Category forcings and generic absoluteness: steps towards a “complete” axiom system for set theory

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Large cardinal axioms turned out to give a close to complete picture of second order arithmetic. This is a consequence of the fact that projective determinacy holds assuming large cardinals and assuming determinacy many nice things happen, for example all sets of reals which are determined are Lebesgue measurable and have the Baire property.

Martin's maximum and other forcing axioms of the same flavor turned out to give a close to complete picture of third order arithmetic. Many mathematical problems formalizable in third order arithmetic get an answer assuming MM, a short list includes:

1. The continuum problem,
2. Whitehead problem on groups,
3. The S-space problem in general topology,
4. The existence of outer automorphisms for the Calkin algebra in C^*-algebra theory,
5. ...

We will argue that Baire's category theorem and forcing axioms of the type of MM are responsible of this phenomenon because they provide a "complete sematics" for second order and third order arithmetic.

A BRIEF REVIEW OF FORCING

Basic idea behind forcing:

Replace Tarski semantics with boolean valued semantics.

The aspect of forcing we want to stress is:

Forcing is a model technique to realize types.

It is particularly effective for the first order theory ZFC.

Baire's category theorem is the key ingredient to apply forcing to realize *countable* types.

Forcing axioms are the key ingredient to apply forcing to realize *uncountable* types.

We shall argue that under favourable circumstances (which often occur) for many first order structures

$$\langle A^V, R_i^V : i \in I \rangle \prec \langle A^{V[G]}, R_i^{V[G]} : i \in I \rangle$$

where G is V-generic for a suitable complete boolean algebra $\mathbb{B} \in V$.

Given a signature

$$\tau = \{R_i, f_j, c_k : i, j, k \in I\}$$

for a first order language and a boolean algebra $\mathbb{B}$

$$(M, R_i^M, f_j^M, c_k^M : i, j, k \in I)$$

is a boolean valued model for τ such that:

- If R_i is a n-ary relation symbol,

$$R_i^M : M^n \to \mathbb{B}.$$

- If f_j is a n-ary function symbol

$$f_i^M : M^n \to M.$$

- If c_k is a constant symbol, $c_k^M \in M$.

R_i^M gives a meaning to atomic formulae.

If ϕ is a formula in the signature τ we can now use the boolean operations and a certain degree of completeness of $\mathbb{B}$ to assign a boolean value $[\![\phi]\!]_{M,\mathbb{B}}$ to ϕ:

- $[\![R(a_1, \ldots, a_m)]\!] = R_i^M(a_1, \ldots, a_n)$,
- $[\![\phi \wedge \psi]\!] = [\![\phi]\!] \wedge [\![\psi]\!]$,
- $[\![\neg\phi]\!] = \neg[\![\phi]\!]$,
- $[\![\exists x\phi(x, \vec{a})]\!] = \bigvee_{b \in M}[\![\phi(b, \vec{a})]\!]$ (here we use the completeness of $\mathbb{B}$, actually it suffices the completeness of $\mathbb{B}$ with respect to subsets definable in M).

These assignment must respect the intended meaning of equality, thus we require that they satisfy, for example:

- $[\![\phi(a) \wedge (a = b)]\!] \leq [\![\phi(b)]\!]$,
- $\bigwedge_{i \leq n} [\![(a_i = b_i)]\!] \leq [\![f_i^M(a_0, \ldots, a_n) = f_i^M(b_0, \ldots, b_n)]\!]$,
- $\ldots$

If G is an ultrafilter on $\mathbb{B}$ we can define

$$M/G$$

a first order structure for τ whose elements are the equivalence classes

$$[a] = \{b \in M : [\![a = b]\!] \in G\},$$

letting

$$R_i^{M/G}([a_0], \dots, [a_n]) \text{ iff } [\![R_i^M(a_0, \dots, a_n)]\!] \in G.$$

M is a *full* boolean valued model for $\mathbb{B}$ if for all $\phi(x, \vec{y}) \in \tau$ and $\vec{b} \in M^{<\omega}$ there is $a \in M$ such that

$$[\![\exists x \phi(x, \vec{b})]\!] = [\![\phi(a, \vec{b})]\!].$$

Theorem (Cohen's forcing theorem)

Assume M is a full boolean valued model for $\mathbb{B}$ and G is an ultrafilter for $\mathbb{B}$. Then $M/G \models \phi$ iff $\phi \in G$.

Forcing is an algorithm which takes as inputs

- A (transitive) Tarski model M of ZFC,
- A boolean algebra $\mathbb{B} \in M$ (which M models to be complete),
- An ultrafilter G on $\mathbb{B}$,

and produce as outputs a boolean valued model $M^{\mathbb{B}}$, and its quotient $M^{\mathbb{B}}/G$ which is a standard Tarski model for the language of set theory.

Theorem

$M^{\mathbb{B}}$ is a full boolean valued model and $[\![\phi]\!]_{\mathbb{B},M} = 1_{\mathbb{B}}$ for all axioms ϕ of ZFC*.*
$M^{\mathbb{B}}/G \models$ ZFC*.*

This theorem uses specific features of ZFC and makes forcing a peculiar technique which is not so easy to declinate for other first order theories.

Many ϕ which are not theorems of ZFC can be shown to be independent finding a $\mathbb{B}$ and a Tarski model M of ZFC such that

$$1_{\mathbb{B}} > [\![\phi]\!]_{\mathbb{B},M} > 0_{\mathbb{B}}.$$

If G is an ultrafilter on $\mathbb{B}$ with $[\![\phi]\!]_{\mathbb{B},M} \in G$

$$M^{\mathbb{B}}/G \models \phi.$$

Similarly if $[\![\neg\phi]\!]_{\mathbb{B},M} \in G$.

The above framework was essentially developed independently on one side by Vopenka, Scott, and others (this is a totally inaccurate account of the discovery of forcing) and on the other side by Cohen.
May be what led Cohen to succeed in proving the independence of CH was that he was able to merge these arguments with the Baire's category theorem.

Theorem (Baire's category)

Assume X is a compact Hausdorff space. Let $\{D_i : i \in \mathbb{N}\}$ be a countable family of open dense sets of X.
Then $\bigcap_{i \in \mathbb{N}} D_i$ is non-empty.

Theorem (Stone representation)

Assume $\mathbb{B}$ is a boolean algebra. Let $St(\mathbb{B})$ be its space of ultrafilters endowed with the topology generated by the sets

$$N_b = \{G \in St(\mathbb{B}) : b \in G\}$$

as b ranges in $\mathbb{B}$.
Then $St(\mathbb{B})$ is compact and Hausdorff and $\mathbb{B}$ is isomorphic to the algebra of clopen subset of $St(\mathbb{B})$.

Assume $\mathbb{B}$ is the family of clopen sets of $2^{\mathbb{N}}$ with the product topology. Then $St(\mathbb{B})$ is homeomorphic to $2^{\mathbb{N}}$.

Let M be a *countable* transitive model of ZFC. Then $2^{\mathbb{N}} \in M$ and the family of dense open subsets of $2^{\mathbb{N}}$ which are definable in M is countable.

By Baire's category theorem there is $a \in 2^{\mathbb{N}}$ meeting all these dense sets.

Cohen showed that

$$M^{\mathbb{B}}/a$$

is a well founded transitive model of ZFC.

We can identify it with its transitive collapse $M[a]$ and check that it it is the minimal model of ZFC which contains M and to which a belongs.

Now observe that for all $b \in 2^{\mathbb{N}} \cap M$

$$D_b = \{c \in 2^{\mathbb{N}} : c \neq b\}$$

is open dense in $2^{\mathbb{N}}$.

Thus $a \in D_b$ for all $b \in 2^{\mathbb{N}} \cap M$ if $a \in 2^{\mathbb{N}}$ is M-generic for the boolean algebra $Cl(2^{\mathbb{N}})$.

In particular $M[a]$ realizes the type

$$\{x \neq b \wedge x \in 2^{\mathbb{N}} : b \in 2^{\mathbb{N}} \cap M\}.$$

So forcing is a powerful technique leading from

$$M \mapsto M[a]$$

which realizes in the *transitive well founded* model $M[a]$ of ZFC a type which is defined in the *transitive well-founded* model M of ZFC.

Baire's category theorem is the key tool to produce this more "saturated" well-founded extension $M[a]$ of M which realizes countable types of M.

Boolean valued models and sheaves

There is a natural duality which identifies $2^{\mathbb{N}} \cap M[a]$ with the set

$$\{h(a) : h \in M,\ h : 2^{\mathbb{N}} \to 2^{\mathbb{N}} \text{ is continuous on a comeager set}\}$$

In particular $M[a] \cap 2^{\mathbb{N}}$ can be identified with the stalk given by a of the sheaf of continuous functions $h : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ in M defined on a comeager set.

Assume $R \in M$ is a clopen relation on $(2^{\mathbb{N}})^2$.
Then Shoenfield's absoluteness gives:

$$\langle 2^{\mathbb{N}} \cap M, R \cap M \rangle \prec_{\Sigma_2} \langle 2^{\mathbb{N}} \cap M[a], R \cap M[a] \rangle$$

for *any* $a \in 2^{\mathbb{N}}$ which is M-generic.

Generic absoluteness, a model theoretic approach:

Let (X, τ) be a Hausdorff topological space.

$A \subset X$ is meager iff there are countably many dense open sets $D_n \in \tau$ such that

$$A \cap \bigcap_{n \in \mathbb{N}} D_n = \emptyset.$$

$A \subset X$ has the Baire property if for some U open in τ $A \Delta U$ is meager.

We shall focus on a polish space, for the sake of simplicity let's take $2^{\mathbb{N}}$, any other will do.

Definition (Magidor, Feng, Woodin)

$A \subset (2^{\mathbb{N}})^n$ is universally Baire if for all compact Hausdorff spaces (X, τ) and all continuous

$$f : X \to (2^{\mathbb{N}})^n$$

$f^{-1}[A]$ has the Baire property.

Let (X,τ) and (Y,σ) be any pair of locally compact Hausdorff space.

Definition

$$Sh(X, Y) = \{f : X \to Y : f \text{ is continuous on a comeager subset of } X\}$$

is the sheaf of continuous modulo meager functions from X to Y.

Let G be a ultrafilter on the complete boolean algebra given by the regular open subsets of X.
$[f]_G \in Sh(X, Y)/G$ if

$$[f]_G = \{h \in Sh(X, Y) : f \restriction O =^* h \restriction O \text{ for some } O \in G\}$$

where $f \restriction O =^* h \restriction O$ if they differ on a meager subset of X.
For $A \subset Y^n$

$$([f_1]_G, \ldots, [f_n]_G) \in A_G$$

if and only if there is $O \in G$ such that $(f_1(x), \ldots, f_n(x)) \in A$ for all but meager many $x \in O$.

Remark:
An ultrafilter G on $RO(X)$ determines a point of X which is the unique element of

$$x_G = \bigcap \{\overline{O} : O \in G\}$$

(here we use that X is locally compact and Hausdorff).
However there is much more information in G than in the point x_G. For example $[f]_G$ is well defined even if x_G is not in the domain of f.

key point 1:

There is a natural duality which identifies the elements of $Sh(X, Y)$ with the $RO(X)$-names for elements of Y.

In the case of $f : X \to 2^\omega$ continuous on a comeager set:

Let for $i < 2$, $n \in \mathbb{N}$:

$$N_{n,i} = \{a \in 2^{\mathbb{N}} : a(n) = i\},$$

then

$$f \mapsto \tau_f = \{\langle(\check{n}, \check{i}), f^{-1}[N_{n,i}]\rangle : n \in \mathbb{N}, i < 2\}$$

and if τ is a $RO(X)$-name for an element of $2^{\mathbb{N}}$

$$\tau \mapsto f_\tau$$

where for τ $RO(X)$-name for an element of $2^{\mathbb{N}}$, f_τ is defined by the requirement:

$$f_\tau^{-1}[N_{n,i}] = [\![\tau(n) = i]\!]_{RO(X)}$$

It is not hard to check that $f_\tau : X \to Y$ is defined and continuous on a comeager subset of X.

key point 2:
Assume $A \subset (2^{\mathbb{N}})^n$ is universally Baire.
Then its preimage under any continuous $h : X^n \to (2^{\mathbb{N}})^n$ can be identified with the regular open set $O(A, h) \subset X^n$ with which it has a meager difference, moreover $O(A^c, h)$ is the boolean complement in $RO(X)$ of $O(A, h)$.

A_G is a well defined n-ary relation on $Sh(X, 2^{\mathbb{N}})/G$ for all G ultrafilter on the regular open subsets of X:
$h = (f_1, \ldots, f_n) : X^n \to (2^{\mathbb{N}})^n$ is continuous, thus:

There is a unique $O(A, h)$ open regular set with a meager difference with A,
Thus we have that

$$([f_1]_G, \ldots, [f_n]_G) \in A_G \text{ iff } O(A, h) \in G.$$

Notice that this is well defined even if the point $x_G \in X$ determined by G is neither in $O(A, h)$ nor in $O(A^c, h)$.
Consider $A =<_{lex}$ to be the lexicographic order relation on $2^{\mathbb{N}}$, to appreciate what's going on.

Woodin's generic absoluteness reformulated

Theorem

Assume there are class many Woodin cardinals.
Let $A \subset (2^{\mathbb{N}})^n$ be any universally Baire relation. Let (X, τ) be Hausdorff and compact.
Then for any ultrafilter G on the family of regular open subsets of X

$$\langle 2^{\mathbb{N}}, =, A \rangle \prec \langle Sh(X, 2^{\mathbb{N}})/G, =_G, A_G \rangle$$

Generic absoluteness for more complex structures

There are serious natural limitations to generalize this result to more complex first order structures.
Nonetheless I think I have a (next to) optimal generalization of Woodin's result for first order structures which can be formalized in third order number theory.
For the sake of completeness this is a possible formulation of the result:

Theorem

Assume there are class many superhuge cardinals and MM^{+++}.
Let $A \subset (2^{\mathbb{N}})^n$ *be* ***almost any*** *relation (for example definable with parameters and using the axiom of choice in* H_{ω_2}*).*
Let (X, τ) *be compact and Hausdorff such that its algebra of regular open sets is an* SSP *forcing notion which preserves* MM^{+++}.
Then for all ultrafilters G *on the family of regular open subsets of* X

$$\langle 2^{\mathbb{N}}, =, A\rangle \prec \langle Sh(X, 2^{\mathbb{N}})/G, =_G, A_G\rangle$$

In this more complex framework one can also handle metric spaces and structures which are naturally axiomatized in continuous logic (C^*-algebras, Von Neumann rings, Banach spaces) and many set theoretic consequences of MM on these type of structures are essentially obtained using the fact that suitable forcing notions produce *elementary* superstructures of the given structure (Banach space C^*-algebra, etc....) realizing types with the desired properties.

Forcing axioms and MM^{+++}

Definition

Assume X is a compact Hausdorff space and λ is an uncountable cardinal.

$\mathsf{FA}_\lambda(X)$ holds if

$$\bigcap_{i<\lambda} D_i \neq \emptyset$$

whenever

$$\{D_i : i < \lambda\}$$

is a family of dense open sets of X.

$\mathsf{FA}_\lambda(\mathbb{B})$ stands for $\mathsf{FA}_\lambda(St(\mathbb{B}))$.

There are some things to point out:

- While all compact Hausdorff spaces satisfy the Baire category theorem, not all compact Hausdorff spaces X satisfy $\mathrm{FA}_{\aleph_1}(X)$.

  For example any compactification X of ω_1^ω endowed with the product topology does not satisfy $\mathrm{FA}_{\aleph_1}(X)$.
- Shelah isolated a necessay condition on $\mathbb{B}$ in order that $\mathrm{FA}_{\aleph_1}(\mathbb{B})$ holds: $\mathbb{B}$ has to be stationary set preserving (SSP).

  Shelah, Magidor, and Foreman proved that it is consistent relative to large cardinal axioms that such a condition is also sufficient and called Martin maximum MM this statement, i.e.:

  MM $\equiv \mathrm{FA}_{\aleph_1}(\mathbb{B})$ *for all* $\mathbb{B} \in$ SSP.

Category forcings

- If $i : \mathbb{B} \to \mathbb{Q}$ is an homomorphism, $\pi_i : St(\mathbb{Q}) \to St(\mathbb{B})$ is a continuous map sending $G \mapsto \{b : i(b) \in G\}$.
- If $i : \mathbb{B} \to \mathbb{Q}$ is also a *complete* homomorphism and $\mathbb{B}$, $\mathbb{Q}$ are complete, π_i is a clopen map and $\pi_i^{-1}[D]$ is open dense in $St(\mathbb{Q})$ for all D open dense of $St(\mathbb{B})$.

  In this case we get that $\mathsf{FA}_\lambda(\mathbb{Q})$ implies $\mathsf{FA}_\lambda(\mathbb{B})$.

The latter statement leads to category forcings:

Definition

Let Γ be a class of complete boolean algebras and Θ be a class of complete homomorphisms between elements of Γ and closed under composition and identity maps.
We say that:

- $\mathbb{B} \leq_\Theta \mathbb{Q}$ if there is a complete homomorphism $i : \mathbb{B} \to \mathbb{Q}$ in Θ.
- $\mathbb{B} \leq^*_\Theta \mathbb{Q}$ if there is a complete and *injective* homomorphism $i : \mathbb{B} \to \mathbb{Q}$ in Θ.

With these definition $(\Gamma, \leq_\Theta)$ and $(\Gamma, \leq^*_\Theta)$ are class partial orders.

For those more averted, forcing is usually presented as a tool to use partial orders to produce new models of ZFC.

In particular we can now look at these class partial orders as forcing notions, and check whether they are interesting forcing notions.

The order $\leq^*_\Theta$ is the one we use to study iterated forcing and captures the notion of complete embedding for partial orders.

$\leq_\Theta$ has been neglected so far but is sufficient to grant that whenever $i : \mathbb{B} \to \mathbb{Q}$ witnesses $\mathbb{Q} \leq_\Theta \mathbb{B}$ and G is V-generic for $\mathbb{Q}$, then $i^{-1}[G]$ is V-generic for $\mathbb{B}$.

Theorem

The following holds:

- ***Woodin:*** *Assume there are class many Woodin cardinals. Then the family of presaturated towers is dense in* $(\Omega, \leq_\Omega)$ *where* Ω *stands for the class of all complete boolean algebras and all complete homomorphisms between them.*
- ***Woodin:*** *Assume there are class many Woodin cardinals. Then Martin's maximum is equivalent to the assertion that the family of presaturated towers is dense in* $(\mathsf{SSP}, \leq_\Omega)$*.*
- ***V.:*** *Assume there are class many Woodin cardinals Then* MM^{++} *(a strong form of* MM*) is equivalent to the assertion that the family of presaturated towers is dense in* $(\mathsf{SSP}, \leq_{\mathsf{SSP}})$*, where* $\mathbb{B} \geq_{\mathsf{SSP}} \mathbb{Q}$ *iff there is* $i : \mathbb{B} \to \mathbb{Q}$ *complete homomorphism such that*

$$[\![\mathbb{Q}/i[\dot{G}_{\mathbb{B}}] \in \mathsf{SSP}]\!]_{\mathbb{B}} = 1_{\mathbb{B}}.$$

Definition (V.)

MM^{+++} holds if the class of *strongly* presaturated towers is dense in $(\mathrm{SSP}, \leq_{\mathrm{SSP}})$.

Fact

$\mathrm{MM}^{+++} \rightarrow \mathrm{MM}^{++} \rightarrow \mathrm{MM}$.

Theorem (V.)

MM^{+++} *is consistent relative to the existence of a huge cardinal.*

Intermezzo – Why I like the category forcing $(\mathsf{SSP}, \leq_{\mathsf{SSP}})$:

This category has many surprising and nice features:

Theorem (V.)

Assume that δ is supercompact. Then $(\mathsf{SSP} \cap V_\delta, \leq_{\mathsf{SSP}} \restriction V_\delta)$ is an SSP *partial order $\mathbb{U}_\delta$.*
Moreover:

- $\mathbb{B} \geq_{\mathsf{SSP}} \mathbb{U}_\delta \restriction \mathbb{B}$ *for all* $\mathbb{B} \in \mathsf{SSP} \cap V_\delta$.
- $\mathbb{U}_\delta$ *forces* MM^{++}.

Theorem (V.)

Assume δ is a reflecting cardinal and MM^{+++} holds (i.e. there are densely many strongly presaturated towers in SSP*).*
Then $\mathbb{U}_\delta$ is itself a strongly presaturated tower.

In general $\mathbb{U}_\delta$ is a universal object for the category up to δ which as a partial order in the category SSP acquires most of the first order properties whose extension is a dense subset of $\mathbb{U}_\delta$.

Generic absoluteness: standard formulation

Assume $T \subseteq \text{ZFC}$, Θ is a family of sentences in the language of T, Γ is a class of forcing notions.

Θ is Γ-generically invariant for T if for all $S \supseteq T$ and any $\phi \in \Theta$ any of the following three equivalent conditions holds:

1. $S \vdash \phi$
2. $S \vdash$ there is $\mathbb{B} \in \Gamma$ such that $[\![\phi]\!] = 1_{\mathbb{B}}$ and $[\![T]\!] = 1_{\mathbb{B}}$.
3. $S \vdash [\![\phi]\!] = 1_{\mathbb{B}}$ for all $\mathbb{B} \in \Gamma$ such that $[\![T]\!] = 1_{\mathbb{B}}$.

The class of boolean valued models of the form $V^{\mathbb{B}}$ is a complete semantic for Θ if our metatheory is S and V is our universe of sets.

Theorem (Woodin)

Let Θ be second order number theory and $T = \text{ZFC}$+large cardinals. Then Θ is Γ-generically invariant for T where Γ is the class of all posets.

Fact (Shelah)

Assume Θ is third order number theory and $T \supseteq \text{ZFC}$. Then Θ cannot be Γ-generically invariant for T unless Γ is contained in SSP.

The theory with parameters of $H_{\aleph_1}$ is an equivalent reformulation of second order number theory.

The theory with parameters of $H_{\aleph_2}$ is a very large fragment of third order number theory and Shelah's fact applies also to this theory. Moreover all results on third order number theory listed before (and many others) are formalizable in the first order theory with parameters of $H_{\aleph_2}$ (or in a theory which fall under the scope of provable strenghtenings of the theorem below).

Theorem (V.)

Let Θ be the theory with parameters of $H_{\aleph_2}$ and

$$T = \text{ZFC} + \textit{ large cardinals } + \text{MM}^{+++}.$$

Then Θ is SSP-generically invariant for T.

THANKS FOR YOUR PATIENCE AND ATTENTION