Quotients of strongly proper posets, and related topics

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Joint work with John Krueger.

A conjecture of Viale-Weiss

The principle $\mathsf{ISP}(\omega_2)$:

- introduced by Weiss
- follows from PFA (Viale-Weiss), and many consequences of PFA factor through $\mathsf{ISP}(\omega_2)$.
- **Conjecture (Viale-Weiss):** $\mathsf{ISP}(\omega_2)$ is consistent with large continuum (i.e. $> \omega_2$).

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- **Conjecture (Viale-Weiss):** $\text{ISP}(\omega_2)$ is consistent with large continuum (i.e. $> \omega_2$).

Theorem (C.-Krueger 2014)

Proved the conjecture of Viale-Weiss. Developed general theory of **quotients** *of strongly proper forcings.*

Outline

Approximation property

Definition (Hamkins)

Let (W, W') be transitive models of set theory such that:

- $W \subset W'$
- μ is regular in W

We say (W, W') has the μ-approximation property iff whenever:

1. $X \in W'$;
2. X is a bounded subset of W;
3. $\forall z \in W \; |z|^W < \mu \implies z \cap X \in W$

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We will focus on the case $\mu = \omega_1$ throughout this talk.

The class G_{ω_1}

Definition (Viale-Weiss)

M is ω_1-guessing, denoted $M \in \mathrm{G}_{\omega_1}$, iff $|M| = \omega_1 \subset M$ and (H_M, V) has the ω_1-approximation property (where H_M is transitive collapse of M).

Definition (Viale-Weiss)

$\mathrm{ISP}(\omega_2)$ is the statement: for all regular $\theta \geq \omega_2$:

$$G_{\omega_1} \cap P_{\omega_2}(H_\theta) \text{ is stationary}$$

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Generalization of theorems of Baumgartner, Krueger

Consequences of PFA that factor through ISP

- $TP(\omega_2)$
- Every tree of height and size ω_1 has at most ω_1 many cofinal branches (in particular no Kurepa trees)
  - together with $2^{\omega_1} = \omega_2$ this yields $\diamondsuit^+(S^2_1)$ (Todorcevic, Baumgartner)
- Failure of $\square(\theta)$ for all $\theta \geq \omega_2$ (Weiss; actually failure of weaker forms of square)
- SCH (Viale)
- $\mathrm{IA}_{\omega_1} \neq^* \mathrm{Unif}_{\omega_1}$ and stronger separations (Krueger)
- Laver Diamond at ω_2 (Viale from PFA, Cox from ISP plus $2^\omega = \omega_2$)

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Even more consequences of *PFA* factor through "specialized" ISP; more on that later.

Example: $ISP(\omega_2)$ implies $TP(\omega_2)$

Let T be a tree of height ω_2 and width $< \omega_2$. By stationarity of G_{ω_1} there is an $M \in G_{\omega_1}$ such that $M \prec (H_{\omega_3}, \in, T)$. Let $\sigma : H_M \to M \prec H_{\omega_3}$ be inverse of collapsing map of M; let

$$\alpha := M \cap \omega_2 = \mathrm{crit}(\sigma) \text{ and } T_M := \sigma^{-1}(T)$$

Our goal is to prove that $H_M \models$ "T_M has a cofinal branch".

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Since (H_M, V) has the ω_1-approximation property, it suffices to find (in V) a cofinal b through T_M such that every proper initial segment of b is an element of H_M. But since T is thin, then $T_M = T|\alpha$. Pick any t on the α-th level of T; then $t\downarrow$ is a cofinal branch through $T_M = T|\alpha$ and every proper initial segment is of course in H_M.

Outline

Review of forcing quotients

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Important variation: "$\mathbb{P}$ is regular in $\mathbb{Q}$ below q"

Strongly proper forcing

The following notion is due to Mitchell.

Definition

Given a poset $\mathbb{P}$ and a model M, a condition $p \in \mathbb{P}$ is an $(M, \mathbb{P})$ strong master condition iff "$M \cap \mathbb{P}$ is a regular suborder of $\mathbb{P}$ below p".

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"$\mathbb{P}$ is strongly proper": defined similarly to properness, using strong master condition instead of master condition.

Examples and properties of strongly proper forcings

Examples:

- Todorcevic's finite $\in$-collapse
- Baumgartner's adding a club with finite conditions
- adding any number of Cohen reals
- Various (pure) side condition posets of Mitchell, Friedman, Neeman, Krueger, and others.

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- $(V, V^{\mathbb{P}})$ has the ω_1-approximation property

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Key properties (Mitchell):

- absorbs Add(ω)
- $(V, V^{\mathbb{P}})$ has the ω_1-approximation property

Remark: To get ω_1 approx, suffices to be strongly proper wrt *stationarily many* countable models.

Sketch of ω_1-approx property from strong properness

Suppose $1_{\mathbb{P}}$ forces that $\dot{b}$ is a **new** subset of θ and that $z \cap \dot{b} \in V$ for every V-countable set z. Let $M \prec (H_{\theta^+}, \in, \dot{b}, \dots)$ be countable and let p be a strong master condition for M. Since M is countable then by assumption $\check{M} \cap \dot{b}$ is forced to be in the ground model. Let $p' \leq p$ decide this value.

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Let $p'|M$ be a reduct of p' into $M \cap \mathbb{P}$. Since $\dot{b}$ is forced to be new and $\dot{b}, p'|M \in M$, then there are $r, s \in M$ below $p'|M$ which disagree about some member of M being an element of $\dot{b}$. Then clearly they cannot both be compatible with a condition which decides $\check{M} \cap \dot{b}$. In particular they cannot both be compatible with p'. Contradiction.

Quotients of strongly proper forcings

Question

Suppose $\mathbb{Q}$ is strongly proper and $\mathbb{P}$ is a regular suborder. When does the quotient $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ have the following properties?

- *strongly proper "wrt V models"?*
- *ω_1-approximation property?*

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Remark: There are well-known examples of quotients of proper forcings that aren't proper.

The star condition

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Definition (Krueger)

Assume $\mathbb{P}$ is a suborder of $\mathbb{Q}$.
$\star(\mathbb{P}, \mathbb{Q})$ denotes the statement: whenever $p \in \mathbb{P}$ and $q_1, q_2 \in \mathbb{Q}$ and p, q_1, q_2 are **pairwise** compatible, then there is a lower bound for all three.
$\star(\mathbb{Q})$ is the stronger statement that $\star(\mathbb{Q}, \mathbb{Q})$ holds.

Examples where $\star(\mathbb{Q})$ holds:

- $\mathrm{Col}(\mu, \theta)$
- Todorcevic's $\in$-collapse
- Krueger's adequate set forcing

Key properties of $\star(\mathbb{P}, \mathbb{Q})$

Lemma

Assume $\star(\mathbb{P}, \mathbb{Q})$ and let $G_{\mathbb{P}}$ be generic for $\mathbb{P}$. Then in $V[G_{\mathbb{P}}]$:

$$\big(\forall q_1, q_2 \in \mathbb{Q}/G_{\mathbb{P}}\big) \quad \big(q_1 \parallel_{\mathbb{Q}} q_2 \implies q_1 \parallel_{\mathbb{Q}/\dot{G}_{\mathbb{P}}} q_2\big)$$

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Proof: let $q_1, q_2 \in \mathbb{Q}/G_{\mathbb{P}}$ and suppose $q_1 \wedge q_2 \neq 0$ in $\mathbb{Q}$; we will prove that $q_1 \wedge q_2 \in \mathbb{Q}/G_{\mathbb{P}}$, i.e. that $q_1 \wedge q_2$ is compatible with every member of $G_{\mathbb{P}}$. Let $p \in G_{\mathbb{P}}$. Then $q_1 \wedge p \neq 0 \neq q_2 \wedge p$. By $\star(\mathbb{P}, \mathbb{Q})$ we have $q_1 \wedge q_2 \wedge p \neq 0$.

$\star(\mathbb{P}, \mathbb{Q})$ implies strong master conditions survive in the quotient

Lemma

Suppose $\star(\mathbb{P}, \mathbb{Q})$ holds and q is $(M, \mathbb{Q})$ strong master condition. Then

$$\Vdash_{\mathbb{P}} \check{q} \in \mathbb{Q}/\dot{G}_{\mathbb{P}} \implies \check{q} \text{ is } (M[\dot{G}_{\mathbb{P}}], \mathbb{Q}/\dot{G}_{\mathbb{P}}) \text{ s.m.c.}$$

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Proof sketch: Suppose $p \in \mathbb{P}$ forces that $\check{q} \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$ (i.e. $\check{q} \parallel \dot{G}_{\mathbb{P}}$). Then p must force that $M[\dot{G}_{\mathbb{P}}] \cap V = M$; otherwise there is some $p' \leq p$ forcing $M \subsetneq M[\dot{G}_{\mathbb{P}}] \cap V$, but p' still forces $\check{q} \in \mathbb{Q}/\dot{G}_{\mathbb{P}}$. So let $G_{\mathbb{P}} * H$ be generic (in the 2-step iteration) with $(p', q) \in G_{\mathbb{P}} * H$. But q is in particular an $(M,\mathbb{Q})$ master condition, so $M = M[G_{\mathbb{P}} * H] \cap V \supset M[G_{\mathbb{P}}] \cap V$. Contradiction.

Recall q is $(M, \mathbb{Q})$ strong master condition, and we showed that if $q \in \mathbb{Q}/G_{\mathbb{P}}$ then in particular $\mathbb{Q} \cap M = \mathbb{Q} \cap M[G_{\mathbb{P}}] =: \mathbb{Q}_M$. Now $\mathbb{Q}_M$ is regular in $\mathbb{Q}$ below q (this is Σ_0 statement).

Suppose $q' \leq q$, where $q' \in \mathbb{Q}/G_{\mathbb{P}}$. Let $q'|M$ be a reduct of q' into $\mathbb{Q}_M$. We need to see that:

- $q'|M \parallel G_{\mathbb{P}}$; this is straightforward, especially if $q'|M \geq q'$ as is usually the case; and
- any extension of $q'|M$ in $\mathbb{Q}_M/G_{\mathbb{P}}$ is compatible with q' in $\mathbb{Q}/G_{\mathbb{P}}$. Suppose q'' is such a condition; so $q'' \parallel G_{\mathbb{P}}$ and is $\mathbb{Q}$-compatible with q'. By the previous lemma (using the $\star(\mathbb{P}, \mathbb{Q})$ assumption), q' and q'' are compatible in $\mathbb{Q}/G_{\mathbb{P}}$.

A sufficient condition

Theorem (C.-Krueger)

Suppose:

- $\mathbb{Q}$ *is well-met;*
- *There is a stationary set* $\mathcal{S}$ *of countable models* M *for which* $\mathbb{Q}$ *has universal strong master conditions;*
- $\mathbb{P}$ *is a regular suborder of* $\mathbb{Q}$ *(possibly "below a condition")*
- $\star(\mathbb{P}, \mathbb{Q})$ *holds*

Then $\mathbb{P}$ *forces that* $\mathbb{Q}/\dot{G}_{\mathbb{P}}$ *is strongly proper for the stationary set of models of the form* $M[\dot{G}_{\mathbb{P}}]$ *where* $M \in \mathcal{S}$*. In particular, the quotient has the* ω_1 *approximation property.*

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REMARK: universality isn't needed if you only want ω_1-approx property.

A counterexample

Quotients of strongly proper posets may fail to have the ω_1-approximation property:

Theorem (Krueger, Mota)

Assume $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Let $\mathbb{Q}$ be the forcing with coherent adequate *sets of countable submodels of H_{ω_3}. Then $\mathbb{Q}$ has the following properties:*

- *$\mathbb{Q}$ is strongly proper and ω_2-cc;*
- *$\mathbb{Q}$ forces CH*
- *$\mathbb{Q}$ adds a Kurepa tree on ω_1 with ω_3 many cofinal branches*
- *There is a regular suborder $\mathbb{P}$ of size ω_2 such that*

$$\Vdash_{\mathbb{P}} \mathbb{Q}/\dot{G}_{\mathbb{P}} \text{ fails to have the } \omega_1 \text{ approximation property}$$

Outline

ISP and large continuum

Recall Viale-Weiss:

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Theorem (C.-Krueger)

Assume κ is a supercompact cardinal and $\theta \geq \kappa$ arbitrary. Let:

- *$\mathbb{P}$ be "adequate set forcing" to turn κ into $\aleph_2$; (or Neeman's side condition forcing; or Friedman's; ...)*
- *$\mathbb{Q} = Add(\omega, \theta)$*

Then $V^{\mathbb{P}\times\mathbb{Q}} \models ISP(\omega_2)$ and $2^\omega = \theta$.

Proof outline

Let $G \times H$ be generic for $\mathbb{P} \times \mathbb{Q}$. Let $\theta \geq \omega_2 = \kappa$ be regular and $\mathfrak{A} = (H_\theta[G \times H], \in, \dots)$ be an algebra.

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Back in V let $j : V \to N$ be sufficiently supercompact with $\mathrm{crit}(j) = \kappa$ so that $j[H_\theta] \in N$. $\mathbb{P} \times \mathbb{Q}$ is κ-cc and $\mathrm{crit}(j) = \kappa$, so $j : \mathbb{P} \times \mathbb{Q} \to j(\mathbb{P} \times \mathbb{Q})$ is a regular embedding; so we can force with the quotient

$$j(\mathbb{P} \times \mathbb{Q})/j[G \times H] \tag{1}$$

and lift j to

$$j : V[G \times H] \to N[G' \times H']$$

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N believes that $j(\mathbb{P} \times \mathbb{Q})$ is strongly proper and the pair

$$j[\mathbb{P} \times \mathbb{Q}], j(\mathbb{P} \times \mathbb{Q})$$

satisfies the star property. So $N\big[j[G \times H]\big]$ believes that the quotient in (1) has the ω_1-approximation property; so $(H_\theta^V[G \times H], N[G' \times H'])$ has ω_1-a.p., and also $j\big[H_\theta^V[G \times H]\big] \prec j(\mathfrak{A})$. Then use elementarity of j.

Outline

What Viale-Weiss really proved

Definition

Let's call M a *specialized* ω_1 *guessing model*, and write $M \in \mathrm{sG}_{\omega_1}$, iff a certain tree related to M is specialized; **in particular** $M \in G_{\omega_1}$ **and remains so in any outer model with the same** ω_1.

They proved that under PFA, $\mathrm{sG}_{\omega_1} \cap P_{\omega_2}(H_\theta)$ $(\cap \mathrm{IC}_{\omega_1})$ is stationary for all $\theta \geq \omega_2$.

Consequences of PFA which factor through specialized guessing models

- If T is a tree of height and size ω_1 then forcing with T collapses ω_1 (Baumgartner)
- (together with assumption $2^\omega = \omega_2$) Every forcing which adds a new subset of ω_1 either adds a real or collapses ω_2 (Todorcevic)

Sketch of proof

In V consider the stationary set $S := sG_{\omega_1} \cap P_{\omega_2}(H_{\omega_2})$. Using stationarity of S and the assumption that $2^\omega = \omega_2$, fix a $\subset$-increasing (non-continuous) chain $\langle M_\alpha \mid \alpha < \omega_2 \rangle$ of elements of S whose union contains H_{ω_1}.

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Suppose W is an outer model of V which adds a new subset b of ω_1, and doesn't add a real. Then it doesn't add new subsets of countable ordinals either, so for all $\xi < \omega_1$ we have

$$b \cap \xi \in H^V_{\omega_1} \subset \bigcup_{\alpha<\omega_2} M_\alpha$$

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In W define a function $f : \omega_1 \to \omega_2^V$ by sending ξ to the least α such that $b \cap \xi \in M_\alpha$. This is a cofinal map from $\omega_1 \to \omega_2^V$ since for any $\alpha < \omega_2$, since $b \notin M_\alpha$ and M_α is $G^W_{\omega_1}$ then there is some $\xi < \omega_1$ such that $b \cap \xi \notin M_\alpha$.

A new question

Our model of ISP(ω_2) plus large continuum is NOT a model of the "specialized" version (because it has a tree of height and size ω_1 whose forcing doesn't collapse ω_1).

This suggests a natural modification of the Viale-Weiss question:

Question

Assume "specialized" ISP(ω_2); i.e. suppose sG_{ω_1} is stationary for all $P_{\omega_2}(H_\theta)$. Does this imply $2^\omega = \omega_2$?