ is a $P(\lambda, \eta)$-name. By the large cardinal properties of κ, we may choose such η which is inaccessible. As η is regular after forcing with $P(\lambda, \eta)$, it follows that $\dot{C}_\eta$ cannot have a $P(\lambda, \eta)$-name, as otherwise it would have to have order-type $\eta > \lambda$, contradicting that $\dot{C}$ is a P-name for a $\square_\lambda$-sequence. Pick $t_0 \perp t_1$ in P and $\xi < \eta$ with $t_0 \restriction \eta = t_1 \restriction \eta$ such that t_0 and t_1 disagree about whether $\xi \in \dot{C}_\eta$.

Proof of Solovay's theorem continued

We have $t_0 \perp t_1$ in P and $\xi < \eta$ with $t_0 \upharpoonright \eta = t_1 \upharpoonright \eta$ such that t_0 and t_1 disagree about whether $\xi \in \dot{C}_\eta$. Let $\{t_0, t_1, P, \dot{C}, \xi, \eta\} \subseteq M \prec H_\theta$ with M countable, for some large, regular θ. We can construct $s_0 \leq t_0$ and $s_1 \leq t_1$ both (M, P)-generic such that $s_0 \upharpoonright \eta = s_1 \upharpoonright \eta$ (BLACKBOARD). Now both s_0 and s_1 force that $\delta = \sup(M \cap \eta) \in \mathrm{Lim}(\dot{C}_\eta)$. Using that $\dot{C}$ is a P-name for a $\Box_\lambda$-sequence, we thus obtain that both s_0 and s_1 force that

$$\xi \in \dot{C}_\eta \iff \xi \in \dot{C}_\delta.$$

This is a contradiction as $\dot{C}_\delta$ has a $P(\lambda, \eta)$-name and $s_0 \upharpoonright \eta = s_1 \upharpoonright \eta$, hence s_0 and s_1 agree about whether $\xi \in \dot{C}_\delta$. $\Box$

The problem with Local Club Condensation

We would of course like to do the same proof again, simply replacing the Lévy collapse by the forcing that I described earlier, which forces Local Club Condensation at κ while performing the above collapse. However the crucial step of extending t_0 and t_1 while keeping them equal (or at least compatible) below η does not work as above anymore. Essentially, the problem is that the forcings below and above η (or rather below and above λ) are not sufficiently independent (unlike the simpe Lévy collapse forcing).

The key trick (due to Liuzhen Wu) is to construct generic conditions for two different $M_0, M_1 \prec H_\theta$, with the key property that $\sup(M_0 \cap M_1 \cap \eta) < \sup(M_0 \cap \eta) = \sup(M_1 \cap \eta)$. We can obtain such structures if η is ω-Erdős. Thus our consistency assumption will be a stationary limit of ω-Erdős cardinals.

ω-Erdős cardinals

Definition

If $\mathcal{A}$ is a structure for a countable language with κ a subset of its domain then $I \subseteq \kappa$ is a sequence of *good indiscernibles* for $\mathcal{A}$ if whenever φ is an n-ary formula in the language of $\mathcal{A}$, $\vec{\gamma}$ and $\vec{\gamma}'$ are increasing k-ary sequences from I, $k \leq n$ and $\vec{\xi}$ is an $(n-k)$-ary sequence of ordinals with $\max \vec{\xi} < \min \vec{\gamma}, \min \vec{\gamma}'$, then

$$\mathcal{A} \models \varphi(\vec{\xi}, \vec{\gamma}) \iff \mathcal{A} \models \varphi(\vec{\xi}, \vec{\gamma}').$$

Definition

κ is ω-Erdős if for every structure $\mathcal{A} = \langle A, \in, \ldots \rangle$ for a countable language with $\kappa \subseteq A$ and every club $C \subseteq \kappa$ there is a sequence $I \subseteq C$ of order-type ω of good indiscernibles for $\mathcal{A}$.

Interleaving structures

Lemma

Assume $\theta \geq \eta$ is regular, η is ω-Erdős and $\mathcal{A}$ is a structure for a countable language with domain H_θ. Then there is a pair of countable substructures M_0 and M_1 of $\mathcal{A}$ such that

$$\sup(M_0 \cap M_1 \cap \eta) < \sup(M_0 \cap \eta) = \sup(M_1 \cap \eta).$$

Proof: We may assume that $\mathcal{A}$ is Skolemized. Let I be an ω-sequence of good indiscernibles for $\mathcal{A}$. Let M_0 be the Skolem Hull in $\mathcal{A}$ of the indiscernibles with even index in I, let M_1 be the Skolem Hull in $\mathcal{A}$ of the indiscernibles with odd index in I. □

We also call M_0 and M_1 as above *interleaving structures* and we say that η has the *interleaving substructures property* (ISS(η)) if such structures exist for any such $\mathcal{A}$.

The Consistency Strength of ISS(κ)

We could now rephrase the lemma from the previous slide as follows:

Lemma

If κ is ω-Erdős, then ISS(κ) *holds.*

However the reversal does not hold true by the following.

Lemma

Assume κ is ω-Erdős and $\lambda < \kappa$ is regular uncountable.

- *After Lévy collapsing κ to become λ^+,* ISS(κ) *holds.*
- *After forcing κ to become the least Mahlo cardinal (by iterating with Easton support to shoot a club through $\nu \cap \mathrm{Sing}$ for every Mahlo cardinal $\nu < \kappa$),* ISS(κ) *holds. Moreover the least Mahlo is never ω-Erdős.*

Open Questions

This leads to some obvious open questions. For example:

Question

If ISS(κ) holds, is κ ω-Erdős in L?

Question

Is it consistent that ISS(κ) holds for the least inaccessible cardinal κ?

The Main Theorem

I should finally state the main theorem:

Theorem

Assuming GCH *and the consistency of a stationary limit of* ω*-Erdős cardinals above some regular* $\kappa \geq \omega_1$*, it is consistent that Local Club Condensation at* κ^+ *holds while* $\Box_\kappa$ *fails.*

Of course, one could ask the following:

Question

What is the consistency strength of having a regular uncountable κ such that Local Club Condensation for κ^+ holds while $\Box_\kappa$ fails?

Chang's Conjecture and Condensation

Definition

$(\alpha, \beta) \twoheadrightarrow (\gamma, \delta)$ is the statement that for every countable language $\mathcal{L}$ with a unary predicate $A \in \mathcal{L}$ and every $\mathcal{L}$-structure $\mathcal{M} = (M, A^{\mathcal{M}}, \ldots)$ with $|M| = \alpha$ and $|A^{\mathcal{M}}| = \beta$, there exists a substructure $\mathcal{N} = (N, A^{\mathcal{N}}, \ldots)$ of $\mathcal{M}$ such that $|N| = \gamma$ and $|A^{\mathcal{N}}| = \delta$.
We let $\mathrm{CC}(\kappa)$ denote the statement that for every $\lambda < \kappa$, $(\kappa, \lambda) \twoheadrightarrow (\omega_1, \omega)$.

Theorem

- LCC *at* ω_2 *refutes* $\mathrm{CC}(\omega_2)$.
- *Strong Condensation for* κ *refutes* $\mathrm{CC}(\kappa)$ *for any* $\kappa \geq \omega_2$.
- *Assume* GCH. *If* κ *is* ω_1*-Erdős and* $\omega_2 \leq \lambda < \kappa$ *is regular, we may force to make* $\kappa = \lambda^+$*, preserve all cardinals* $\leq \lambda$ *and obtain* $\mathrm{CC}(\kappa)$ *and Local Club Condensation at* κ.

Thank you.