Selective properties of ideals

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Selective properties of ideals

If $\mathcal{I}$ is an ideal on ω, then by $\mathcal{I}^+ = \mathcal{P}(\omega) \setminus \mathcal{I}$ we denote the family of all $\mathcal{I}$-positive subsets of ω.

Definition

An ideal $\mathcal{I}$ on ω is:

- **selective** *if every partition $(X_n)_{n\in\omega}$ of ω such that $\bigcup_{m\geq n} X_m \notin \mathcal{I}$ for each n, has an $\mathcal{I}$-positive selector;*
- **weakly selective** *if every partition $(X_n)_{n\in\omega}$ of ω with at most one element not in $\mathcal{I}$ and such that $\bigcup_{m\geq n} X_m \notin \mathcal{I}$ for each n, has an $\mathcal{I}$-positive selector;*
- **locally selective** *if every partition $(X_n)_{n\in\omega}$ of ω into elements belonging to $\mathcal{I}$, has an $\mathcal{I}$-positive selector.*

selective $\Rightarrow$ weakly selective $\Rightarrow$ locally selective

For maximal ideals all the above notions coincide.

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Selectivity versus density

Recall that an ideal $\mathcal{I}$ on ω is **dense** (also the term **tall** is commonly used) if for any infinite $A \subseteq \omega$ one can find infinite $B \subseteq A$ with $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is selective if and only if $\mathcal{I}^+$ is a happy family.

Theorem (Mathias, 1977)

Selectivity and density exclude each other in the case of definable ideals, i.e., any analytic (or coanalytic) selective ideal is not dense.

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The "combinatorial distance"

Following Laflamme we say that an ideal $\mathcal{I}$ on ω is ω-**diagonalizable** if there is a sequence $(A_n)_{n\in\omega}$ of infinite subsets of ω such that

$$\forall A \in \mathcal{I} \quad \exists n \quad A \cap A_n = \emptyset.$$

If, moreover, for a certain family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ all A_n's are members of $\mathcal{A}$, then we say that $\mathcal{I}$ is ω-**diagonalizable by** $\mathcal{A}$.

Theorem (K. and Zakrzewski)

If $\mathcal{I}$ is an analytic (or coanalytic) ideal on ω, then:

(a) *$\mathcal{I}$ is selective if and only if $\mathcal{I}$ is ω-diagonalizable by every ultrafilter $\mathcal{U} \subseteq \mathcal{I}^+$;*

(b) *$\mathcal{I}$ is not dense if and only if $\mathcal{I}$ is ω-diagonalizable by any filter on ω.*

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The "topological distance"

Given a nonprincipal ultrafilter $\mathcal{U}$ on ω, we define the so-called $\mathcal{U}$**-topology on** $[\omega]^\omega$ (for details see S. Todorčević, *Introduction to Ramsey Spaces*). This is a slight modification of the Ellentuck's topology.

Theorem (K. and Zakrzewski)

If $\mathcal{I}$ is an analytic (or coanalytic) ideal on ω, then:

(a) *$\mathcal{I}$ is selective if and only if $\mathcal{I}$ is nowhere dense in the $\mathcal{U}$-topology for every ultrafilter $\mathcal{U} \subseteq \mathcal{I}^+$;*

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Topological representations

Definition

Suppose that X is a separable metrizable space and J is a σ-ideal on X containing all singletons. Given a dense countable set $D \subseteq X$, we define the ideal

$$\mathcal{I}_J = \{A \subseteq D : cl(A) \in J\}.$$

Given an ideal $\mathcal{I}$ on a countable set E, we say that it **admits a topological representation** *if there are X, J, D as above and a bijection $\rho\colon E \to D$ such that $\mathcal{I} = \{A \subseteq E : \rho[A] \in \mathcal{I}_J\}$.*

Example

$NWD(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : A \textit{ is nowhere dense}\}$
$NULL(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : cl(A) \textit{ is of Lebesgue measure zero}\}$

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A conjecture

Conjecture (Sabok and Zapletal, 2011)

For any ideal $\mathcal{I}$ on a countable set TFAE:

(a) *$\mathcal{I}$ admits a topological representation;*

(b) *$\mathcal{I}$ is dense $\mathbf{\Pi}^0_3$ and weakly selective.*

The following definition is a variation of Todorčević's notion of countably separated gaps.

Definition

We say that an ideal $\mathcal{I}$ on a countable set D is **countably separated** *if there is a countable family $\{X_n : n \in \omega\}$ of subsets of D such that for any $A, B \subseteq D$ with $A \notin \mathcal{I}$ and $B \in \mathcal{I}$ there is $n \in \omega$ with $A \cap X_n \notin \mathcal{I}$ and $B \cap X_n = \emptyset$.*

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Characterization

Theorem (K. and Sabok)

For any ideal $\mathcal{I}$ TFAE:

(a) *$\mathcal{I}$ admits a topological representation;*

(b) *$\mathcal{I}$ is dense and countably separated.*

Theorem (K. and Sabok)

Any analytic ideal which admits a topological representation is $\mathbf{\Pi}^0_3$-complete.

Theorem (K. and Sabok)

For any coanalytic ideal $\mathcal{I}$ TFAE:

(a) *$\mathcal{I}$ is weakly selective;*

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Method (the case of Borel ideals)

Consider the following game $G(\mathcal{I})$ invented by Laflamme: Player I in his n-th turn picks $C_n \in \mathcal{I}$ and Player II responds with a point x_n such that $x_n \notin C_n$. Player I wins if $\{x_n : n \in \omega\} \in \mathcal{I}$. Otherwise, Player II wins.

1. Martin's Theorem on Borel determinacy: $G(\mathcal{I})$ is a determined game.
2. Laflamme's characterization of $G(\mathcal{I})$: Player I has a winning strategy in $G(\mathcal{I})$ if and only if $\mathcal{I}$ is not weakly Ramsey.
3. It can be shown that weak selectivity implies weak Ramseyness. Therefore, Player II has a winning strategy.
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More about weak Ramseyness

Recall that $\mathcal{I}$ is:

- weakly selective if every partition $(X_n)_{n\in\omega}$ with at most one element not in $\mathcal{I}$ and such that $\bigcup_{m\geq n} X_m \notin \mathcal{I}$ for each n, has an $\mathcal{I}$-positive selector;
- locally selective if every partition $(X_n)_{n\in\omega} \subseteq \mathcal{I}$ has an $\mathcal{I}$-positive selector.

Proposition (Grigorieff, 1971)

$\mathcal{I}$ is weakly selective if and only if for every partition $(X_n)_{n\in\omega}$ with at most one element not in $\mathcal{I}$ and such that $\bigcup_{m\geq n} X_m \notin \mathcal{I}$ for each n, there exists a strictly increasing function $f:\omega\to\omega$, with $f[\omega]\notin\mathcal{I}$ and such that $f(n+1)\in\bigcup_{i>f(n)} X_i$ for each $n\in\omega$.

Definition

*$\mathcal{I}$ is **weakly Ramsey** if for every partition $(X_n)_{n\in\omega}\subseteq\mathcal{I}$, there exists a strictly increasing function $f:\omega\to\omega$, with $f[\omega]\notin\mathcal{I}$ and such that $f(n+1)\in\bigcup_{i>f(n)} X_i$ for each $n\in\omega$.*

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selective ⇒ weakly selective ⇒ weakly Ramsey ⇒ locally selective

Theorem (K.)

The implications cannot be reversed!

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Critical ideal for weak Ramseyness

We write $\mathcal{I} \leq_K \mathcal{J}$ if there is $f\colon \bigcup\mathcal{J} \to \bigcup\mathcal{I}$ such that $f^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$. If f is a bijection, then we write $\mathcal{I} \sqsubseteq \mathcal{J}$.

Theorem (K.)

There is a critical (in the sense of $\leq_K$ and $\sqsubseteq$) ideal for weak Ramseyness, i.e., an ideal $\mathcal{WR}$ such that for any ideal $\mathcal{I}$ TFAE:

(a) *$\mathcal{I}$ is weakly Ramsey;*

(b) *$\mathcal{WR} \not\leq_K \mathcal{I}$;*

(c) *$\mathcal{WR} \not\sqsubseteq \mathcal{I}$.*

If, moreover, $\mathcal{I}$ is coanalytic, then the above conditions are equivalent to the following:

(d) $\mathcal{I}$ is ω-diagonalizable by $\mathcal{I}^+$ (there is a sequence $(A_n)_{n\in\omega} \subseteq \mathcal{I}^+$ such that for each $A \in \mathcal{I}$ there is n with $A \cap A_n = \emptyset$).

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(d) $\mathcal{I}$ is ω-diagonalizable by $\mathcal{I}^+$ (there is a sequence $(A_n)_{n\in\omega} \subseteq \mathcal{I}^+$ such that for each $A \in \mathcal{I}$ there is n with $A \cap A_n = \emptyset$).

Critical ideal for weak Ramseyness

We write $\mathcal{I} \leq_K \mathcal{J}$ if there is $f\colon \bigcup \mathcal{J} \to \bigcup \mathcal{I}$ such that $f^{-1}[A] \in \mathcal{J}$ for all $A \in \mathcal{I}$. If f is a bijection, then we write $\mathcal{I} \sqsubseteq \mathcal{J}$.

Theorem (K.)

There is a critical (in the sense of $\leq_K$ and $\sqsubseteq$) ideal for weak Ramseyness, i.e., an ideal $\mathcal{WR}$ such that for any ideal $\mathcal{I}$ TFAE:

(a) *$\mathcal{I}$ is weakly Ramsey;*

(b) *$\mathcal{WR} \not\leq_K \mathcal{I}$;*

(c) *$\mathcal{WR} \not\sqsubseteq \mathcal{I}$.*

If, moreover, $\mathcal{I}$ is coanalytic, then the above conditions are equivalent to the following:

(d) $\mathcal{I}$ is ω-diagonalizable by $\mathcal{I}^+$ (there is a sequence $(A_n)_{n\in\omega} \subseteq \mathcal{I}^+$ such that for each $A \in \mathcal{I}$ there is n with $A \cap A_n = \emptyset$).

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