Ramsey type properties of definable ideals

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A family $\mathcal{I}$ of subsets of a (countable) set $X$ is an **ideal** if it is
- closed under subsets
- closed under finite unions
- $X \not\in \mathcal{I}$ and
- it contains all singletons of $X$.

Dually, a family $\mathcal{F}$ of subsets of $X$ is a **filter** if it is (1) closed under supersets, (2) closed under finite intersections (3) $\emptyset \not\in \mathcal{F}$ and (4) it contains all co-finite subsets of $X$.

For an ideal $\mathcal{I}$ on $X$,
- $\mathcal{I}^* = \{X \setminus I : I \in \mathcal{I}\}$ is the **dual** filter (and the same for filters),
- $\mathcal{I}^+$ denotes $\mathcal{P}(X) \setminus \mathcal{I}$ (for filters $\mathcal{F}^+ = \mathcal{P}(X) \setminus \mathcal{F}^*$).

An ideal $\mathcal{I}$ on $\omega$ is **meager, Borel, analytic,***... if it is meager, Borel, analytic,... as a subspace of $\mathcal{P}(\omega) \simeq 2^\omega$. 
An ideal $\mathcal{I}$ on $\omega$ is

- **tall** if for every infinite $A \subseteq \omega$ there is an $I \in \mathcal{I}$ such that $|A \cap I|$ is infinite,

- **$\omega$-hitting** if for every $\langle A_n : n \in \omega \rangle \subseteq [\omega]^\omega$ there is an $I \in \mathcal{I}$ such that $A_n \cap I$ is infinite for all $n \in \omega$,

- a **$P$-ideal** if for every $\langle I_n : n \in \omega \rangle \subseteq \mathcal{I}$ there is an $I \in \mathcal{I}$ such that $I_n \setminus I$ is finite for all $n \in \omega$,

- a **$P^+$-ideal** if for every decreasing sequence $\langle X_n : n < \omega \rangle$ of $\mathcal{I}$-positive sets there is an $\mathcal{I}$-positive set $X$ such that $X \subseteq^* X_n$, for all $n < \omega$.

- a **$Q^+$-ideal** if for every partition $\langle F_n : n < \omega \rangle$ of an $\mathcal{I}$-positive set into finite sets there is an $\mathcal{I}$-positive set $X$ such that $|X \cap F_n| \leq 1$, for all $n < \omega$.

Every $\omega$-hitting ideal is tall, and every tall $P$-ideal is $\omega$-hitting.
An ultrafilter $U$ on $\omega$ is

- **selective** if for every partition $\{I_n : n \in \omega\}$ of $\omega$ into sets not in $U$ there is $U \in U$ such that $|U \cap I_n| = 1$ for every $n \in \omega$.
- a **P-point** if for every partition $\{I_n : n \in \omega\}$ of $\omega$ into sets not in $U$ there is $U \in U$ such that $|U \cap I_n|$ is finite for every $n \in \omega$.
- a **Q-point** if for every partition $\{I_n : n \in \omega\}$ of $\omega$ into finite sets there is $U \in U$ such that $|U \cap I_n| = 1$ for every $n \in \omega$.

An ultrafilter $U$ is selective iff it is both a P-point and a Q-point.
Ultrafilters and analytic ideals

Theorem (Mathias 1977)

Let $\mathcal{U}$ be an ultrafilter on $\omega$. Then, $\mathcal{U}$ is selective if and only if $\mathcal{U} \cap I \neq \emptyset$ for every analytic tall ideal $I$ on $\omega$.

Theorem (Zapletal 2008)

Let $\mathcal{U}$ be a free ultrafilter on $\omega$. Then the following are equivalent:

1. $\mathcal{U}$ is a P-point.
2. For every analytic tall ideal $I$ disjoint from $\mathcal{U}$ there is an $F_\sigma$-ideal $J$ disjoint from $\mathcal{U}$ containing $I$. 
Ramsey ideals

**Definition**

An ideal $\mathcal{I}$ is Ramsey($\omega$) if for every coloring $\varphi : [\omega]^2 \rightarrow 2$ there is an $\mathcal{I}$-positive set $X$ which is $\varphi$-homogeneous, i.e.

$$\omega \longrightarrow (\mathcal{I}^+)^2.$$

$\mathcal{I}$ is Ramsey if for every $\mathcal{I}$-positive set $X$ and every coloring $\varphi : [X]^2 \rightarrow 2$ there is an $\mathcal{I}$-positive subset $Y$ of $X$ which is $\varphi$-homogeneous, i.e.

$$\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2.$$

- If $\mathcal{I}$ is both a $P^+$ and $Q^+$-ideal then $\mathcal{I}^+ \longrightarrow (\mathcal{I}^+)^2$.
- While $Q^+$ is necessary, $P^+$ is not.
Observation

No tall analytic ideal is both $P^+$ and $Q^+$.

- Assume not, i.e. $I$ is tall analytic, and both $P^+$ and $Q^+$.
- Force with $P(\omega)/I$ and let $U$ be the generic ultrafilter.
- In $V[U]$, $U$ is a selective ultrafilter disjoint from $I$. Contradiction!

Question

Is there a tall Borel (analytic) Ramsey ideal?
Definition

Let $I$ and $J$ be ideals on $\omega$.

- (Katětov order) $I \leq_K J$ if there is a function $f : \omega \to \omega$ such that $f^{-1}[I] \in J$, for all $I \in I$.

- (Katětov-Blass order) as above with $f$ finite-to-one.

The following are fundamental questions about the Katětov order:

- Is there a tall Borel ideal Katětov-minimal among tall Borel ideals?
- Is there a Borel tall ideal $J$ such that for every Borel tall ideal $I$ there is an $I$-positive set $X$ such that $J \leq_K I | X$?
- Let $\mathcal{R}$ be the ideal generated by the cliques and free sets of the random graph. Is $\mathcal{R}$ such an ideal? Equivalently, is $\mathcal{I}^+ \not\rightarrow (\mathcal{I}^+)_2^2$ true for every tall Borel ideal?
Five ideals

- \( \mathcal{ED} = \{ A \subseteq \omega \times \omega : (\exists m, n \in \omega) (\forall k > n) (|\{l : \langle k, l \rangle \in A\}| \leq m)\} \).
- \( \mathcal{ED}_{\text{fin}} = \mathcal{ED} \upharpoonright \triangle \), where \( \triangle = \{ \langle m, n \rangle : n \leq m \} \).
- \( \text{fin} \times \text{fin} = \{ A \subseteq \omega \times \omega : \{ n : \{ m : (n, m) \in A \} \notin \text{fin} \} \in \text{fin} \} \).
- conv is the ideal on \( \mathbb{Q} \cap [0, 1] \) generated by sequences in \( \mathbb{Q} \cap [0, 1] \) convergent in \( [0, 1] \).
- \( \mathcal{R} \) is the ideal on \( \omega \) generated by the homogeneous sets (cliques and free sets) in Rado’s random graph.

\[ \omega \rightarrow (\mathcal{I}^+)^2 \quad \text{if and only if} \quad \mathcal{I} \not\gtrless_K \mathcal{R}. \]
An ideal $\mathcal{I}$ is $Q^+$ iff $\forall X \in \mathcal{I}^+ \mathcal{E}_{\text{fin}} \not\subseteq_{KB} \mathcal{I} \upharpoonright X$.

**Theorem**

*For any analytic ideal $\mathcal{I}$ the following conditions are equivalent*

1. $\mathcal{I}$ is a $Q^+$-ideal,
2. $\forall X \in \mathcal{I}^+ \mathcal{E}_{\text{fin}} \not\subseteq_{KB} \mathcal{I} \upharpoonright X$
3. $\forall X \in \mathcal{I}^+ \mathcal{I} \upharpoonright X$ is not an $\omega$-hitting ideal.
P$^+$-ideals

**Definition**

An ideal $\mathcal{I}$ is a

- **$P^+$-ideal** if for every decreasing sequence $\{X_n : n \in \omega\} \subseteq \mathcal{I}^+$ there is an $\mathcal{I}$-positive set $X$ such that $X \subseteq^* X_n$, for all $n$.

- a **$P^+_{\text{tower}}$-ideal** ($P^+$ according to Grigorieff) if decreasing sequences $\{X_n : n \in \omega\}$ of $\mathcal{I}$-positive sets such that $X_n \setminus X_{n+1} \in \mathcal{I}$ for all $n$, have $\mathcal{I}$-positive pseudointersections.

- $\mathcal{I}$ is $P^+_{\text{tower}}$ if and only if $\text{fin} \times \text{fin} \not\leq_K \mathcal{I} \upharpoonright X$ for some $X \in \mathcal{I}^+$.

- $\mathcal{I}$ is a $P^+$-ideal if and only if $\mathcal{I}$ is $P^+_{\text{tower}}$ and $\mathcal{P}(\omega)/\mathcal{I}$ is $\sigma$-closed.

- $\mathcal{P}(\omega)/\mathcal{I}$ is $\sigma$-closed if and only if $\mathcal{I}$ is indecomposable.

An ideal $\mathcal{I}$ is *decomposable* if there is a partition $\{X_n : n \in \omega\}$ of $\omega$ such that $I \in \mathcal{I}$ iff $I \cap X_n \in \mathcal{I}$ for all $n \in \omega$. 

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Ramsey type properties of definable ideals
**P+ according to Laflamme and Fσ ideals**

**Definition**

Given a family $\mathcal{X}$ of infinite subsets of $\omega$, we call a tree $T \subseteq ([\omega]^{<\omega})^{<\omega}$ an $\mathcal{X}$-tree of finite sets if for each $s \in T$ there is an $X_s \in \mathcal{X}$ such that $s \upharpoonright a \in T$ for each $a \in [X_s]^{<\omega}$.

An ideal $\mathcal{I}$ on $\omega$ is a $P^+_\text{tree}$-ideal ($P^+$ according to Laflamme) if every $\mathcal{I}^+$-tree of finite sets has a branch whose union is in $\mathcal{I}^+$.

$P^+_\text{tree} \Rightarrow P^+ \Rightarrow P^+_\text{tower}$

**Theorem**

Let $\mathcal{I}$ be an analytic ideal on $\omega$. Then:

- (H.-Meza) $\mathcal{I}$ is $P^+_\text{tree}$ iff it is $F_\sigma$.
- (HMTU) $\mathcal{I}$ is $P^+$ iff $\mathcal{I}$ is indecomposable and locally $F_\sigma$, i.e. every $\mathcal{I}$-positive $X$ set contains an $\mathcal{I}$-positive set $Y$ such that $\mathcal{I} \upharpoonright Y$ is $F_\sigma$.
- (Laczkovich-Recław + Solecki) $\mathcal{I}$ is $P^+_\text{tower}$ iff $\mathcal{I}$ can be $F_\sigma$-separated from $\mathcal{I}^*$.
Theorem
An analytic ideal $\mathcal{I}$ is $P^+$ iff it is indecomposable and locally $F_\sigma$.

Proof. Assume $\mathcal{I}$ is analytic and $P^+$, hence indecomposable. We shall show that it is locally $F_\sigma$.

- Let $X$ be an $\mathcal{I}$-positive set, and let $\mathcal{U}$ be the $\mathcal{P}(\omega)/\mathcal{I}$-generic ultrafilter on $\omega$ containing $X$. Then, as $\mathcal{I}$ is $P^+$, in $V[\mathcal{U}]$, $\mathcal{U}$ is a $P$-point disjoint from $\mathcal{I}$.

- By Zapletal's theorem there is, in $V[\mathcal{U}]$, an $F_\sigma$ ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$ and $\mathcal{J} \cap \mathcal{U} = \emptyset$.

- As $\mathcal{P}(\omega)/\mathcal{I}$ is $\sigma$-closed, $\mathcal{J}$ is in $V$, and there is a $Y \in \mathcal{U}$ (in particular $Y \in \mathcal{I}^+$), $Y \subseteq X$, such that $Y \vdash \langle \mathcal{I} \subseteq \mathcal{J} \rangle$ and $\mathcal{J} \cap \mathcal{U} = \emptyset$.

- To finish the argument it suffices to see that $\mathcal{I} \upharpoonright Y = \mathcal{J} \upharpoonright Y$. If not there is a $Z \subseteq Y$, $Z \in \mathcal{J} \setminus \mathcal{I}$. Then, however, $Z \in \mathcal{I}^+$ and $Z \vdash \langle Z \in \mathcal{U} \cap \mathcal{J} \rangle$, which is a contradiction.
Question

- Is there a tall Borel Ramsey ideal?
- I.e. is $\mathcal{I}^+ \not\rightarrow (\mathcal{I}^+)^2_2$ true for every tall Borel ideal?
- Equivalently, is $\mathcal{R}$ such that for every Borel tall ideal $\mathcal{I}$ there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{R} \leq_K \mathcal{I} \upharpoonright X$?

Theorem (H.-Meza-Thümmel-Uzcategui)

- Let $\mathcal{I}$ be a tall Borel ideal on $\omega$ such that $\mathcal{P}(\omega)/\mathcal{I}$ is proper. Then there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{I} \upharpoonright X \geq_K \mathcal{R}$. In particular, 
  - For every $F_{\sigma}$ ideal $\mathcal{I}$, there is an $\mathcal{I}$-positive set $X$ such that $\mathcal{I} \upharpoonright X \geq_K \mathcal{R}$.
  - There is a tall co-analytic $\mathcal{I}$ such that $\mathcal{I}^+ \rightarrow (\mathcal{I}^+)^2_2$. 
Theorem (H.-Meza-Thümmel-Uzcategui)

Let $I$ be a tall Borel ideal on $\omega$ such that $\mathcal{P}(\omega)/I$ is proper. Then there is an $I$-positive set $X$ such that $I \upharpoonright X \succcurlyeq_{K} \mathcal{R}$, i.e. $I^{+} \not\rightarrow (I^{+})_{2}^{2}$.

Case 1. $\mathcal{P}(\omega)/I$ adds reals

- $\text{conv} \leq_{K} I$ iff there is a countable family $\mathcal{X} \subseteq [\omega]^{\omega}$ such that for every $Y \in I^{+}$ there is $X \in \mathcal{X}$ such that $|X \cap Y| = |Y \setminus X| = \aleph_{0}$.
- Let $I$ be an ideal on $\omega$ such that $\mathcal{P}(\omega)/I$ is proper and adds a new real. Then there is an $I$-positive set $X$ such that $I \upharpoonright X \succcurlyeq_{K} \text{conv}$.
- $\mathcal{R} \leq_{K} \text{conv}$.

Case 2. $\mathcal{P}(\omega)/I$ does not add reals.

- Assume $I^{+} \rightarrow (I^{+})_{2}^{2}$. Let $\mathcal{U}$ be a $\mathcal{P}(\omega)/I$ generic ultrafilter.
- As before, derive contradiction with Mathias’ Theorem in $V[\mathcal{U}]$. 
There is a tall co-analytic ideal such that $\mathcal{I}^+ \to (\mathcal{I}^+)_2^2$.

- There is a Borel function $F : [\omega]^\omega \times 2^{[\omega]^2} \to [\omega]^\omega$ such that $F(A, \varphi)$ is a $\varphi$-homogeneous infinite subset of $A$.

- There is a continuous function $\psi : [\omega]^\omega \times 2^\omega \to [\omega]^\omega$ such that for every infinite $A \subseteq \omega$, the collection $\{\psi(A, x) : x \in 2^\omega\}$ is an almost disjoint family of infinite subsets of $A$.

Moreover, for all infinite $A \subseteq \omega$ there is an infinite $B \subseteq A$ such that $B \cap \psi(A, x) = \emptyset$ for all $x \in 2^\omega$.

There is a tall $F_\sigma$ ideal such that $\omega \to (\mathcal{I}^+)_2^2$. 

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Given an ideal $\mathcal{I}$ let

$$\widetilde{\mathcal{I}} = \{ A \subseteq \omega \times \omega : \exists k \in \omega (\forall i < k (A)_i \in \mathcal{I}) \ & \ (\forall i > k |(A)_i| < k)\}.$$ 

It is clear that if $\mathcal{I}$ is a Borel ideal then $\widetilde{\mathcal{I}}$ is a Borel ideal too. In fact, if $\mathcal{I}$ is $F_\sigma$ then so is $\widetilde{\mathcal{I}}$.

**Theorem**

If $\mathcal{I}^+ \rightarrow (<\omega, \mathcal{I}^+)^2$ then $\omega \rightarrow (\widetilde{\mathcal{I}}^+)^2$.
Ramsey\( (\omega) \) ideal which is \( F_\sigma \)

**Theorem**

\[ I^+ \rightarrow (\omega, I^+) \frac{2}{2} \text{ then } \omega \rightarrow (\overline{I}^+) \frac{2}{2}. \]

**Theorem**

The ideal \( \mathcal{ED} \) has the following properties.

1. \( \omega \not\rightarrow (\mathcal{ED}^+) \frac{2}{2}. \)
2. \( \mathcal{ED}^+ \not\rightarrow (\omega, \mathcal{ED}^+) \frac{2}{2}. \)
3. \( \omega \rightarrow (\omega, \mathcal{ED}^+) \frac{2}{2}. \)
4. \( \mathcal{ED}^+ \rightarrow (\omega, \mathcal{ED}^+) \frac{2}{2}. \)

**Corollary.**

\( \overline{\mathcal{ED}} \) is an \( F_\sigma \) ideal such that \( \omega \rightarrow (\overline{\mathcal{ED}}^+) \frac{2}{2} \) but \( \omega \not\rightarrow (\overline{\mathcal{ED}}^+) \frac{2}{3}. \)
Questions

- Is there a Borel tall ideal $\mathcal{I}$ on $\omega$ satisfying $\mathcal{I}^+ \rightarrow (\mathcal{I}^+)_2^2$?
- Is there a (locally) $\leq_K$-minimal ideal $\mathcal{I}$ among tall Borel ideals?
- Does every tall Borel ideal $\mathcal{I}$ contain a tall $F_\sigma$ ideal?
- Is there an ideal such that $\mathcal{I}^+ \rightarrow (\omega, \mathcal{I}^+_+)_2^2$ and $\mathcal{I}^+ \not\rightarrow (\mathcal{I}^+)_2^2$?
- Is there a Borel such ideal?

A standard “stepping up” type argument shows that $\mathcal{I}^+ \rightarrow (\mathcal{I}^+)_2^2$ implies the stronger statement $\mathcal{I}^+ \rightarrow (\mathcal{I}^+)_n^k$ for all $n, k > 0$. Baumgartner and Taylor showed that $\mathcal{I}^+ \rightarrow (4, \mathcal{I}^+_+)_2^3$ iff $\mathcal{I}^+ \rightarrow (\mathcal{I}^+)_2^2$.

- What more is there to say about higher dimensions?

Let $\Omega = \{U \in Clp(2^\omega) : \mu(U) = 1/2\}$, and let

$S = \{A \subseteq \Omega : \exists F \in [2^\omega]^{<\omega} \forall U \in A \ U \cap F \neq \emptyset\}$.

- Does $\Omega \rightarrow (S^+_+)_2^2$?