The higher sharp

Yizheng Zhu

Universität Münster

8th Young Set Theory Workshop
October 2015
Projective sets

\( \mathbb{R} = \omega^\omega \), the Baire space. \( \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^\omega \).

Definition

\( \mathbb{R} = \omega^\omega \), the Baire space. \( \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^\omega \).
**Projective sets**

\( \mathbb{R} = \omega^\omega \), the Baire space. \( \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^\omega \).

**Definition**

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma^1_n \)-definable over \( (V_{\omega+1}; \in) \).
Projective sets

\( \mathbb{R} = \omega^\omega \), the Baire space. \( \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^\omega \).

Definition

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma_n \)-definable over \( (V_{\omega+1}; \in) \).
- Given \( x \in \mathbb{R} \), \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n(x) \) iff \( A \) is \( \Sigma_n \)-definable over \( (V_{\omega+1}; \in, x) \).
Projective sets

\( \mathbb{R} = \omega^\omega \), the Baire space. \( \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^\omega \).

Definition

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma_n \)-definable over \( (V_{\omega+1}; \in) \).
- Given \( x \in \mathbb{R} \), \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n(x) \) iff \( A \) is \( \Sigma_n \)-definable over \( (V_{\omega+1}; \in, x) \).
- \( A \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma^1_n(x) \) for some \( x \in \mathbb{R} \).
Projective sets

\( \mathbb{R} = \omega^\omega \), the Baire space. \( \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^\omega \).

**Definition**

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma_n \)-definable over \((\mathbb{V}_{\omega+1}; \in)\).
- Given \( x \in \mathbb{R} \), \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n(x) \) iff \( A \) is \( \Sigma_n \)-definable over \((\mathbb{V}_{\omega+1}; \in, x)\).
- \( A \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma^1_n(x) \) for some \( x \in \mathbb{R} \).

Equivalently,

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_1 \) iff there is a Borel set \( B \subseteq \mathbb{R}^2 \) such that \( x \in A \iff \exists y \ (x, y) \in B \).
Projective sets

\( \mathbb{R} = \omega^\omega \), the Baire space. \( \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^\omega \).

Definition

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma_n \)-definable over \( (V_{\omega+1}; \in) \).
- Given \( x \in \mathbb{R} \), \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n(x) \) iff \( A \) is \( \Sigma_n \)-definable over \( (V_{\omega+1}; \in, x) \).
- \( A \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma^1_n(x) \) for some \( x \in \mathbb{R} \).

Equivalently,

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_1 \) iff there is a Borel set \( B \subseteq \mathbb{R}^2 \) such that \( x \in A \iff \exists y \ (x, y) \in B \).
- \( A \subseteq \mathbb{R} \) is \( \Pi^1_n \) iff \( \mathbb{R} \setminus A \) is \( \Sigma^1_n \).
Projective sets

\( \mathbb{R} = \omega^\omega \), the Baire space. \( \mathbb{R} \cong \mathbb{R}^2 \cong \mathbb{R}^\omega \).

**Definition**

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma^n_n \)-definable over \( (V_{\omega+1}; \in) \).
- Given \( x \in \mathbb{R} \), \( A \subseteq \mathbb{R} \) is \( \Sigma^1_n(x) \) iff \( A \) is \( \Sigma^n_n \)-definable over \( (V_{\omega+1}; \in, x) \).
- \( A \) is \( \Sigma^1_n \) iff \( A \) is \( \Sigma^1_n(x) \) for some \( x \in \mathbb{R} \).

Equivalently,

- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_1 \) iff there is a Borel set \( B \subseteq \mathbb{R}^2 \) such that \( x \in A \iff \exists y \ (x, y) \in B \).
- \( A \subseteq \mathbb{R} \) is \( \Pi^1_n \) iff \( \mathbb{R} \setminus A \) is \( \Sigma^1_n \).
- \( A \subseteq \mathbb{R} \) is \( \Sigma^1_{n+1} \) iff there is a \( \Pi^1_n \) set \( B \subseteq \mathbb{R}^2 \) such that \( x \in A \iff \exists y \ (x, y) \in B \).

\( A \) is **projective** iff \( A \) is \( \Sigma^1_n \) for some \( n < \omega \).
Regularity properties of sets of reals

Assume Projective Determinacy (PD). Then

- Every projective set of reals is Lebesgue measurable
Regularity properties of sets of reals

Assume Projective Determinacy (PD). Then

- Every projective set of reals is Lebesgue measurable
- Every projective set of reals has the Baire property
Assume Projective Determinacy (PD). Then

- Every projective set of reals is Lebesgue measurable
- Every projective set of reals has the Baire property
- Every projective set of reals is either countable or equinumerous with $\mathbb{R}$ via a projective bijection (effective CH)
Regularity properties of sets of reals

Assume Projective Determinacy (PD). Then

▶ Every projective set of reals is Lebesgue measurable
▶ Every projective set of reals has the Baire property
▶ Every projective set of reals is either countable or equinumerous with $\mathbb{R}$ via a projective bijection (effective CH)
▶ ...

The influence of large cardinals on the reals

▶ (Martin-Steel) Suppose there are $n$ Woodin cardinals below a measurable cardinal, then $\Pi_{n+1}^1$ sets are determined.
Regularity properties of sets of reals

Assume Projective Determinacy (PD). Then

▶ Every projective set of reals is Lebesgue measurable
▶ Every projective set of reals has the Baire property
▶ Every projective set of reals is either countable or equinumerous with $\mathbb{R}$ via a projective bijection (effective CH)
▶ ...

The influence of large cardinals on the reals

▶ (Martin-Steel) Suppose there are $n$ Woodin cardinals below a measurable cardinal, then $\Pi_{n+1}^1$ sets are determined.
▶ (Woodin) Suppose there are $\omega$ Woodin cardinals below a measurable cardinal, then $L(\mathbb{R}) \models \text{AD}$.

In this talk, we assume $\text{ZFC} + L(\mathbb{R}) \models \text{AD}$. 
Suslin representation

A tree on $\omega \times X$ is a subset of $\omega^{<\omega} \times X^{<\omega}$, closed under initial segments.
Suslin representation

A tree on \( \omega \times X \) is a subset of \( \omega^{<\omega} \times X^{<\omega} \), closed under initial segments. If \( T \) is a tree on \( \omega \times X \), then

\[ [T] = \{(x, \vec{\alpha}) \in \omega^\omega \times X^\omega : \forall n < \omega \ (x \upharpoonright n, \vec{\alpha} \upharpoonright n) \in T \}, \]

**Definition**

1. Suppose \( \lambda \) is an ordinal. A \( \subseteq \mathbb{R} \) is \( \lambda \)-Suslin iff \( A = [T] \) for some tree \( T \) on \( \omega \times \lambda \).

2. A \( \subseteq \mathbb{R} \) is Suslin iff \( A \) is \( \lambda \)-Suslin for some ordinal \( \lambda \).

\( \Sigma^1_1 \) sets are \( \omega \)-Suslin. \( \Sigma^1_1 \) sets are effectively \( \omega \)-Suslin.

(AC) Every subset of reals is 2\( ^{\aleph_0} \)-Suslin.

**Question**

What about the effective version?
Suslin representation

A tree on $\omega \times X$ is a subset of $\omega^{<\omega} \times X^{<\omega}$, closed under initial segments. If $T$ is a tree on $\omega \times X$, then

- $[T] = \{(x, \vec{\alpha}) \in \omega^\omega \times X^\omega : \forall n < \omega \ (x \upharpoonright n, \vec{\alpha} \upharpoonright n) \in T\}$,
- $p[T] = \{x \in \omega^\omega : \exists \vec{\alpha} \ (x, \vec{\alpha}) \in [T]\}$.

**Definition**

1. Suppose $\lambda$ is an ordinal. $A \subseteq \mathbb{R}$ is $\lambda$-Suslin iff $A = p[T]$ for some tree $T$ on $\omega \times \lambda$. 

Σ₁¹ sets are $\omega$-Suslin. Σ₁¹ sets are effectively $\omega$-Suslin.

(AC) Every subset of reals is $2^{\aleph_0}$-Suslin.

Question: What about the effective version?
A tree on $\omega \times X$ is a subset of $\omega^{<\omega} \times X^{<\omega}$, closed under initial segments. If $T$ is a tree on $\omega \times X$, then

- $[T] = \{(x, \vec{\alpha}) \in \omega^\omega \times X^\omega : \forall n < \omega \ (x \upharpoonright n, \vec{\alpha} \upharpoonright n) \in T\}$,
- $p[T] = \{x \in \omega^\omega : \exists \vec{\alpha} \ (x, \vec{\alpha}) \in [T]\}$.

Definition

1. Suppose $\lambda$ is an ordinal. $A \subseteq \mathbb{R}$ is $\lambda$-Suslin iff $A = p[T]$ for some tree $T$ on $\omega \times \lambda$.

2. $A \subseteq \mathbb{R}$ is Suslin iff $A$ is $\lambda$-Suslin for some ordinal $\lambda$. 

Σ₁¹ sets are $\omega$-Suslin. Σ₁¹ sets are effectively $\omega$-Suslin.

(AC) Every subset of reals is $2^{\aleph_0}$-Suslin.

Question

What about the effective version?
Suslin representation

A tree on \( \omega \times X \) is a subset of \( \omega^{<\omega} \times X^{<\omega} \), closed under initial segments. If \( T \) is a tree on \( \omega \times X \), then

- \( [T] = \{(x, \vec{\alpha}) \in \omega^\omega \times X^\omega : \forall n < \omega \ (x \upharpoonright n, \vec{\alpha} \upharpoonright n) \in T\} \),
- \( p[T] = \{x \in \omega^\omega : \exists \vec{\alpha} \ (x, \vec{\alpha}) \in [T]\} \).

Definition

1. Suppose \( \lambda \) is an ordinal. \( A \subseteq \mathbb{R} \) is \( \lambda \)-Suslin iff \( A = p[T] \) for some tree \( T \) on \( \omega \times \lambda \).
2. \( A \subseteq \mathbb{R} \) is Suslin iff \( A \) is \( \lambda \)-Suslin for some ordinal \( \lambda \).

\( \Sigma_1^1 \) sets are \( \omega \)-Suslin. \( \Sigma_1^1 \) sets are effectively \( \omega \)-Suslin.

(AC) Every subset of reals is \( 2^{\aleph_0} \)-Suslin.
Suslin representation

A tree on $\omega \times X$ is a subset of $\omega^{<\omega} \times X^{<\omega}$, closed under initial segments. If $T$ is a tree on $\omega \times X$, then

- $[T] = \{(x, \vec{\alpha}) \in \omega^\omega \times X^\omega : \forall n < \omega (x \upharpoonright n, \vec{\alpha} \upharpoonright n) \in T\}$,
- $p[T] = \{x \in \omega^\omega : \exists \vec{\alpha} (x, \vec{\alpha}) \in [T]\}$.

Definition

1. Suppose $\lambda$ is an ordinal. $A \subseteq \mathbb{R}$ is $\lambda$-Suslin iff $A = p[T]$ for some tree $T$ on $\omega \times \lambda$.
2. $A \subseteq \mathbb{R}$ is Suslin iff $A$ is $\lambda$-Suslin for some ordinal $\lambda$.

$\Sigma^1_1$ sets are $\omega$-Suslin. $\Sigma^1_1$ sets are effectively $\omega$-Suslin.

(AC) Every subset of reals is $2^{\aleph_0}$-Suslin.

Question

What about the effective version?
Suslin representation

Basis theorems
If a mathematical property about reals is nonempty, how complicated is it to find a witness to this property?

1. Every nonempty $\Sigma^0_1$ set contains a real $\leq T_O$.
2. Every nonempty $\Sigma^0_2$ set contains a $\Delta^0_2$ real.
3. Every nonempty $\Sigma^0_3$ set contains a real $\leq T_M\#_1$.

...
Suslin representation

Basis theorems
If a mathematical property about reals is nonempty, how complicated is it to find a witness to this property?

1. Every nonempty $\Sigma^1_1$ set contains a real $\leq_T \emptyset$.
Suslin representation

Basis theorems
If a mathematical property about reals is nonempty, how complicated is it to find a witness to this property?

1. Every nonempty $\Sigma^1_1$ set contains a real $\leq_T \emptyset$.
2. Every nonempty $\Sigma^1_2$ set contains a $\Delta^1_2$ real.
Suslin representation

Basis theorems

If a mathematical property about reals is nonempty, how complicated is it to find a witness to this property?

1. Every nonempty $\Sigma^1_1$ set contains a real $\leq_T \emptyset$.
2. Every nonempty $\Sigma^1_2$ set contains a $\Delta^1_2$ real.
3. Every nonempty $\Sigma^1_3$ set contains a real $\leq_T M^#$.
4. ...
Suslin representation

Basis theorems
If a mathematical property about reals is nonempty, how complicated is it to find a witness to this property?

1. Every nonempty $\Sigma^1_1$ set contains a real $\leq_T \emptyset$.
2. Every nonempty $\Sigma^1_2$ set contains a $\Delta^1_2$ real.
3. Every nonempty $\Sigma^1_3$ set contains a real $\leq_T M_1^#$.
4. ...

DST way of thinking: Suppose $\emptyset \neq A = p[T]$.

$(x, \bar{\alpha}) \in [T]$ is the leftmost branch of $T$ iff whenever $(y, \bar{\beta}) \in [T]$, $(x(0), \alpha_0, x(1), \alpha_1, \ldots)$ is lexicographically smaller than $(y(0), \beta_0, y(1), \beta_1, \ldots)$. 
Suslin representation

Basis theorems
If a mathematical property about reals is nonempty, how complicated is it to find a witness to this property?

1. Every nonempty $\Sigma^1_1$ set contains a real $\leq_T \emptyset$.
2. Every nonempty $\Sigma^1_2$ set contains a $\Delta^1_2$ real.
3. Every nonempty $\Sigma^1_3$ set contains a real $\leq_T M_1^\#$.
4. ...

DST way of thinking: Suppose $\emptyset \neq A = p[T]$.

- $(x, \vec{\alpha}) \in [T]$ is the leftmost branch of $T$ iff whenever $(y, \vec{\beta}) \in [T]$, $(x(0), \alpha_0, x(1), \alpha_1, \ldots)$ is lexicographically smaller than $(y(0), \beta_0, y(1), \beta_1, \ldots)$.
- If $(x, \vec{\alpha}) \in [T]$ is the leftmost branch of $T$, then $x$ is the leftmost real associated to $T$. 
Effective codes of ordinals (1)

$\Sigma^1_2$ sets are “effectively” $\omega_1$-Suslin.

Codes of ordinals up to $\omega_1$:

- $x \in WO$ iff $x$ codes a wellordering of $\omega$. 

Is there a canonical coding system for ordinals smaller than $\Theta^\text{UB}$?

Next level: $\Sigma^1_3$ sets are “effectively” $\omega$-Suslin.
Effective codes of ordinals (1)

\[ \Sigma_1^2 \text{ sets are “effectively” } \omega_1\text{-Suslin.} \]

Codes of ordinals up to \( \omega_1 \):

- \( x \in \text{WO} \) iff \( x \) codes a wellordering of \( \omega \).
- Every infinite countable ordinal \( \alpha \) is coded by a real \( x \in \text{WO} \).
Effective codes of ordinals (1)

$\Sigma^1_2$ sets are “effectively” $\omega_1$-Suslin.

Codes of ordinals up to $\omega_1$:

- $x \in WO$ iff $x$ codes a wellordering of $\omega$.
- Every infinite countable ordinal $\alpha$ is coded by a real $x \in WO$.
- $WO$ is a $\Pi^1_1$ set.
Effective codes of ordinals (1)

$\Sigma^1_2$ sets are “effectively” $\omega_1$-Suslin.

Codes of ordinals up to $\omega_1$:

- $x \in \text{WO}$ iff $x$ codes a wellordering of $\omega$.
- Every infinite countable ordinal $\alpha$ is coded by a real $x \in \text{WO}$.
- WO is a $\Pi^1_1$ set.

Is there a canonical coding system for ordinals smaller than $\Theta^{UB}$?

Next level: $\Sigma^1_3$ sets are “effectively” $u_\omega$-Suslin.
The level-1 sharp

Assuming large cardinals, there is a club class of indiscernibles for $L$. 

The level-1 sharp

Assuming large cardinals, there is a club class of indiscernibles for $L$. That is, a club class of ordinals $(c_\alpha : \alpha < \infty)$ such that

- $L$ is Skolem generated by $\{c_\alpha : \alpha < \infty\}$,
- for every first-order formula $\varphi$, every $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$,

$$L \models \varphi(c_{\alpha_1}, \cdots, c_{\alpha_n}) \leftrightarrow \varphi(c_{\beta_1}, \cdots, c_{\beta_n}).$$
The level-1 sharp

Assuming large cardinals, there is a club class of indiscernibles for $L$. That is, a club class of ordinals $(c_\alpha : \alpha < \infty)$ such that

- $L$ is Skolem generated by \{c_\alpha : \alpha < \infty\},
- for every first-order formula $\varphi$, every $\alpha_1 < \cdots < \alpha_n$, $\beta_1 < \cdots < \beta_n$,

$$L \models \varphi(c_{\alpha_1}, \cdots, c_{\alpha_n}) \iff \varphi(c_{\beta_1}, \cdots, c_{\beta_n}).$$

$(c_\alpha : \alpha < \infty)$ is called the class of **Silver indiscernibles** for $L$. In particular, every uncountable cardinal is a Silver indiscernible for $L$.

$$0^\# = \bigoplus_{n<\omega} \left\{ \varphi^\frown : \varphi \text{ has } n \text{ free variables, } L \models \varphi(\aleph_1, \ldots, \aleph_n) \right\}.$$

$0^\#$ is the unique wellfounded remarkable EM blueprint.
The EM blueprint formulation of sharps

Definition
An EM blueprint is a complete consistent theory in the language \( \{ \dot{\in}, \dot{c}_n : n < \omega \} \) that extends the following axioms:

- \( ZFC + V = L \). In particular, there is a definable wellordering of the universe.
- \( c_1 < c_2 \)
- For every formula \( \varphi \), every increasing tuple \( k_1 < \cdots < k_m \), \( \varphi(\dot{c}_1, \ldots, \dot{c}_m) \rightarrow \varphi(\dot{c}_{k_1}, \ldots, \dot{c}_{k_m}) \) is an axiom
The EM blueprint formulation of sharps

Definition
An EM blueprint is a complete consistent theory in the language \( \{ \dot{\in}, \dot{c}_n : n < \omega \} \) that extends the following axioms:

- \( \text{ZFC} + V = L \). In particular, there is a definable wellordering of the universe.
- \( c_1 < c_2 \)
- For every formula \( \varphi \), every increasing tuple \( k_1 < \cdots < k_m \), \( \varphi(\dot{c}_1, \ldots, \dot{c}_m) \rightarrow \varphi(\dot{c}_{k_1}, \ldots, \dot{c}_{k_m}) \) is an axiom

An EM blueprint \( \Gamma \) is remarkable iff \( \Gamma \) contains the remarkability axioms:

\[
\tau(\dot{c}_1, \ldots, \dot{c}_{m+n}) < \dot{c}_{m+1} \rightarrow \tau(\dot{c}_1, \ldots, \dot{c}_{m+n}) = \tau(\dot{c}_1, \ldots, \dot{c}_m, \dot{c}_{m+n+1}, \ldots, \dot{c}_{m+2n})
\]

holds for every Skolem term \( \tau \).
Suppose $\Gamma$ is an EM blueprint. For every linear ordering $<^*$ on a set $W$, there is (up to isomorphism) a unique model $\mathcal{M}(\Gamma, <^*)$ and points $\{b_v : v \in W\} \subseteq \mathcal{M}(\Gamma, <^*)$ such that

$$\mathcal{M}(\Gamma, <^*) \models \varphi(b_{v_1}, \ldots, b_{v_n})$$

whenever $\varphi(\dot{c}_1, \ldots, \dot{c}_n)$ is true in $\Gamma$. 
The EM blueprint formulation of sharps

Suppose $\Gamma$ is an EM blueprint. For every linear ordering $<^*$ on a set $W$, there is (up to isomorphism) a unique model $\mathcal{M}(\Gamma, <^*)$ and points $\{b_v : v \in W\} \subseteq \mathcal{M}(\Gamma, <^*)$ such that

$$\mathcal{M}(\Gamma, <^*) \models \varphi(b_{v_1}, \ldots, b_{v_n})$$

whenever $\varphi(\dot{c}_1, \ldots, \dot{c}_n)$ is true in $\Gamma$.

$\mathcal{M}(\Gamma, <^*)$ is called the EM model associated to $\Gamma$ and $<^*$. 
The EM blueprint formulation of sharps

Suppose $\Gamma$ is an EM blueprint. For every linear ordering $<^*$ on a set $W$, there is (up to isomorphism) a unique model $M(\Gamma, <^*)$ and points $\{b_v : v \in W\} \subseteq M(\Gamma, <^*)$ such that

$$M(\Gamma, <^*) \models \varphi(b_{v_1}, \ldots, b_{v_n})$$

whenever $\varphi(\dot{c}_1, \ldots, \dot{c}_n)$ is true in $\Gamma$.

$M(\Gamma, <^*)$ is called the EM model associated to $\Gamma$ and $<^*$. An EM blueprint $\Gamma$ is wellfounded iff for every countable wellordering $<^*$, the EM model $M(\Gamma, <^*)$ is wellfounded.
The EM blueprint formulation of sharps

Suppose $\Gamma$ is an EM blueprint. For every linear ordering $<^*$ on a set $W$, there is (up to isomorphism) a unique model $\mathcal{M}(\Gamma, <^*)$ and points $\{b_v : v \in W\} \subseteq \mathcal{M}(\Gamma, <^*)$ such that

$$\mathcal{M}(\Gamma, <^*) \models \varphi(b_{v_1}, \ldots, b_{v_n})$$

whenever $\varphi(\dot{c}_1, \ldots, \dot{c}_n)$ is true in $\Gamma$.

$\mathcal{M}(\Gamma, <^*)$ is called the EM model associated to $\Gamma$ and $<^*$. An EM blueprint $\Gamma$ is wellfounded iff for every countable wellordering $<^*$, the EM model $\mathcal{M}(\Gamma, <^*)$ is wellfounded. Wellfoundedness is $\Pi_1^1$. 

▶ 0 # is the unique wellfounded remarkable EM blueprint ▶ x # is the unique wellfounded remarkable EM blueprint over x.
Suppose $\Gamma$ is an EM blueprint. For every linear ordering $<^*$ on a set $W$, there is (up to isomorphism) a unique model $\mathcal{M}(\Gamma, <^*)$ and points $\{b_v : v \in W\} \subseteq \mathcal{M}(\Gamma, <^*)$ such that

$$\mathcal{M}(\Gamma, <^*) \models \varphi(b_{v_1}, \ldots, b_{v_n})$$

whenever $\varphi(\dot{c}_1, \ldots, \dot{c}_n)$ is true in $\Gamma$. $\mathcal{M}(\Gamma, <^*)$ is called the EM model associated to $\Gamma$ and $<^*$. An EM blueprint $\Gamma$ is wellfounded iff for every countable wellordering $<^*$, the EM model $\mathcal{M}(\Gamma, <^*)$ is wellfounded. Wellfoundedness is $\Pi^1_2$.

- $0\#$ is the unique wellfounded remarkable EM blueprint
- $\kappa\#$ is the unique wellfounded remarkable EM blueprint over $\kappa$. 
Effective codes of ordinals (2)

Codes of ordinals up to $u_\omega$:

$\gamma$ is a uniform indiscernible iff $\gamma$ is a Silver indiscernible for $L[x]$ for every $x \in \mathbb{R}$. The uniform indiscernibles are enumerated ($u_\alpha : 1 \leq \alpha < \infty$).

$u_1 = \omega_1$. $u_\omega = \sup_{n < \omega} u_n$.

A sharp code is pair $(\lceil \tau \rceil, x^\#)$ where $x \in \mathbb{R}$, $\tau$ is an $n$-ary Skolem term for some $n$, and $\tau^{L[x]}(u_1, \ldots, u_n)$ is an ordinal.
Effective codes of ordinals (2)

Codes of ordinals up to $u_\omega$:

$\gamma$ is a **uniform indiscernible** iff $\gamma$ is a Silver indiscernible for $L[x]$ for every $x \in \mathbb{R}$. The uniform indiscernibles are enumerated $(u_\alpha : 1 \leq \alpha < \infty)$.

$$u_1 = \omega_1, \; u_\omega = \sup_{n<\omega} u_n.$$  

A **sharp code** is pair $(\lceil \tau \rceil, x^\#)$ where $x \in \mathbb{R}$, $\tau$ is an $n$-ary Skolem term for some $n$, and $\tau^{L[x]}(u_1, \ldots, u_n)$ is an ordinal. If $(\lceil \tau \rceil, x^\#)$ is an $n$-ary sharp code, then it codes an ordinal in $u_{n+1}$:

$$\left| (\lceil \tau \rceil, x^\#) \right| = \tau^{L[x]}(u_1, \ldots, u_n).$$
Codes of ordinals up to $u_\omega$:

\[ \gamma \text{ is a uniform indiscernible iff } \gamma \text{ is a Silver indiscernible for } L[x] \text{ for every } x \in \mathbb{R}. \]

The uniform indiscernibles are enumerated ($u_\alpha : 1 \leq \alpha < \infty$).

\[ u_1 = \omega_1. \quad u_\omega = \sup_{n<\omega} u_n. \]

A sharp code is pair $\langle \tau^{-}, x^\# \rangle$ where $x \in \mathbb{R}$, $\tau$ is an $n$-ary Skolem term for some $n$, and $\tau^{L[x]}(u_1, \ldots, u_n)$ is an ordinal. If $\langle \tau^{-}, x^\# \rangle$ is an $n$-ary sharp code, then it codes an ordinal in $u_{n+1}$:

\[ \left| \langle \tau^{-}, x^\# \rangle \right| = \tau^{L[x]}(u_1, \ldots, u_n). \]

- Every ordinal in $u_\omega$ is represented by a sharp code.
Effective codes of ordinals (2)

Codes of ordinals up to $u_\omega$:

$\gamma$ is a **uniform indiscernible** iff $\gamma$ is a Silver indiscernible for $L[x]$ for every $x \in \mathbb{R}$. The uniform indiscernibles are enumerated $(u_\alpha : 1 \leq \alpha < \infty)$.

$$u_1 = \omega_1. \quad u_\omega = \sup_{n < \omega} u_n.$$ 

A **sharp code** is pair $(\lceil \tau \rceil, x^\#)$ where $x \in \mathbb{R}$, $\tau$ is an $n$-ary Skolem term for some $n$, and $\tau^{L[x]}(u_1, \ldots, u_n)$ is an ordinal. If $(\lceil \tau \rceil, x^\#)$ is an $n$-ary sharp code, then it codes an ordinal in $u_{n+1}$:

$$\left| (\lceil \tau \rceil, x^\#) \right| = \tau^{L[x]}(u_1, \ldots, u_n).$$

▶ Every ordinal in $u_\omega$ is represented by a sharp code.
▶ The set of sharp codes is $\Pi^1_2$. 

---
Effective codes of ordinals (2)

Codes of ordinals up to $u_\omega$:

$\gamma$ is a uniform indiscernible iff $\gamma$ is a Silver indiscernible for $L[x]$ for every $x \in \mathbb{R}$. The uniform indiscernibles are enumerated $(u_\alpha : 1 \leq \alpha < \infty)$.

$$u_1 = \omega_1. \ u_\omega = \sup_{n<\omega} u_n.$$  

A sharp code is pair $(\tau^\frown, x^\#)$ where $x \in \mathbb{R}$, $\tau$ is an $n$-ary Skolem term for some $n$, and $\tau^{L[x]}(u_1, \ldots, u_n)$ is an ordinal. If $(\tau^\frown, x^\#)$ is an $n$-ary sharp code, then it codes an ordinal in $u_{n+1}$:

$$\left\vert (\tau^\frown, x^\#) \right\vert = \tau^{L[x]}(u_1, \ldots, u_n).$$

- Every ordinal in $u_\omega$ is represented by a sharp code.
- The set of sharp codes is $\Pi^1_2$.
- The relation “$\nu, \omega$ are sharp codes $\land \left\vert \nu \right\vert = \left\vert \omega \right\vert$” is $\Delta^1_3$. 

Projective well-ordered cardinals

- \( \kappa \) is a cardinal iff for any \( \alpha < \kappa \), there is no surjection \( \pi : \alpha \to \kappa \).
Projective well-ordered cardinals

- $\kappa$ is a cardinal iff for any $\alpha < \kappa$, there is no surjection $\pi : \alpha \rightarrow \kappa$.

- $\kappa$ is a projective wellordered cardinal iff for any $\alpha < \kappa$, there are no projective surjections $\rho : \mathbb{R} \rightarrow \alpha$, $\rho' : \mathbb{R} \rightarrow \kappa$, $\pi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\rho(x) = \rho(y) \rightarrow \rho' \circ \pi(x) = \rho' \circ \pi(y)$.
Projective well-ordered cardinals

- \( \kappa \) is a cardinal iff for any \( \alpha < \kappa \), there is no surjection \( \pi : \alpha \to \kappa \).

- \( \kappa \) is a projective wellordered cardinal iff for any \( \alpha < \kappa \), there are no projective surjections \( \rho : \mathbb{R} \to \alpha \), \( \rho' : \mathbb{R} \to \kappa \), \( \pi : \mathbb{R} \to \mathbb{R} \) such that \( \rho(x) = \rho(y) \to \rho' \circ \pi(x) = \rho' \circ \pi(y) \).

- \( u_1 = \omega_1, u_2, \ldots, u_n, \ldots \) list the first \( \omega \) uncountable projective wellordered cardinals.
Projective well-ordered cardinals

- $\kappa$ is a cardinal iff for any $\alpha < \kappa$, there is no surjection $\pi : \alpha \to \kappa$.

- $\kappa$ is a projective wellordered cardinal iff for any $\alpha < \kappa$, there are no projective surjections $\rho : \mathbb{R} \to \alpha$, $\rho' : \mathbb{R} \to \kappa$, $\pi : \mathbb{R} \to \mathbb{R}$ such that $\rho(x) = \rho(y) \rightarrow \rho' \circ \pi(x) = \rho' \circ \pi(y)$.

- $u_1 = \omega_1, u_2, \ldots, u_n, \ldots$ list the first $\omega$ uncountable projective wellordered cardinals.

- The “projective successor” of $u_\omega$ is $\delta_3^{\frac{1}{\omega}}$.

- IMT provides a deep insight into the structural theory related to the projective well-ordered cardinals.
The Martin-Solovay tree projecting to $\Pi^1_2$

Theorem (Martin-Solovay)

If $A$ is $\Pi^1_2$, then $A = p[T]$ for some $T$ on $\omega \times u_\omega$ such that $T$ is $\Delta^1_3$ in the sharp codes.
The Martin-Solovay tree projecting to $\Pi^1_2$

Theorem (Martin-Solovay)
If $A$ is $\Pi^1_2$, then $A = p[T]$ for some $T$ on $\omega \times u_\omega$ such that $T$ is $\Delta^1_3$ in the sharp codes.

Definition of the Martin-Solovay tree:
If $\sigma : \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ is an order preserving map, then $\sigma$ induces $j^{\sigma} : u_{m+1} \rightarrow u_{n+1}$ by

$$j^{\sigma} (\tau^{L[x]}(u_1, \ldots, u_m)) = \tau^{L[x]}(u_{\sigma(1)}, \ldots, u_{\sigma(m)}).$$
The Martin-Solovay tree projecting to $\Pi^1_2$

Fix an effective list of Skolem terms $(\tau_n)_{n<\omega}$. Suppose $\tau_n$ is $k_n$-ary.
The Martin-Solovay tree projecting to $\Pi^1_2$

Fix an effective list of Skolem terms $(\tau_n)_{n<\omega}$. Suppose $\tau_n$ is $k_n$-ary.

- Let $T \subseteq 2^{<\omega}$ be a recursive tree such that $[T]$ is the set of remarkable EM blueprints over some real.
The Martin-Solovay tree projecting to $\Pi^1_2$

Fix an effective list of Skolem terms $(\tau_n)_{n<\omega}$. Suppose $\tau_n$ is $k_n$-ary.

- Let $T \subseteq 2^{<\omega}$ be a recursive tree such that $[T]$ is the set of remarkable EM blueprints over some real.
- $(s, \bar{\alpha}) \in T_2$ iff $s \in T$ and for every $m, n$, for every order preserving $\sigma : \{1, \ldots, k_m\} \to \{1, \ldots, k_n\}$, if the formula
  \[
  \tau_m(\dot{c}_{\sigma(1)}, \ldots, \dot{c}_{\sigma(k_m)}) = \tau_n(\dot{c}_1, \ldots, \dot{c}_{k_n})
  \]
  is true in $s$, then $j^\sigma(\alpha_m) = \alpha_n$. Similarly for $<$ and $>$. 
The Martin-Solovay tree projecting to $\nabla^1_2$

Fix an effective list of Skolem terms $(\tau_n)_{n<\omega}$. Suppose $\tau_n$ is $k_n$-ary.

- Let $T \subseteq 2^{<\omega}$ be a recursive tree such that $[T]$ is the set of remarkable EM blueprints over some real.
- $(s, \bar{\alpha}) \in T_2$ iff $s \in T$ and for every $m, n$, for every order preserving $\sigma : \{1, \ldots, k_m\} \rightarrow \{1, \ldots, k_n\}$, if the formula
  \[
  \tau_m(\dot{c}_{\sigma(1)}, \ldots, \dot{c}_{\sigma(k_m)}) = \tau_n(\dot{c}_1, \ldots, \dot{c}_{k_n})
  \]
  is true in $s$, then $j^\sigma(\alpha_m) = \alpha_n$. Similarly for $<$ and $>$.
- (Martin-Solovay) $p[T_2] = \{x^\# : x \in \mathbb{R}\}$. 
The Martin-Solovay tree projecting to $\Pi^1_2$

Fix an effective list of Skolem terms $(\tau_n)_{n<\omega}$. Suppose $\tau_n$ is $k_n$-ary.

- Let $T \subseteq 2^{<\omega}$ be a recursive tree such that $[T]$ is the set of remarkable EM blueprints over some real.

- $(s, \vec{\alpha}) \in T_2$ iff $s \in T$ and for every $m, n$, for every order preserving $\sigma : \{1, \ldots, k_m\} \to \{1, \ldots, k_n\}$, if the formula
  \[
  \tau_m(\dot{c}_{\sigma(1)}, \ldots, \dot{c}_{\sigma(k_m)}) = \tau_n(\dot{c}_1, \ldots, \dot{c}_{k_n})
  \]
  is true in $s$, then $j^\sigma(\alpha_m) = \alpha_n$. Similarly for $<$ and $>$. 

- (Martin-Solovay) $p[T_2] = \{x^\# : x \in \mathbb{R}\}$.

- It is easy to define $\widehat{T}_2$, a variant of $T_2$, such that $p[\widehat{T}_2]$ is a good universal $\Pi^1_2$ set.
\[ \hat{T}_2 \text{ is a tree on } \omega \times u_\omega. \text{ It is } \Delta^1_3 \text{ in the codes. } p[\hat{T}_2] \text{ is a good universal } \Pi^1_2 \text{ set.} \]
\( \hat{T}_2 \) is a tree on \( \omega \times u_\omega \). It is \( \Delta^1_3 \) in the codes. \( p[\hat{T}_2] \) is a good universal \( \Pi^1_2 \) set.

**Definition**

\( y^0_3 \) is the leftmost real associated to \( \hat{T}_2 \).
\( \widehat{T}_2 \) is a tree on \( \omega \times u_\omega \). It is \( \Delta^1_3 \) in the codes. \( p[\widehat{T}_2] \) is a good universal \( \Pi^1_2 \) set.

**Definition**

\( y^0_3 \) is the leftmost real associated to \( \widehat{T}_2 \).

**Theorem**

Assume \( \Delta^1_2 \)-determinacy. Then \( y^0_3 \equiv^m \mathcal{M}_1^\# \) (many-one equivalence).
Effective codes of ordinals (3)

Codes of ordinals up to $\delta_3^1$:

Recall: Under AD, $u_\alpha = \aleph_\alpha$, $\lambda_3 = u_\omega = \aleph_\omega$, $\delta_3^1 = \aleph_{\omega+1}$.

- $\alpha \in WO_3$ iff $\alpha$ codes a wellordering of $u_\omega$ via Kunen’s coding of subsets of $u_\omega$. (under AD, there is a surjection $\pi : \mathbb{R} \to \mathcal{P}(u_\omega)$ such that the relation $\alpha \in \pi(\alpha)$ is $\Delta_3^1$)
Effective codes of ordinals (3)

Codes of ordinals up to $\delta_3^1$:

Recall: Under AD, $u_n = \aleph_n$, $\lambda_3 = u_\omega = \aleph_\omega$, $\delta_3^1 = \aleph_\omega + 1$.

$\quad \triangleright$ $x \in WO(3)$ iff $x$ codes a wellordering of $u_\omega$ via Kunen’s coding of subsets of $u_\omega$. (under AD, there is a surjection $\pi : \mathbb{R} \to \mathcal{P}(u_\omega)$ such that the relation $\alpha \in \pi(x)$ is $\Delta_3^1$)

$\quad \triangleright$ Every ordinal in $(u_\omega, \delta_3^1)$ is coded by a real $x \in WO(3)$. 

$\quad \triangleright$ $WO(3)$ is a $\Pi_1^1$ set.
Effective codes of ordinals (3)

Codes of ordinals up to $\delta^1_3$:

Recall: Under AD, $u_n = \aleph_n$, $\lambda_3 = u_\omega = \aleph_\omega$, $\delta^1_3 = \aleph_\omega + 1$.

$\triangleright$ $x \in \text{WO}_3$ iff $x$ codes a wellordering of $u_\omega$ via Kunen’s coding of subsets of $u_\omega$. (under AD, there is a surjection $\pi : \mathbb{R} \rightarrow \mathcal{P}(u_\omega)$ such that the relation $\alpha \in \pi(x)$ is $\Delta^1_3$)

$\triangleright$ Every ordinal in $(u_\omega, \delta^1_3)$ is coded by a real $x \in \text{WO}_3$.

$\triangleright$ $\text{WO}_3$ is a $\Pi^1_3$ set.
Effective codes of ordinals (3)

Codes of ordinals up to $\delta_3^1$:

Recall: Under AD, $u_n = \aleph_n$, $\lambda_3 = u_\omega = \aleph_\omega$, $\delta_3^1 = \aleph_\omega + 1$.

- $x \in \text{WO}(3)$ iff $x$ codes a wellordering of $u_\omega$ via Kunen’s coding of subsets of $u_\omega$. (under AD, there is a surjection $\pi : \mathbb{R} \rightarrow \mathcal{P}(u_\omega)$ such that the relation $\alpha \in \pi(x)$ is $\Delta^1_3$)
- Every ordinal in $(u_\omega, \delta_3^1)$ is coded by a real $x \in \text{WO}(3)$.
- $\text{WO}(3)$ is a $\Pi^1_3$ set.

How to continue?
Effective codes of ordinals (3)

Codes of ordinals up to $\delta^1_3$:

Recall: Under AD, $u_n = \aleph_n$, $\lambda_3 = u_\omega = \aleph_\omega$, $\delta^1_3 = \aleph_{\omega+1}$.

- $x \in WO(3)$ iff $x$ codes a wellordering of $u_\omega$ via Kunen’s coding of subsets of $u_\omega$. (under AD, there is a surjection $\pi : \mathbb{R} \to \mathcal{P}(u_\omega)$ such that the relation $\alpha \in \pi(x)$ is $\Delta^1_3$)

- Every ordinal in $(u_\omega, \delta^1_3)$ is coded by a real $x \in WO(3)$.
- $WO(3)$ is a $\Pi^1_3$ set.

How to continue?
Effective codes of ordinals (3)

Codes of ordinals up to $\delta^1_3$:

Recall: Under AD, $u_n = \aleph_n$, $\lambda_3 = u_\omega = \aleph_\omega$, $\delta^1_3 = \aleph_\omega + 1$.

\begin{itemize}
  \item $x \in WO(3)$ iff $x$ codes a wellordering of $u_\omega$ via Kunen’s coding of subsets of $u_\omega$. (under AD, there is a surjection $\pi : \mathbb{R} \to \mathcal{P}(u_\omega)$ such that the relation $\alpha \in \pi(x)$ is $\Delta^1_3$)
  \item Every ordinal in $(u_\omega, \delta^1_3)$ is coded by a real $x \in WO(3)$.
  \item $WO(3)$ is a $\Pi^1_3$ set.
\end{itemize}

How to continue?

WO or (higher level) Kunen’s coding
Effective codes of ordinals (3)

Codes of ordinals up to $\delta_3^1$:

Recall: Under AD, $u_n = \aleph_n$, $\lambda_3 = u_\omega = \aleph_\omega$, $\delta_3^1 = \aleph_\omega + 1$.

- $x \in WO(3)$ iff $x$ codes a wellordering of $u_\omega$ via Kunen’s coding of subsets of $u_\omega$. (under AD, there is a surjection $\pi : \mathbb{R} \to \mathcal{P}(u_\omega)$ such that the relation $\alpha \in \pi(x)$ is $\Delta^1_3$)
- Every ordinal in $(u_\omega, \delta_3^1)$ is coded by a real $x \in WO(3)$.
- $WO(3)$ is a $\Pi^1_3$ set.

How to continue?

$WO$ or (higher level) Kunen’s coding (higher level) sharp coding
Projective well-ordered cardinalities

Definition
\[ \delta_n^1 = \sup \{ \text{lh}(W) : W \text{ is a } \Delta_n^1 \text{ prewellordering of } \mathbb{R} \}. \]
Projective well-ordered cardinalities

**Definition**

\[ \delta_n^1 = \sup \{ \text{lh}(W) : W \text{ is a } \Delta_n^1 \text{ prewellordering of } \mathbb{R} \} . \]

Assuming large cardinals,

\[ \omega_1 = \delta_1^1 < \delta_2^1 < \cdots < \sup_{n<\omega} \delta_n^1 < \Theta^{L(\mathbb{R})} . \]
Projective well-ordered cardinalities

Definition
$\delta^1_n = \sup \{ \text{lh}(W) : W \text{ is a } \Delta^1_n \text{ prewellordering of } \mathbb{R} \}$.

Assuming large cardinals,

- $\omega_1 = \delta^1_1 < \delta^1_2 < \cdots < \sup_{n<\omega} \delta^1_n < \Theta^{L(\mathbb{R})}$.
- (Moschovakis) Assume ZF+DC+PD. Then every $\Pi^1_{2n+1}$ set is $\delta^1_{2n+1}$-Suslin.

Assuming large cardinals,

- $(\text{Moschovakis})$ Assume ZF+DC+PD. Then every $\Pi^1_{2n+1}$ set is $\delta^1_{2n+1}$-Suslin.
Projective well-ordered cardinalities

Definition
\[ \delta_n^1 = \sup \{ \text{lh}(W) : W \text{ is a } \Delta_n^1 \text{ prewellordering of } \mathbb{R} \} . \]

Assuming large cardinals,
\begin{itemize}
  \item \( \omega_1 = \delta_1^1 < \delta_2^1 < \cdots < \sup_{n<\omega} \delta_n^1 < \Theta^L(\mathbb{R}) . \)
  \item (Moschovakis) Assume ZF+DC+PD. Then every \( \Pi^{2n+1}_1 \) set is \( \delta_{2n+1}^1 \)-Suslin.
  \item (Jackson) Assume ZF+DC+AD. Then \( \delta_{2n+1}^1 = \kappa_{E(2n)+1} \), where \( E(0) = 0, E(n+1) = \omega^{E(n)} \) in ordinal arithmetic. In particular, \( \delta_3^1 = \kappa_{\omega+1}, \delta_5^1 = \kappa_{\omega\omega+1} . \)
\end{itemize}
Projective well-ordered cardinalities

Definition
\( \delta_n^1 = \sup \{ \text{lh}(W) : W \text{ is a } \Delta_n^1 \text{ prewellordering of } \mathbb{R} \} \).

Assuming large cardinals,

- \( \omega_1 = \delta_1^1 < \delta_2^1 < \cdots < \sup_{n < \omega} \delta_n^1 < \Theta^L(\mathbb{R}) \).
- (Moschovakis) Assume ZF+DC+PD. Then every \( \Pi_{2n+1}^1 \) set is \( \delta_{2n+1}^1 \)-Suslin.
- (Jackson) Assume ZF+DC+AD. Then \( \delta_{2n+1}^1 = \aleph_{E(2n)+1} \), where \( E(0) = 0, E(n+1) = \omega^{E(n)} \) in ordinal arithmetic. In particular, \( \delta_3^1 = \aleph_{\omega+1}, \delta_5^1 = \aleph_{\omega^\omega+1} \).
- (Kechris) Assume ZF+DC+AD. Let \( \lambda_{2n+1} = (\delta_{2n+1}^1)^- \). Then every \( \Pi_{2n}^1 \) set is \( \lambda_{2n+1} \)-Suslin.
The model theoretic form of $\Pi^1_3$ sets

The model theoretic form of $\Pi^1_1$ subsets of $\mathbb{R}$

The following are equivalent:

- $A \subseteq \mathbb{R}$ is $\Pi^1_1$.
- For some $\Sigma^1_1$ formula $\varphi$, $x \in A$ iff $L_{\omega^1_1}[x] \models \varphi(x)$.
The model theoretic form of $\Pi^1_3$ sets

The model theoretic form of $\Pi^1_1$ subsets of $\mathbb{R}$

The following are equivalent:

- $A \subseteq \mathbb{R}$ is $\Pi^1_1$.
- For some $\Sigma_1$ formula $\varphi$, $x \in A$ iff $L_{\omega_1^x}[x] \models \varphi(x)$.

Kleene's $O$: $\langle \varphi \rangle \in \mathcal{O}$ iff $\varphi$ is a $\Sigma_1$ sentence and $L_{\omega_1^{\text{ck}}} \models \varphi$. 
The model theoretic form of $\Pi^1_3$ sets

The model theoretic form of $\Pi^1_1$ subsets of $\mathbb{R}$

The following are equivalent:

- $A \subseteq \mathbb{R}$ is $\Pi^1_1$.
- For some $\Sigma^1_1$ formula $\varphi$, $x \in A$ iff $L_{\omega^*_1}[x] \models \varphi(x)$.

Kleene's $O$: $\models \varphi \in O$ iff $\varphi$ is a $\Sigma^1_1$ sentence and $L_{\omega^1_{CK}} \models \varphi$.

The model theoretic form of $\Pi^1_3$ sets of $u_\omega \times \mathbb{R}$

(Kechris-Martin, Becker-Kechris) The following are equivalent:

- $A \subseteq u_\omega \times \mathbb{R}$ is $\Pi^1_3$.
- For some $\Sigma^1_1$ formula $\varphi$, $x \in A$ iff $L_{\kappa^*_3}[T_2, x] \models \varphi(T_2, \alpha, x)$.
  Here $\kappa^*_3$ is the least $(T_2, x)$-admissible.
The model theoretic form of $\Pi^1_3$ sets

The model theoretic form of $\Pi^1_1$ subsets of $\mathbb{R}$

The following are equivalent:

- $A \subseteq \mathbb{R}$ is $\Pi^1_1$.
- For some $\Sigma^1_1$ formula $\varphi$, $x \in A$ iff $L_{\omega^1_1}[x] \models \varphi(x)$.

Kleene’s $O$: $\Gamma^\varphi \in O$ iff $\varphi$ is a $\Sigma^1_1$ sentence and $L_{\omega^1_{CK}} \models \varphi$.

The model theoretic form of $\Pi^1_3$ sets of $u^\omega \times \mathbb{R}$

(Kechris-Martin, Becker-Kechris) The following are equivalent:

- $A \subseteq u^\omega \times \mathbb{R}$ is $\Pi^1_3$.
- For some $\Sigma^1_1$ formula $\varphi$, $x \in A$ iff $L_{\kappa^x_3}[T_2,x] \models \varphi(T_2,\alpha,x)$. Here $\kappa^x_3$ is the least $(T_2,x)$-admissible.

Higher Kleene’s $O$: $(\Gamma^\varphi, \alpha) \in O^{T_2}$ iff $\varphi$ is a $\Sigma^1_1$ formula with one free variable, $\alpha < u^\omega$ and $L_{\kappa^3_3}[T_2] \models \varphi(\alpha)$. 
The level-2 sharp

\[ O^{T_2} \not\in L_{\kappa_3}[T_2]. \] For each \( n < \omega \), \( O^{T_2} \cap (\omega \times u_n) \in L_{\kappa_3}[T_2] \).
The level-2 sharp

$\mathcal{O}^{T_2} \notin L_{\kappa_3}[T_2]$. For each $n < \omega$, $\mathcal{O}^{T_2} \cap (\omega \times u_n) \in L_{\kappa_3}[T_2]$.

**Definition**

- $(0^2\#)_n = \{ (\varphi, \psi) : \varphi, \psi$ are $\Sigma_1$ formulas, $\exists \alpha < u_n$
  \[ ((\varphi, \alpha) \notin \mathcal{O}^{T_2} \land \forall \eta < \alpha (\psi, \eta) \in \mathcal{O}^{T_2}) \}.$

- $\mathcal{O}^{T_2} \cap (\omega \times u_n)$ is squeezed into the real $(0^2\#)_n$ by applying a Boolean operation on the second coordinate.
The level-2 sharp

$\mathcal{O}^{T_2} \notin L_{\kappa_3}[T_2]$. For each $n < \omega$, $\mathcal{O}^{T_2} \cap (\omega \times u_n) \in L_{\kappa_3}[T_2]$.

Definition

- $(0^{2\#})_n = \{ (\varphi, \psi) : \varphi, \psi$ are $\Sigma_1$ formulas, $\exists \alpha < u_n$
  
  $((\varphi, \alpha) \notin \mathcal{O}^{T_2} \land \forall \eta < \alpha \ (\psi, \eta) \in \mathcal{O}^{T_2}) \}.$

- $\mathcal{O}^{T_2} \cap (\omega \times u_n)$ is squeezed into the real $(0^{2\#})_n$ by applying a Boolean operation on the second coordinate.

- $0^{2\#} = \bigoplus_{n<\omega} (0^{2\#})_n.$
The level-2 sharp

$\cal{O}^{T_2} \notin L_{\kappa_3}[T_2]$. For each $n < \omega$, $\cal{O}^{T_2} \cap (\omega \times u_n) \in L_{\kappa_3}[T_2]$.

Definition

- $(0^2\#)_n = \{(\varphi, \psi) : \varphi, \psi \text{ are } \Sigma_1 \text{ formulas, } \exists \alpha < u_n$ \)
  \[
  ((\varphi, \alpha) \notin \cal{O}^{T_2} \land \forall \eta < \alpha (\psi, \eta) \in \cal{O}^{T_2})\}.
\]
- $\cal{O}^{T_2} \cap (\omega \times u_n)$ is squeezed into the real $(0^2\#)_n$ by applying a Boolean operation on the second coordinate.
- $0^2\# = \bigoplus_{n < \omega} (0^2\#)_n$.
- For $x \in \mathbb{R}$, $x^2\#$ is the relativized version.
The level-2 sharp

\(\mathcal{O}^{T_2} \notin L_{\kappa_3}[T_2]\). For each \(n < \omega\), \(\mathcal{O}^{T_2} \cap (\omega \times u_n) \in L_{\kappa_3}[T_2]\).

Definition

\(\vdash (0^{2\#})_n = \{(\varphi, \psi) : \varphi, \psi \text{ are } \Sigma_1 \text{ formulas, } \exists \alpha < u_n ((\varphi, \alpha) \notin \mathcal{O}^{T_2} \land \forall \eta < \alpha (\psi, \eta) \in \mathcal{O}^{T_2})\}\).

\(\vdash \mathcal{O}^{T_2} \cap (\omega \times u_n) \text{ is squeezed into the real } (0^{2\#})_n \text{ by applying a Boolean operation on the second coordinate.}\)

\(\vdash 0^{2\#} = \bigoplus_{n<\omega} (0^{2\#})_n\).

\(\vdash \text{For } x \in \mathbb{R}, x^{2\#} \text{ is the relativized version.}\)

By the previous slide, \(0^{2\#} \equiv_m y_3^0\). It is the model theoretic definition of \(y_3^0\).

Theorem

\(0^{2\#} \equiv_m M^\#_1\). \(x^{2\#} \equiv_m M^\#_1(x)\), the many-one reductions independent of \(x\).
$\Pi^1_3$ sets, the structure of $L[T_3]$

**Definition**

$T_3$ is the tree on $\omega \times \delta^1_3$ associated to a $\Pi^1_3$-scale on a good universal $\Pi^1_3$ set.

**The fine structure of $L[T_3]$**

1. (Becker-Kechris) $L[T_3]$ is invariant of the choice of $T_3$. 
\( \Pi^1_3 \) sets, the structure of \( L[T_3] \)

**Definition**

\( T_3 \) is the tree on \( \omega \times \delta^1_3 \) associated to a \( \Pi^1_3 \)-scale on a good universal \( \Pi^1_3 \) set.

**The fine structure of \( L[T_3] \)**

1. (Becker-Kechris) \( L[T_3] \) is invariant of the choice of \( T_3 \).
2. (Steel) \( L[T_3] \) is a mouse. Let \( j : M^\#_2 \to M^\#_{2,\infty} \) be the direct limit map of all countable iterates of \( M^\#_2 \), \( \delta_\infty \) be the least Woodin of \( M^\#_{2,\infty} \), then
   - \( \delta^1_3 \) is the least \( < \delta_\infty \)-strong in \( M^\#_{2,\infty} \), and
   - \( L[T_3] = L[M^\#_{2,\infty} | \delta^1_3] \).
Π₁³ sets, the structure of $L[T₃]$

Definition
$T₃$ is the tree on $ω × δ₁³$ associated to a Π₁³-scale on a good universal Π₁³ set.

The fine structure of $L[T₃]$

1. (Becker-Kechris) $L[T₃]$ is invariant of the choice of $T₃$.
2. (Steel) $L[T₃]$ is a mouse. Let $j : M₂^# → M₂,∞^#$ be the direct limit map of all countable iterates of $M₂^#$, $δ_∞$ be the least Woodin of $M₂,∞^#$, then
   - $δ₃^1$ is the least $< δ_∞$-strong in $M₂,∞^#$, and

Intuition: $(L[T₃],$ level-$3$ indiscernibles) $∼ M₂^#$. 
Level-3 EM blueprint

The role of level-3 EM blueprints

The (level-1) EM blueprint definition of sharps leads to

- Effective wellordered cardinals \((u_n : n \leq \omega), u_n = \kappa_n^{L(R)}\),
- Effective \((\Delta^1_3)\) coding system of ordinals in \(u_\omega\),
- The Martin-Solovay tree \(T_2\) projecting to \(\Pi^1_2\),
- The model theoretic form of \(\Pi^1_3\) subsets of \(u_\omega \times \mathbb{R}\),
- \(0^{\#} \equiv_m M^\#_1\).
Level-3 EM blueprint

The role of level-3 EM blueprints

The (level-1) EM blueprint definition of sharps leads to

- Effective wellordered cardinals ($u_n : n \leq \omega$), $u_n = \kappa_n^{L(\mathbb{R})}$,
- Effective ($\Delta^1_3$) coding system of ordinals in $u_\omega$,
- The Martin-Solovay tree $T_2$ projecting to $\Pi^1_2$,
- The model theoretic form of $\Pi^1_3$ subsets of $u_\omega \times \mathbb{R}$,
- $0^2# \equiv_m M_1^#$.

The level-3 EM blueprint definition of level-3 sharps leads to

- More effective wellordered cardinals ($u_{\xi;3} : \xi \leq \omega^{\omega\omega}$),
  $u_{\xi;3} = \kappa_{\omega+1+\xi}^{L(\mathbb{R})}$,
- Effective ($\Delta^1_5$) coding system of ordinals in $u_{\omega\omega\omega;3}$,
- The level-4 Martin-Solovay tree $T_4$ projecting to $\Pi^1_4$,
- The model theoretic form of $\Pi^1_5$ subsets of $u_{\omega\omega\omega;3} \times \mathbb{R}$,
- $0^4# \equiv_m M_3^#$. 
Level-3 indiscernibles

What does a level-3 EM blueprint $\Gamma$ look like?

A level-3 tree of uniform cofinalities is a finite function on $V_\omega$ with some combinatorial properties.
Level-3 indiscernibles

What does a level-3 EM blueprint $\Gamma$ look like?

A level-3 tree of uniform cofinalities is a finite function on $V_\omega$ with some combinatorial properties. $\Gamma$ is a function on the set of level-3 trees. For each level-3 tree $R$, $\Gamma(R)$ is intended to be the theory of $L_{\delta^1_3}[T_3]$ with level-3 indiscernibles of type $R$. 
Level-3 indiscernibles

What does a level-3 EM blueprint $\Gamma$ look like?

A level-3 tree of uniform cofinalities is a finite function on $V_\omega$ with some combinatorial properties. $\Gamma$ is a function on the set of level-3 trees. For each level-3 tree $R$, $\Gamma(R)$ is intended to be the theory of $L_{\delta_3^1}[T_3]$ with level-3 indiscernibles of type $R$.

- $\mathcal{L}^R$ is the language $\{\dot{\in}, \dot{c}_s : s \in \text{dom}(R)\}$. 
Level-3 indiscernibles

What does a level-3 EM blueprint \( \Gamma \) look like?

A level-3 tree of uniform cofinalities is a finite function on \( V_\omega \) with some combinatorial properties. \( \Gamma \) is a function on the set of level-3 trees. For each level-3 tree \( R \), \( \Gamma(R) \) is intended to be the theory of \( L_{\delta_3}^1[T_3] \) with level-3 indiscernibles of type \( R \).

- \( \mathcal{L}^R \) is the language \( \{ \dot{\varepsilon}, \dot{c}_s : s \in \text{dom}(R) \} \).

- \( \Gamma(R) \) is a complete consistent \( \mathcal{L}^R \) theory containing the following axioms:
  - \( ZFC + V = K + \text{“I am a 2-small mouse”} \).
  - (\( \varphi_\Sigma, \varphi_\Pi \) are fixed formulas) \( \varphi_\Sigma, \varphi_\Pi \) define class trees \( T_\Sigma, T_\Pi \) resp. In any set generic extension, \( \check{T}_\Sigma, \check{T}_\Pi \) project to the good universal \( \Sigma_3^1 \) set and its complement.
  - \( V \) is closed under the \( M_1^# \) operator, as certified by \( T_\Pi \).
  - \( \ldots \) (The order relation of those \( c_s \)'s)…

\( \mathcal{M}(\Gamma, R) \) is (up to isomorphism) the unique model of \( \Gamma(R) \) that is Skolem generated by \( (\dot{c}_s^{\mathcal{M}(\Gamma, R)} : s \in \mathbb{R}) \). \( \mathcal{M}(\Gamma, R) \) is called the level-3 EM model associated to \( \Gamma \) and \( R \).
What does a level-3 EM blueprint $\Gamma$ look like?

- $\Gamma$ is coherent. In particular, if $R$ is a subfunction of $R'$, then $\Gamma(R) \subseteq \Gamma(R')$, so that $\mathcal{M}(\Gamma, R)$ elementarily embeds into $\mathcal{M}(\Gamma, R')$ in a unique way.
What does a level-3 EM blueprint $\Gamma$ look like?

- $\Gamma$ is coherent. In particular, if $R$ is a subfunction of $R'$, then $\Gamma(R) \subseteq \Gamma(R')$, so that $\mathcal{M}(\Gamma, R)$ elementarily embeds into $\mathcal{M}(\Gamma, R')$ in a unique way.

- Remarkability. Generalization of the level-1 remarkability axiom.
Level-3 EM blueprint

What does a level-3 EM blueprint $\Gamma$ look like?

- **Coherence.** In particular, if $R$ is a subfunction of $R'$, then $\Gamma(R) \subseteq \Gamma(R')$, so that $\mathcal{M}(\Gamma, R)$ elementarily embeds into $\mathcal{M}(\Gamma, R')$ in a unique way.

- **Remarkability.** Generalization of the level-1 remarkability axiom.

- **Iterability.** If a tower of level-3 trees $(R_n : n < \omega)$ is $\Pi^1_3$-wellfounded, then the direct limit of $(\mathcal{M}(\Gamma, R_n) : n < \omega)$ is a $\Pi^1_3$-iterable mouse.

**Iterability is $\Pi^1_4$.**

**Theorem**

*Assume $\Pi^1_3$-determinacy. Then there is a unique iterable remarkable level-3 EM blueprint.*
The level-3 sharp

**Definition**
Assume $\Pi^1_3$-determinacy.

- $0^3#$ is the unique level-3 EM blueprint.
- For $x \in \mathbb{R}$, $x^3#$ is the unique level-3 EM blueprint over $x$. 
The level-3 sharp

Definition
Assume $\Pi^1_3$-determinacy.
- $0^3#$ is the unique level-3 EM blueprint.
- For $x \in \mathbb{R}$, $x^3#$ is the unique level-3 EM blueprint over $x$.

Theorem
Assume $\Pi^1_3$-determinacy. Then $x^3# \equiv_m M^#_2(x)$, the many-one reductions independent of $x$. 
The level-3 sharp

Definition
Assume $\Pi^1_3$-determinacy.

- $0^3#$ is the unique level-3 EM blueprint.
- For $x \in \mathbb{R}$, $x^3#$ is the unique level-3 EM blueprint over $x$.

Theorem
Assume $\Pi^1_3$-determinacy. Then $x^3# \equiv_m M_2^#(x)$, the many-one reductions independent of $x$.

The proof uses the equivalence $x^2# \equiv_m M_1^#(x)$. 
Level-3 EM blueprint

The role of level-3 EM blueprints

The level-3 EM blueprint definition of level-3 sharps leads to

- More effective wellordered cardinals ($u_{\xi;3} : \xi \leq \omega^{\omega\omega}$),

$$u_{\xi;3} = \aleph_{L(\mathbb{R})}^{\omega+1+\xi} : \text{level-3 uniform indiscernibles}$$
Level-3 EM blueprint

The role of level-3 EM blueprints

The level-3 EM blueprint definition of level-3 sharps leads to

- More effective wellordered cardinals ($u_{\xi;3} : \xi \leq \omega^{\omega^\omega}$),
  
  $u_{\xi;3} = \aleph^L(\mathbb{R})_{\omega+1+\xi}$: level-3 uniform indiscernibles

- Effective ($\Delta^1_5$) coding system of ordinals in $u_{\omega^{\omega^\omega};3}$: every ordinal in $u_{\omega^{\omega^\omega};3}$ is coded by a triple ($R, \tau^\downarrow, x^3#$), $R$ a level-3 tree, $\tau$ a Skolem term, $x \in \mathbb{R}$.
Level-3 EM blueprint

The role of level-3 EM blueprints

The level-3 EM blueprint definition of level-3 sharps leads to

- More effective wellordered cardinals ($u_{\xi;3} : \xi \leq \omega^{\omega^{\omega}}$),
  \[ u_{\xi;3} = \kappa^{L(R)}_{\omega+1+\xi} : \text{level-3 uniform indiscernibles} \]

- Effective ($\Delta^1_5$) coding system of ordinals in $u_{\omega^{\omega^{\omega}};3}$: every ordinal in $u_{\omega^{\omega^{\omega}};3}$ is coded by a triple $(R, \tau^\frown, x^3\#)$, $R$ a level-3 tree, $\tau$ a Skolem term, $x \in R$.

- The level-4 Martin-Solovay tree $T_4$ projecting to $\Pi^1_4$. 
Level-3 EM blueprint

The role of level-3 EM blueprints

The level-3 EM blueprint definition of level-3 sharps leads to

- More effective wellordered cardinals \( (u_{\xi;3} : \xi \leq \omega^\omega) \),
  \[ u_{\xi;3} = \kappa^{L(\mathbb{R})}_{\omega+1+\xi} : \text{level-3 uniform indiscernibles} \]
- Effective \( (\Delta^1_5) \) coding system of ordinals in \( u_{\omega^\omega;3} \): every ordinal in \( u_{\omega^\omega;3} \) is coded by a triple \( (R, [\tau], x^3\#) \), \( R \) a level-3 tree, \( \tau \) a Skolem term, \( x \in \mathbb{R} \).
- The level-4 Martin-Solovay tree \( T_4 \) projecting to \( \Pi^1_4 \).
- The model theoretic form of \( \Pi^1_5 \) subsets of \( u_{\omega^\omega;3} \times \mathbb{R} \),
- \[ 0^4\# \equiv_m M^\#_3. \]

WO or (higher level) Kunen’s coding (higher level) sharp coding
The model theoretic form of \( \Pi^1_5 \) sets

The model theoretic form of \( \Pi^1_5 \) subsets of \( u_{\omega\omega;3} \times \mathbb{R} \)

The following are equivalent:

1. \( A \subseteq u_{\omega\omega;3} \times \mathbb{R} \) is \( \Pi^1_5 \).

2. For some \( \Sigma_1 \) formula \( \varphi \), \( x \in A \) iff \( L_{\kappa_5^x}[T_4, x] \models \varphi(T_4, \alpha, x) \).
The model theoretic form of $\Pi^1_5$ subsets of $u_{\omega\omega} \times \mathbb{R}$

The following are equivalent:

1. $A \subseteq u_{\omega\omega} \times \mathbb{R}$ is $\Pi^1_5$.

2. For some $\Sigma_1$ formula $\varphi$, $x \in A$ iff $L_{\kappa_5}[T_4, x] = \varphi(T_4, \alpha, x)$.

Level-4 Kleene’s $O$: $(\neg \varphi, \alpha) \in O^{T_4}$ iff $\varphi$ is a $\Sigma_1$ formula with one free variable, $\alpha < u_{\omega\omega}$ and $L_{\kappa_5}[T_4] = \varphi(\alpha)$. 
The model theoretic form of $\Pi^1_5$ sets

The model theoretic form of $\Pi^1_5$ subsets of $u_{\omega^\omega;3} \times \mathbb{R}$

The following are equivalent:

1. $A \subseteq u_{\omega^\omega;3} \times \mathbb{R}$ is $\Pi^1_5$.

2. For some $\Sigma_1$ formula $\varphi$, $x \in A$ iff $L_{\kappa^5_5}[T_4, x] \models \varphi(T_4, \alpha, x)$.

Level-4 Kleene’s O: $(\overline{\varphi}, \alpha) \in O^{T_4}$ iff $\varphi$ is a $\Sigma_1$ formula with one free variable, $\alpha < u_{\omega^\omega;3}$ and $L_{\kappa^5_5}[T_4] \models \varphi(\alpha)$.

$0^4#$ is obtained by squeezing $O^{T_4}$ into a real. It is many-one equivalent to $M_3^#$. 
0# \equiv_m M_1^{\#} \quad 0^3# \equiv_m M_2^{\#} \quad 0^4# \equiv_m M_3^{\#} \quad 0^5# \equiv_m M_4^{\#}

Question

1. How to continue this analysis in $L(\mathbb{R})$ and beyond? I.e., an effective coding of ordinals in $\Theta^{L(\mathbb{R})}$, etc.

2. Is there a purely inner model theoretic approach? Test question: What is the fine structure of $L_{\kappa_3}[T_2]$?
Theory of the mouse $L_{\delta_{2n+1}^1 \left[ T_{2n+1} \right]}$ with level-$(2n + 1)$ indiscernibles

$0^\# \equiv_m M_1^\#$

$0^3^\# \equiv_m M_2^\#$

$0^5^\# \equiv_m M_4^\#$

$0^2^\# \equiv_m M_1^\#$

$0^4^\# \equiv_m M_3^\#$

Question

1. How to continue this analysis in $L(\mathbb{R})$ and beyond? I.e., an effective coding of ordinals in $\Theta^{L(\mathbb{R})}$, etc.

2. Is there a purely inner model theoretic approach? Test question: What is the fine structure of $L_{\kappa_3}[T_2]$?
Continue!

Theory of the mouse $L_{\delta_{2n+1}^1}[T_{2n+1}]$ with level-$(2n + 1)$ indiscernibles

0#

$0^3# \equiv_m M_2^#$

$0^5# \equiv_m M_4^#$

0#

$0^2# \equiv_m M_1^#$

$0^4# \equiv_m M_3^#$

Truth values over $L_{\kappa_{2n+1}}[T_{2n}]$, the least admissible set over $T_{2n}$

Question

1. How to continue this analysis in $L(\mathbb{R})$ and beyond? I.e., an effective coding of ordinals in $\Theta^{L(\mathbb{R})}$, etc.

2. Is there a purely inner model theoretic approach? Test question: What is the fine structure of $L_{\kappa_3}[T_2]$?
Applications

Theorem
The following are equivalent.

1. $\Pi^1_{2n+1}$ and $\Pi^1_{2n+2}$ sets are determined.

2. There is a countably iterable $M^\#_{2n+1}$.

The $n = 0$ case is proved by Neeman and Woodin. The equivalence of $0^{(n+1)}#$ and $M^\#_n$ allows the full generalization. (Open for $n > 0$) What if we replace $(2n + 1, 2n + 2)$ by $(2n, 2n + 1)$?

The big hope
Every theorem in DST at the level of $\Sigma^1_1$ under the assumption of sharps will generalize to projective sets under PD. E.g., the classification of thin projective equivalence relations on $\mathbb{R}$. 
The first half of “the higher sharp” will be available on arXiv by the end of 2015.
The first half of “the higher sharp” will be available on arXiv by the end of 2015.

Thank you!