A Panoramic View of Inner Model Theory for not so Small/Large Cardinals

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I would like to thank Menachem Magidor for suggesting this great title.

The First Inner Model

Gödel 1938: The Constructible Universe $\mathbf{L}$.
Cumulative Hierarchy L_α:

$$
\begin{aligned}
L_0 &= \varnothing \\
L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\
L_\alpha &= \bigcup_{\bar\alpha<\alpha} L_{\bar\alpha} \quad \text{limit } \alpha
\end{aligned}
$$

$$\mathbf{L} = L_\infty = \bigcup_{\alpha<\infty} L_\alpha.$$

Jensen hierarchy J_α:

$$J_{\alpha+1} = \mathrm{rud}(J_\alpha).$$

Rudimentary Functions

$$\begin{aligned}
F_0(x,y) &= \in \cap (x \times x) \\
F_1(x,y) &= x \setminus y \\
F_2(x,y) &= \{x,y\} \\
F_3(x,y) &= x \times y \\
F_4(x,y) &= \bigcup x \\
F_5(x,y) &= \mathrm{rng}(x) \\
F_6(x,y) &= \{\langle u,v,w\rangle \mid \langle u,v\rangle \in x \;\&\; w \in y\} \\
F_7(x,y) &= \{\langle v,w,v\rangle \mid \langle u,v\rangle \in x \;\&\; w \in y\} \\
F_8(x,y) &= \{x''z \mid z\rangle \in y\}
\end{aligned}$$

$$\langle x,y,z\rangle = \langle x,\langle y,z\rangle\rangle$$

$\text{rud}(x)$ = the closure of $x \cup \{x\}$ under rudimentary functions

RELATIVIZATION:

$$F_9(x, y) = A \cap x$$

$\text{rud}_A(x)$ = the closure of $x \cup \{x\}$ under A-rudimentary functions

$$J^A_{\alpha+1} = \text{rud}_A(J^A_\alpha)$$

At limit steps, the unions of relativized hierarchy are model theoretic: We take the union of the predicates of structures J^A_α.

$$\mathbf{L}[A] = J^A_\infty = \bigcup_{\alpha<\infty} J^A_\alpha$$

CAUTION: In general, $A \notin \mathbf{L}[A]$. In fact, it may happen that $\mathbf{L}[A] = L$ for some $A \not\subseteq \mathbf{L}$.
However, if $A \subseteq \mathbf{On}$ is a set then $A \in L[A]$.

Basic Properties of the J-hierarchy

(a) There is a uniformly Σ_1-definable partial surjection $f : [\alpha]^{<\omega} \to J_\alpha$.

(b) There is a uniformly Σ_1-definable well-ordering $<_\alpha$ of J_α.

(c) (b) implies the AC in **L**, in fact the Σ_1-definable global form

(d) Every $a \in \mathcal{P}(\alpha) \cap L$ is an element of some J_β where $\beta < \alpha^{+\mathbf{L}}$

(e) (d) implies GCH in **L**.

(f) There is a uniformly Σ_1-definable Skolem function $h : \omega \times J_\alpha \to J_\alpha$.

(g) (Condensation) If $\sigma : M \to J_\alpha$ is a Σ_1-preserving map where M is a transitive structure then $M = J_{\bar{\alpha}}$ for some $\bar{\alpha} \leq \alpha$.

"Uniformly Σ_1-definable" means: There is a single Σ_1-formula which defines f over J_α without parameters, and this formula works for **every** α.

Basic Properties of the J^A-hierarchy

(a) There is a uniformly Σ_1-definable partial surjection $f : [\alpha]^{<\omega} \to J^A_\alpha$.

(b) There is a uniformly Σ_1-definable well-ordering $<^A_\alpha$ of J^A_α.

(c) (b) implies the AC in $\mathbf{L}[A]$, in fact the Σ_1-definable global form

(d) Not in general.

(e) Not in general.

(f) There is a uniformly Σ_1-definable Skolem function $h : \omega \times J^A_\alpha \to J^A_\alpha$.

(g) (Condensation) If $\sigma : M \to J^A_\alpha$ is a Σ_1-preserving map where M is transitive then $M = J^{\bar{A}}_{\bar{\alpha}}$ for some $\bar{A}$ and $\bar{\alpha} \le \alpha$.

<u>DEFINABILITY IN J^A-HIERARCHY</u> is in the language with additional unary predicate symbol $\dot{A}$ denoting A. Under J^A_α we mean the structure

$$(|J^A_\alpha|, A \cap |J^A_\alpha|)$$

where $|J^A_\alpha|$ is the domain of the structure and the symbol $\dot{A}$ is interpreted as $A \cap |J^A_\alpha|$.

- The structure J^A_α is **amenable**, that is, $A \cap x \in J^A_\alpha$ whenever $x \in J^A_\alpha$. Due to amenability we get (a),(b) and (f).
- A new feature in comparison with the J-hierarchy: Under condensation, the symbol $\dot{A}$ may get a new interpretation. That is,
$$\dot{A}^M = \bar{A}.$$
- $\dot{A}^{\mathbf{L}[A]} = A \cap \mathbf{L}[A]$.
- So $A \cap \mathbf{L}[A] \in A$ and again $A \in \mathbf{L}[A]$ if $A \subseteq \mathbf{On}$ is a set.
- In general, $\mathbf{L}[A] = \mathbf{L}[A \cap \mathbf{L}[A]]$.

The model $\mathbf{L}$ has rich combinatorics (diamonds, squares, morasses), on the other hand

- the internals structure of $\mathbf{L}$ is too rigid,
- $\mathbf{L}$ is inconsistent with most forcing axioms and large cardinals,
- forcing gives us a much more flexible way of constructing models of set theory and do relative consistency arguments.

Question

So why do we care about $\mathbf{L}$*?*

An argument in favor of $\mathbf{L}$: By relating $\mathbf{L}$ to $\mathbf{V}$ we can obtain "absolute" results.

Example 1: Perfect Set Property (PSP).

Theorem (Gödel)

Assume $\mathbf{V} = \mathbf{L}$. *Then there is a* Π^1_1*-set without* PSP.

Proof.

This set in picks one code in $\mathbb{R}$ for each countable ordinal. □

Corollary (of the proof, Relativized Version)

If $\mathbf{V} = \mathbf{L}[x]$ *then there is a* $\Pi^1_1(x)$*-set of reals without* PSP.

In the early years of set theory, people tried to prove the Continuum Hypothesis by considering sets of low complexity and gradually generalizing the proof for more complex sets. Classical result along these lines is:

Theorem (Cantor-Bendixson)

The perfect set property holds for closed sets.

The Cantor-Bendixson theorem can be generalized to analytic sets. Recall: analytic sets are precisely those set which are $\Sigma^1_1(x)$ for some real x.

Theorem

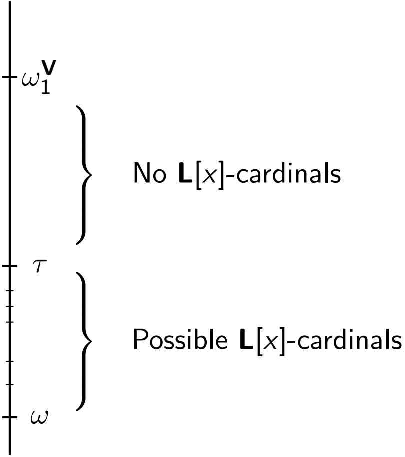
The perfect set property holds for analytic sets.

Gödel's theorem shows that PSP cannot be proved in ZFC and in fact, attempts at proving the PSP hit a wall already at the coanalytic stage. By bringing metamathematics into play, his theorem give us more.

- Assume $x \in \mathbb{R}$ and ω_1^V is a cardinal successor in $\mathbf{L}[x]$. Say $\omega_1^V = \tau^{+\mathbf{L}[x]}$ where $\tau < \omega_1^V$.
- In $\mathbf{V}$ find a surjection $f : \omega \to \tau$. Then

$$a_x = \{2^i \cdot 3^j \mid i, j \in \omega \text{ and } f(i) \in f(j)\}$$

  is a subset of ω which codes the same information as f.
- So $\omega_1^{\mathbf{L}[y]} = \omega_1^{\mathbf{V}}$ where $y = x \oplus a_x$.
- By Gödel's theorem and absoluteness of Π^1_1, in $\mathbf{V}$ there is a $\Pi^1_1(y)$-set of reals without the PSP.


$\omega_1^{\mathbf{V}}$
No $\mathbf{L}[x]$-cardinals
τ
Possible $\mathbf{L}[x]$-cardinals
ω

$\mathbf{L}[x]$

Corollary

If all coanalytic sets have the PSP *then for every $x \in \mathbb{R}$, the ordinal $\omega_1^{\mathbf{V}}$ is a limit cardinal in $\mathbf{L}[x]$.*

But $\omega_1^{\mathbf{V}}$ is regular in $\mathbf{V}$, and hence is regular in $\mathbf{L}[x]$; moreover GCH holds in $\mathbf{L}[x]$ whenever $x \in \mathbb{R}$.

Recall: A cardinal κ is **strongly inaccessible** iff κ is regular and strong limit; the latter means: $\mu < \kappa \implies 2^\mu < \kappa$. So:

- If all coanalytic sets have the PSP then for every $x \in \mathbb{R}$, the ordinal $\omega_1^{\mathbf{V}}$ is a strongly inaccessible cardinal in $\mathbf{L}[x]$.

In particular:

$$\mathsf{Con}(\mathsf{ZFC} + \mathsf{PSP}) \implies \mathsf{Con}(\mathsf{ZFC} + \mathsf{I})$$

where "I" stands for "There is a strongly inaccessible cardinal". By Solovay, the converse also holds, wo we have an equiconsistency.

Measurable Cardinals

Recall: A **normal measure** on a cardinal κ is an ultrafilter U on κ which is

- **uniform**, that is, every co-bounded subset of κ is in U, and
- **normal**, that is, every regressive function $f : A \to \kappa$ such that $A \in U$ is constant on some set in U.

Here "regressive" means that $f(\xi) < \xi$ for all $\xi \in A$ such that $\xi \neq 0$.

A measurable κ is strongly inaccessible (and more), and any normal measure is κ-**complete**, that is, the intersection of less than κ many sets in U is again in U.

M-ULTRAFILTERS

Given is a transitive structure M and an ordinal $\kappa \in M$.

- A filter U on κ is an M-**ultrafilter**, or an **ultrafilter over** M iff U is an ultrafilter on the Boolean algebra $\mathcal{P}(\kappa) \cap M$.
- An M-ultrafilter U is **normal/κ-complete** iff U is normal/κ-complete with respect to functions in M.
- An M-ultrafilter is **countably complete** iff for every sequence $\langle x_n \mid n \in \omega \rangle \in \mathbf{V}$ such that each $x_n \in U$: $\bigcap_{n\in\omega} x_n \neq \varnothing$.

A normal uniform M-ultrafilter is called a **measure** over M.

If a measure over M is on κ then κ is regular, but in general may not have any large cardinal properties, even in the sense of M.

The term "countably complete" always refers to $\mathbf{V}$, so if $M \neq \mathbf{V}$ then "countably complete" and "ω_1-complete" may have different meanings.

Ultrapower Construction

M-ultrafilters can be used to build M-**ultrapowers**. An M-ultrapower is built in the same way as usual ultrapowers with the only difference that the functions are taken from the structure M. The M-ultrapower by U is denoted by $\mathrm{Ult}(M, U)$. So: Given functions $f, g \in M$ with $\mathrm{dom}(f) = \kappa = \mathrm{dom}(g)$, we define

- $f \sim_U g$ iff $\{\xi < \kappa \mid f(\xi) = g(\xi)\} \in U$.

Then $\sim_U$ is an equivalence relation on $M^\kappa \cap M$, and we let $[f]_U$ be the equivalence class of f with respect to $\sim_U$. We can then define

- $[f]_U\, E_U [g]_U$ iff $\{\xi < \kappa \mid f(\xi) \in g(\xi)\} \in U$,
- $[f]_U \in A_U$ iff $\{\xi < \kappa \mid f(\xi) \in A\} \in U$ whenever A is a predicate of M. Notice that this can only be defined if the set on the RHS is in M, as U measures only those sets.

This determines the ultrapower $\mathrm{Ult}(M, U)$.

We get the Łoś theorem: For any Σ_0-formula $\varphi(v_1, \ldots, v_\ell)$ and any $f_1, \ldots, f_\ell$, TFAE:

- $\mathrm{Ult}(M, U) \models \varphi([f_1]_U, \ldots, [f_\ell]_U)$
- $\{\xi < \kappa \mid M \models \varphi(f_1(\xi), \ldots, f_\ell(\xi))\} \in U$.

The "membership relation" E_U is extensional (by Łoś theorem) and set-like (easy counting). Thus, if the E_U is well-founded then $\mathrm{Ult}(M, E)$ can be transitivised and in this case E_U is isomorphically mapped to true membership relation.

If $\mathrm{Ult}(M, U)$ is well-founded we automatically consider it transitive. Otherwise there is a longest initial segment w.r.t. E_U which is well-founded. This initial segment is called the **well-founded part** of $\mathrm{Ult}(M, U)$ and we always consider this on transitive.

The ultrapower embedding/map $i_U : M \to \mathrm{Ult}(M, U)$ is defined by

$$a \mapsto [c_a]_U$$

where c_a is the constant function with value a.

- i_U is Σ_0-preserving and cofinal, hence is Σ_1-preserving.
- If M has good closure properties, i_U may have higher preservation degree. For instance, if $M \models \mathrm{ZFC}^-$ then i_U is fully elementary.

The κ-completeness of U yields

- κ is the first ordinal moved by i_U; this ordinal is called the **critical point** of i_U, denoted by $\mathrm{cr}(i_U)$.
- $i_U(\kappa) > \kappa$.

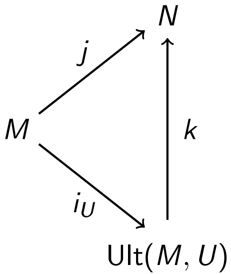
The reverse procedure

Given a Σ_0-preserving map $j : M \to N$ where M, N are transitive such that $j \notin \mathrm{id}$ one can show that j must move an ordinal. The least such ordinal κ is the critical point of j. We can then define the **measure derived from** j on κ by letting:

$$x \in U_j \iff \kappa \in j(x)$$

whenever $x \subseteq \kappa$.

- U_j is a normal uniform M-ultrafilter, hence a measure over M.
- $\mathrm{Ult}(M, U)$ is well-founded because we have the following commutative diagram where all maps are Σ_0-preserving.



The map k is defined by

- $k : [f]_U \mapsto j(f)(\kappa)$

So the following are equivalent.

- κ is measurable.
- There is an elementary embedding $j : \mathbf{V} \to N$ with critical point κ.

Back to $\mathbf{L}$

- There is a Σ_1-sentence $\sigma^{\mathbf{L}}$ such that for every transitive structure N,

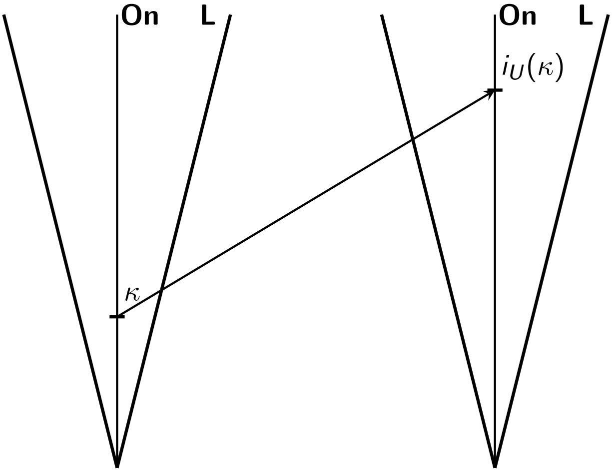
$$N \models \sigma^{L} \iff N = J_\alpha \text{ for some } \alpha \in \mathbf{On}$$

So if $j : \mathbf{L} \to N$ is a Σ_1-preserving map then $N = \mathbf{L}$. This gives

Theorem (Scott)

There are no measurable cardinals in $\mathbf{L}$.

Proof.
If κ were the least measurable cardinal in $\mathbf{L}$ then, picking some normal measue U on κ and using elementarity, $i_U(\kappa) > \kappa$ would be the least measurable cardinal in $\mathbf{L}$. □


On
L
On
L
$i_U(\kappa)$
κ

However, this does not rule out the existence of a normal measure over $\mathbf{L}$ which is not an element of $\mathbf{L}$!

If κ is measurable and U^* is a normal measure on κ in $\mathbf{V}$ then

$$U = U^* \cap \mathbf{L}$$

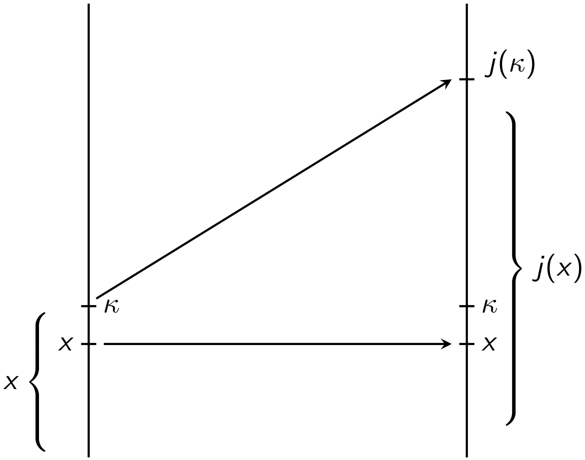
is a normal measure on κ over $\mathbf{L}$.

In general, if $j : M \to N$ and $\mathrm{cr}(j) = \kappa$ then

$$\mathcal{P}(\kappa) \cap M \subseteq \mathcal{P}(\kappa) \cap N,$$

as for every $x \in \mathcal{P}(\kappa) \cap M$ we have

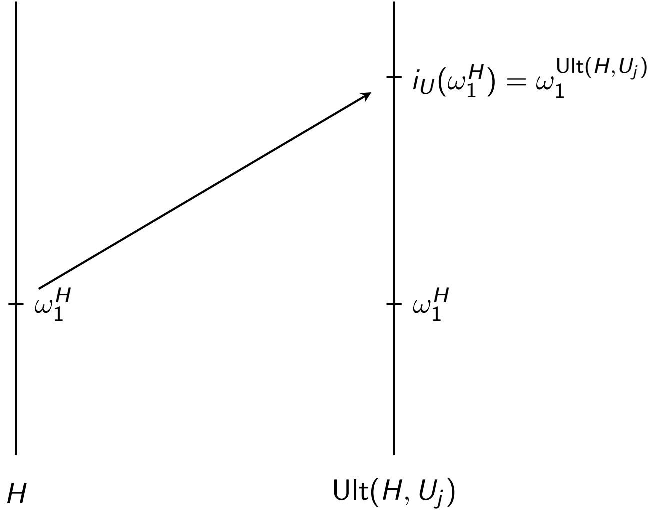
$$x = j(x) \cap \kappa \in N.$$


$j(\kappa)$
$j(x)$
κ
κ
x
x
x

Extreme case

Ult(M, U) may add many subsets of the critical point.

- H_θ is the collection of all sets of cardinality hereditarily $< \theta$.
- $H_\theta \models \mathrm{ZFC}^-$ whenever θ is regular.
- Let X be a countable elementary substructure of H_θ and H be the transitive collapse of X.
- $j : H \to H_\theta$ be the inverse of the collapsing map.
- Then $\mathrm{cr}(j) = \omega_1^H$.
- So U_j, the derived measure over H, is a measure on ω_1^H.
- Then $\mathcal{P}(\omega_1^H) \cap H \subsetneq \mathcal{P}(\omega_1^H) \cap \mathrm{Ult}(H, U_j)$.


$i_U(\omega_1^H) = \omega_1^{\mathrm{Ult}(H,U_j)}$
ω_1^H
ω_1^H
H
$\mathrm{Ult}(H, U_j)$

Definition

And M-ultrafilter on κ is **weakly amenable** with respect to M iff M and $\mathrm{Ult}(M, U)$ have the same subsets of κ.

Definition

A structure (M, A) where M is transitive and $A \subseteq M$ is **amenable** iff $x \cap A \in M$ whenever $x \in M$.

<u>SOME FACTS:</u> If U on κ is weakly amenable w.r.t. M then

- κ is inaccessible, weakly compact ... (and more) but not necessarily measurable in M.
- $U \cap x \in M$ whenever $x \in M$ is of cardinality $\leq \kappa$ in M, so (M, U) is an amenable structure.
- if $f \in M$ is a function with domain κ then $\{\xi < \kappa \mid f(\xi) \in U\} \in M$.

The last clause tells us that it makes sense to form $\mathrm{Ult}((M, U), U)$, so we get the ultrapower embedding

$$i_U : (M, U) \to (M', U').$$

Iterating Ultrapowers I: A Single Measure

We consider structures of the form $\mathcal{M} = (M, U)$ such that

- M is a transitive structure satisfying some reasonable fragment of ZFC.
- U is a normal measure over M.
- U is weakly amenable with respect to M.

Of particular interest are structures $\mathcal{M}$ as above where

- $M \models \mathrm{ZFC}^-$.
- $\kappa = \mathrm{cr}(U)$ is the largest cardinal of M, that is, for every $x \in M$ there is a surjection $f : \kappa \to x$ such that $f \in M$.

Such a structure $\mathcal{M}$ is the simplest nontrivial example of a **coarse premouse**. Here "coarse" refers to the fact that we omit the fine structure of $\mathcal{M}$ (to be introduced later).

Apart from coarse premice we will deal with **fine structural** premice.

The notion "premouse" is often used in a sloppy way, assuming that it is clear from the context whether the coarse or the fine structural version is meant. In situations where there may be a danger of confusion it will be explicitly said which version is meant.

Definition

Let $\mathcal{M} = (M, U)$ be a premouse. A (linear) iteration of $\mathcal{M}$ of length λ is a sequence

$$\langle \mathcal{M}_i, \pi_{i,j} \mid i \leq j \leq \lambda \rangle$$

such that:

- $\mathcal{M}_0 = \mathcal{M} = (M, U)$
- $\mathcal{M}_i = (M_i, U_i)$ is determined inducetively as follows.
- $\mathcal{M}_{i+1} = \mathrm{Ult}(\mathcal{M}_i, U_i)$ and $\pi_{i,i+1} : \mathcal{M}_i \to \mathcal{M}_{i+1}$ is the ultrapower embedding. That is,

$$\pi_{i,i+1} : (M_i, U_i) \to (M_{i+1}, U_{i+1}).$$

- The maps $\pi_{i',j'} : \mathcal{M}_{i'} \to \mathcal{M}_{j'}$ are defined in the natural way so that they all commute.

Definition (Continued)

- If $i > 0$ is a limit ordinal then

$$\langle \mathcal{M}_i, \pi_{i',i} \mid i' < i \rangle$$

is the direct limit of the diagram

$$\langle \mathcal{M}_{i'}, \pi_{i',j'} \mid i' \leq j' < i \rangle.$$

- All structures M_i, $i < \lambda$ are transitive.

A **putative** iteration is defined similarly as itertion, with the exception that there may be a last model that is ill-founded.

Definition

A premouse $\mathcal{M}$ is κ-**iterable** iff every iteration of length $< \kappa$ can be continued. $\mathcal{M}$ is **iterable** iff $\mathcal{M}$ is ∞-iterable. An iterable premouse is a **mouse**.

The main challenge in proving iterability is guaranteeing the well-foundedness of the models on iterations.

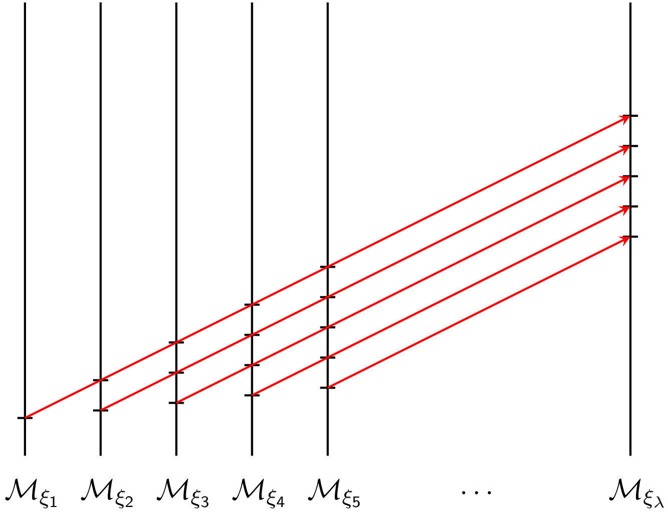
Fact

If $\langle \mathcal{M}_i, \pi_{ij} \mid i \leq j < \lambda \rangle$ is an iteration and $\mathrm{cf}(\lambda)$ is uncountable then the direct limit $\mathcal{M}_\lambda$ is well-founded.

Proof.

If $(a_i \mid i \in \omega)$ is an infinite descending chain with respect to $\in^{\mathcal{M}_\lambda}$ then $a_i = \pi_{\xi_i,\lambda}(\bar{a}_i)$ for some $\xi_i < \lambda$ and $\bar{a}_i \in M_{\xi_i}$; wlog $\xi_i < \xi_{i+1}$. Since $\mathrm{cf}(\lambda) > \omega$ we can find some $\bar{\lambda} < \lambda$ above all ξ_i. Then $(\pi_{\xi_i,\bar{\lambda}}(a_i) \mid i \in \omega)$ is an infinite descending chain with respect to $\in^{\mathcal{M}_{\bar{\lambda}}}$. □

Typical Witness of Ill-Foundedness



In particular:

- If $\langle \mathcal{M}_i \mid i < \infty \rangle$ is an iteration then $\mathcal{M}_\infty$ is well-founded. Its membership relation $\in^{\mathcal{M}_\infty}$ is also extensional, but $\mathcal{M}_\infty$ may not always be transitivised as it is not set-like: Its ordinals may be longer than $\mathbf{On}^{\mathbf{V}}$.
- The problematic stages in proving iterability are the successor steps and limit steps of cofinality ω.

However, the proof does not require to distinguish cofinalities.

The following two lemmata are useful.

The Shift Lemma

Assume $\sigma : (\bar{M}, \bar{U}) \to (M, U)$ is Σ_0-preserving. Then there is a unique Σ_0-preserving map $\sigma' : \mathrm{Ult}(\bar{M}, \bar{U}) \to \mathrm{Ult}(M, U)$ such that $\sigma' \restriction (\kappa + 1) = \sigma \restriction (\kappa + 1)$ and the following diagram commutes.

$$\begin{array}{ccc} (M, U) & \xrightarrow[i_{\bar{U}}]{} & (M', U') = \mathrm{Ult}((M, U), U) \\ \sigma \uparrow & & \uparrow \sigma' \\ (\bar{M}, \bar{U}) & \xrightarrow[i_{\bar{U}}]{} & (\bar{M}', \bar{U}') = \mathrm{Ult}((\bar{M}, \bar{U}), \bar{U}) \end{array}$$

The map σ' is defined by

$$\sigma' : [f]_{\bar{U}} \mapsto [\sigma(f)]_U.$$

THE COPY CONSTRUCTION

Assume $\mathcal{M} = (M, U)$ and $\bar{\mathcal{M}} = (\bar{M}, \bar{U})$ are premice and $\sigma : \bar{\mathcal{M}} \to \mathcal{M}$ is a Σ_0-preserving map. If $\mathcal{M}$ is sufficiently iterable then iterations of $\bar{\mathcal{M}}$ can be copied onto iterations of $\mathcal{M}$ as follows.

.

$$
\begin{array}{ccccccccc}
 & \mathcal{M}_0 & \xrightarrow{i_{0,1} = i_{U_0}} & \mathcal{M}_1 & \xrightarrow{i_{1,2} = i_{U_1}} & \mathcal{M}_2 & \xrightarrow{i_{2,3} = i_{U_2}} & & \\
\sigma = \sigma_0 & \uparrow & & \sigma_1 \uparrow & & \sigma_2 \uparrow & & & \cdots \\
 & \bar{\mathcal{M}}_0 & \xrightarrow[\bar{i}_{0,1} = i_{\bar{U}_0}]{} & \bar{\mathcal{M}}_1 & \xrightarrow[\bar{i}_{1,2} = i_{\bar{U}_1}]{} & \bar{\mathcal{M}}_2 & \xrightarrow[\bar{i}_{2,3} = i_{\bar{U}_2}]{} & &
\end{array}
$$

Theorem

Assume $\bar{\mathcal{M}}$ is a premouse, $\mathcal{M}$ is a λ-iterable premouse and $\sigma : \bar{\mathcal{M}} \to \mathcal{M}$ is Σ_0-elementary. Then $\bar{\mathcal{M}}$ is λ-iterable.

Proof.

Any putative iteration $\bar{\mathcal{I}}$ on $\bar{\mathcal{M}}$ can be copied via the copy construction onto some iteration $\mathcal{I}$ of $\mathcal{M}$, because $\mathcal{M}$ is iterable. The last copy map embeds the last model of $\bar{\mathcal{I}}$ elementarily into the last model of $\mathcal{I}$, so the last model of $\mathcal{I}$ must be well-founded. □

Theorem

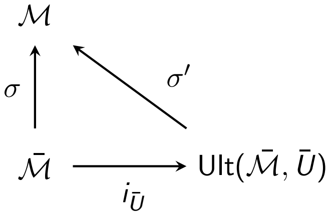
If a premouse $\mathcal{M}$ is countably iterable, i.e. λ-iterable for each $\lambda < \omega_1$, then $\mathcal{M}$ is iterable.

- If not, we have a putative iteration $\mathcal{I} = \langle \mathcal{M}_i, \pi_{ij} \mid i \leq j < \lambda + 1 \rangle$ with last ill-founded model $\mathcal{M}_\lambda$.
- Pick a regular θ large enough that $\mathcal{I} \in H_\theta$. Let X be a countable elementary substructure of H_θ such that $\mathcal{I} \in X$, let H be the transitive collapse of X and $\sigma : H \to H_\theta$ be the inverse of the collapsing map.
- Let $\bar{\mathcal{I}} = \sigma^{-1}(\mathcal{I})$ and $\bar{\mathcal{M}} = \sigma^{-1}(\mathcal{M})$. Then by the elementarity of σ, the model H believes that $\bar{\mathcal{I}}$ is a putative iteration of $\bar{\mathcal{M}}$ with last ill-founded model.
- Since all notions we refer to are sufficiently absolute, **V** also believes that $\bar{\mathcal{I}}$ is a putative iteration of $\bar{\mathcal{M}}$ with last ill-founded model.
- Now $\sigma : \bar{\mathcal{M}} \to \mathcal{M}$ is fully elementary, so $\bar{\mathcal{M}}$ must be iterable by the previous theorem. Contradiction.

Iterability from Countable Completeness

Theorem

Assume $\mathcal{M} = (M, U)$ is a premouse such that U is countably complete. If $\bar{\mathcal{M}}$ is countable and $\sigma : \bar{\mathcal{M}} \to \mathcal{M}$ is Σ_0-preserving then there is a Σ_0-preserving map $\sigma' : \mathrm{Ult}(\bar{\mathcal{M}}, \bar{U}) \to \mathcal{M}$ such that the following diagram commutes.



Theorem (Continued)

The map σ' is defined by

$$\sigma' : [f]_{\bar{U}} \mapsto \sigma(f)(\zeta)$$

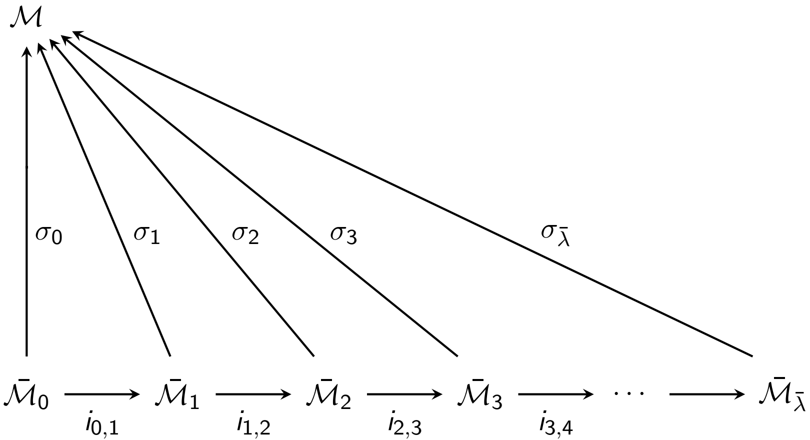
where $\zeta \in \bigcap \sigma[\bar{U}]$ is arbitrary, fixed in advance.

Theorem

If $\mathcal{M} = (M, U)$ is a premouse such that U is countably complete then $\mathcal{M}$ is iterable.

Proof.

If not, there is a putative iteration $\mathcal{I}$ on $\mathcal{M}$ with last ill-founded model. Using a reflection argument as before we obtain a countable $\bar{\mathcal{M}}$, a countable putative iteratioin $\bar{\mathcal{I}} = \langle \mathcal{M}_i \mid i < \bar{\lambda} \rangle$ with last ill-founded model, and an elementary embedding $\sigma : \bar{\mathcal{M}} \to \mathcal{M}$. By iterative application of the previous theorem we get the following diagram which yields a contradiction. □



$\mathcal{M}$
σ_0
σ_1
σ_2
σ_3
$\sigma_{\bar{\lambda}}$
$\bar{\mathcal{M}}_0$
$\bar{\mathcal{M}}_1$
$\bar{\mathcal{M}}_2$
$\bar{\mathcal{M}}_3$
$\cdots$
$\bar{\mathcal{M}}_{\bar{\lambda}}$
$i_{0,1}$
$i_{1,2}$
$i_{2,3}$
$i_{3,4}$

Getting Iterable Structures

If U comes from a measurable cardinal then U is countably complete, hence iterable, more precisely:

Assume $\tilde{U}$ is a normal measure on κ in $\mathbf{V}$.

- The structure $(H_{\kappa^+}, \tilde{U})$ is iterable.
- Let $\tau = \kappa^{+\mathbf{L}}$ and $U = \tilde{U} \cap \mathbf{L} = \tilde{U} \cap J_\tau$. Then

$$\mathcal{M} = (J_\tau, U)$$

  is iterable.

A Canonical Mouse

Let $\mathcal{M} = (J_\tau, U)$ be as above.

- U is weakly amenable w.r.t. **L**, hence w.r.t. J_τ.
- So $\mathcal{M}$ is an amenable structure.
- Recall that amenable structures have uniformly Σ_1-definable Σ_1-Skolem functions. Let $h_{\mathcal{M}}$ be such a Skolem fcn. for $\mathcal{M}$.
- Then $h_M(\omega)$ determines a Σ_1 elementary substructure of $\mathcal{M}$; by transitivising this substructure we obtain a Σ_1-elementary map

$$\sigma : \mathcal{M}^* \to \mathcal{M}$$

where $\mathcal{M}^*$ is the transitive collapse in question.

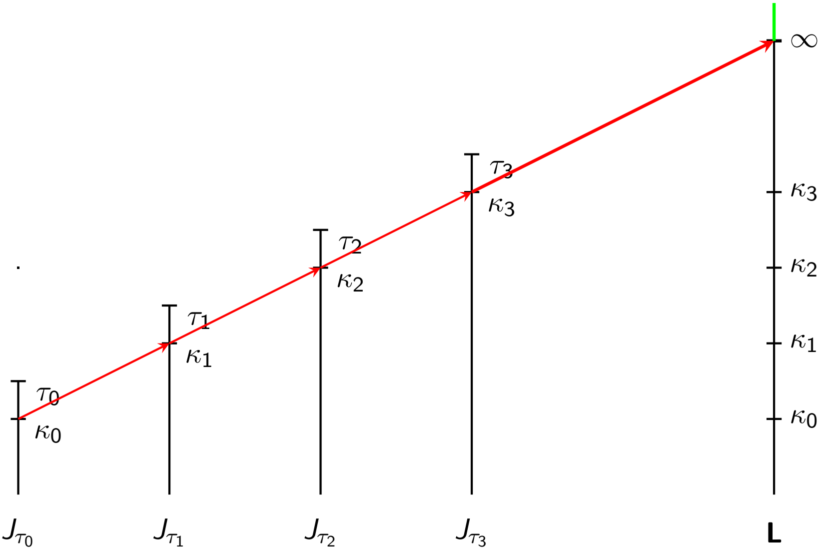
Then:

- $\mathcal{M}^*$ is a premouse.
- $h_{\mathcal{M}^*}(\omega) = \mathcal{M}^*$.
- $\mathcal{M}^*$ is iterable.

Some facts:

- $\mathcal{M}^*$ is of the form (J_{τ^*}, U^*) where U^* is a normal measure on κ^* over J_{τ^*} where κ^* is the largest cardinal in J_{τ^*}.
- $\mathcal{M}^*$ is a countable in an "effective" way: The Σ_1-Skolem function is a partial surjection of $\omega \times \omega$ onto $\mathcal{M}^*$ which is Σ_1-definable over $\mathcal{M}^*$. So $h_{\mathcal{M}^*}$ can be used to define a real coding $\mathcal{M}^*$.
- $\mathcal{P}(\kappa^*) \cap \mathcal{M}^* = \mathcal{P}(\kappa^*) \cap \mathbf{L}$ hence $\tau^* = (\kappa^*)^{+\mathbf{L}}$
- $\mathcal{M}^*$ codes the complete truth in $\mathbf{L}$.

Reconstructing **L** from $\mathcal{M}^*$



- Each U_i is weakly amenable with respect to J_{τ_i} so no subset of κ^* is added. Similarly for all κ_i.
- So in fact, $\tau_i = \kappa_i^{\mathbf{L}}$.
- Since all iteration maps are continuous at τ_i we see that each τ_i is ω-cofinal. (In fact, with a bit more work one can show that all cardinal successors of **L** are ω-cofinal in **V**.)
- The sequence of critical points $\langle \kappa_i \mid i < \infty \rangle$ is a closed proper class containing all uncountable **V**-cardinals. So every uncountable **V**-cardinal is strongly inaccessible, weakly compact,... but not measurable in **L**.
- Moreover, $J_{\kappa_i} \prec J_{\kappa_j}$ whenever $i < j$. So $J_{\kappa_i} \prec \mathbf{L}$. In particular, $J_{\kappa^*} \prec \mathbf{L}$. So:

$$\{\sigma \mid \sigma \text{ is a sentence and } J_{\kappa^*} \models \sigma\} \in \mathbf{L}$$

  is the set of all sentences true in **L**, and is Σ_1-definable inside **L** from the parameter κ^*. But more is true.

For every formula $\varphi(v_1, \ldots, v_\ell)$ in the language of set theory and every tuple $a_1, \ldots, a_\ell \in \mathbf{L}$ there is a formula $\varphi^*(v_1, \ldots, v_\ell)$ in the language of set theory with additional relational symbol $\dot{U}$ and a tuple $g_1, \ldots, g_\ell \in \mathcal{M}^*$ such that

$$\mathbf{L} \models \varphi(a_1, \ldots, a_\ell) \iff \mathcal{M}^* \models \varphi^*(g_1, \ldots, g_\ell).$$

Because

- $\mathbf{L} \models \varphi(a_1, \ldots, a_\ell) \iff \mathcal{M}_i \models$ "$J_{\kappa_i} \models \varphi(a_1, \ldots, a_\ell)$" for sufficiently large i.
- Using Łoś theorem and properties of direct limits one can translate the truth between $\mathcal{M}_i$ and $\mathcal{M}^* = \mathcal{M}_0$.

The translation of the truth between $\mathbf{L}$ and $\mathcal{M}^*$ is "effective".

Theorem

$\mathcal{M}^*$ is unique.

Proof.

If $\mathcal{M}^k$, $k = 0, 1$ are two premice having the same properties as $\mathcal{M}^*$, form iterations $\mathcal{M}^k$ of length $\omega_1 + 1$ of each. Under ω_1 we mean $\omega_1^{\mathbf{V}}$. The last model $\mathcal{M}^k_{\omega_1}$ is of the form

$$\mathcal{M}^k_{\omega_1} = (J_{\tau'}, U^k_{\omega_1})$$

where $\tau' \stackrel{\text{def}}{=} \tau_{\omega_1} = (\omega_1^{\mathbf{V}})^{+\mathbf{L}}$.

- Each $C^k = \{\kappa^k_i \mid i < \omega_1\}$, $k = 0, 1$ is a club subset of ω_1, so $C^0 \cap C^1$ is a club in ω_1.
- A set $a \in J_{\tau'}$ is in $U^k_{\omega_1}$ iff a contains a tail-end of C^k. So $U^0_{\omega_1} = U^1_{\omega_1}$, hence
$$\mathcal{M}^0_{\omega_1} = \mathcal{M}^1_{\omega_1};$$
denote this structure by $\mathcal{M}'$.

To be continued...

Continued.

- The iteration map $i^k : \mathcal{M}^k \to \mathcal{M}'$ is Σ_1-preserving.
- $\mathcal{M}^k = h_{\mathcal{M}^k}(\omega)$, so $\mathrm{rng}(i^k) = h_{\mathcal{M}'}(\omega)$.

It follows that

$$\mathcal{M}^0 = \text{ the transitive collapse of } h_{\mathcal{M}'}(\omega) \ = \mathcal{M}^1.$$

□

Definition

The mouse $\mathcal{M}^*$ is called $0^\#$.

$0^\#$ was introduced by Silver earlier using the notion of **L**-indiscernibles as the theory of the first ω many remarkable **L**-indiscernibles. The above definition is equivalent to Silver's.

Theorem (Kunen,Silver,...)

The following are equivalent.

- $0^{\#}$ *exists.*
- *There is a remarkable class of* **L**-*indiscernibles.*
- *There is a non-trivial elementary embedding* $j : \mathbf{L} \to \mathbf{L}$.

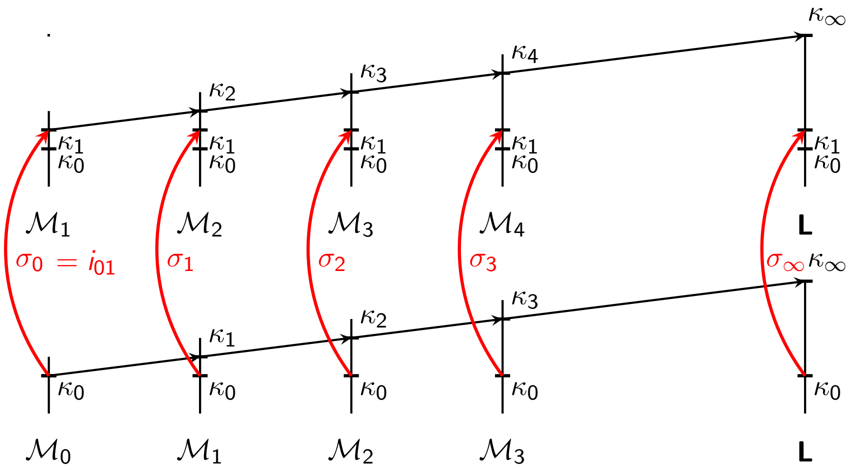
A class C is a **remarkable class of L-indiscernibles** iff for every

- formula $\varphi(v_1, \dots, v_\ell, u_1, \dots, u_k)$ in the language of set theory,
- pair of tuples $\xi_1 < \xi_2 < \cdots < \xi_\ell$ and $\xi'_1 < \xi'_2 < \cdots < \xi'_\ell$ in C,
- a tuple of ordinals $\alpha_1, \dots, \alpha_k < \min(\xi_1, \xi'_1)$,

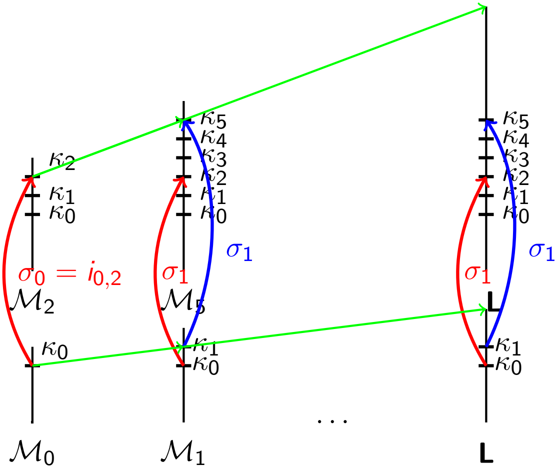
$$\mathbf{L} \models \varphi(\xi_1, \dots, \xi_\ell, \alpha_1, \dots, \alpha_k) \iff \mathbf{L} \models \varphi(\xi'_1, \dots, \xi'_\ell, \alpha_1, \dots, \alpha_k)$$

Nontrivial Embedding of L via Shift Lemma.

$\sigma_\infty : \mathbf{L} \to \mathbf{L}$, $\quad \mathrm{cr}(\sigma_\infty) = \kappa_0 = \kappa^*$, $\quad \sigma_\infty(\kappa_0) = \kappa_1$



Remarkable Indescernibility Property



Singular Cardinal Hypothesis, SCH

Conjecture (Solovay, An instance)

If κ is a strong limit cardinal then $2^\kappa = \kappa^+$.

Theorem (Silver, late 60's or early 70's)

If κ is supercompact then there is a generic extension where κ is strong limit (in fact $2^\mu = \mu^{++}$ for all $\mu < \kappa$) and $2^\kappa = \kappa^{++}$.

Theorem (Magidor, early 70's)

Assume κ is supercompact. Then there is a generic extension $\mathbf{V}[G]$ such that $\kappa = (\aleph_\omega)^{\mathbf{V}[G]}$, is strong limit in $\mathbf{V}[G]$, and

$$\mathbf{V}[G] \models 2^{\aleph_\omega} > \aleph_{\omega+1}.$$

In early 70's there was a belief that une can ultimately remove large cardinals from the construction of a model where SCH fails.

The following result was completely unexpected.

Theorem (Silver, 1974)

If κ is a singular cardinal of uncountable cofinality and $2^\mu = \mu^+$ for stationarily many $\mu < \kappa$ then $2^\kappa = \kappa^+$.

Jensen's response:

Theorem (Covering Theorem, Jensen, 1974)

Assume $0^\#$ does not exist. Then for every set $X \subseteq \mathbf{On}$ there is a set $Y \in \mathbf{L}$ such that

- $X \subseteq Y$
- $\mathrm{card}^{\mathbf{V}}(Y) \leq \mathrm{card}^{\mathbf{V}}(X) + \aleph_1$.

Corollary

Assume $0^{\#}$ does not exist. Let λ be a singular cardinal in $\mathbf{V}$*. Then*

- $\lambda^{+\mathbf{L}} = \lambda^{+}$.
- λ *is singular in* $\mathbf{L}$.
- *If* λ *is strong limit then* $2^{\lambda} = \lambda^{+}$.

Proof.

If $\lambda^{+\mathbf{L}} < \lambda^{+}$, in $\mathbf{V}$ we have a cofinal set $X \subseteq \lambda^{+\mathbf{L}}$ of order-type $< \lambda$. From Covering Theorem we get a set $Y \supseteq X$ of order-type $< \lambda$, contradicting the fact that $\lambda^{+\mathbf{L}}$ is regular in $\mathbf{L}$.

Similarly, a set of small order-type singularizing λ can be covered by a set of small ordertype in $\mathbf{L}$, thereby witnessing the singularity of λ in $\mathbf{L}$. □

To be continued...

Continued.

If λ is strong limit of cofinality $\gamma < \lambda$ then $2^\lambda = \lambda^\gamma$. By Covering each subset of λ of size γ can be covered by a set in **L** of order-type $< \lambda$, so counting the number of elements in λ^γ boils down to counting the number subsets of λ of size $\gamma + \aleph_1$ which are in **L**, where we use GCH in **L**, combined with counting the size $\mathcal{P}(\gamma + \aleph_1)$, where we use the assumption that λ is strong limit. □

So we have a dichotomy:

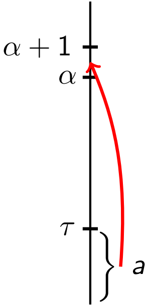
- Either $0^\#$ exists; in this case **L** is very "thin" in comparison to **V**: All cardinal successors of **L** have countable cofinality in **V**, every uncountable **V**-cardnal is strongly inaccessible (and more) in **L**, etc.
- Or else, $0^\#$ does not exist, in which case **L** is very close to **V** from global point of view: cardnal successors and cofinalities are often correctly computed by **L**, or at least closely aporoximated, SCH holds,...

Basic Notions of Fine Structure Theory

- **Acceptability**. Each J_β is acceptable, that is: If $\tau < \beta$, $a \subseteq \tau$ is a set in J_β and $\alpha < \beta$ is such that $a \in J_{\alpha+1} \setminus J_\alpha$ then there is a surjection

$$f : \tau \to J_\alpha$$

such that $f \in J_{\alpha+1}$.



- Acceptability is a very strong form of GCH.
- Notice that the "converse" holds automatically, that is, if f is as above the we obtain a set $a \subseteq \tau$ which is not in J_α by Cantor diagonal argument:

$$\xi \in a \iff \xi \notin f(\xi)$$

Can one strengthen this further?

Notation. $\Sigma_n(M)$ is the collection of all sets definable over M without parameters by a Σ_n-formula. $\boldsymbol{\Sigma}_n(M)$ is the collection of all sets definable over M with parameters by a Σ_n-formula. The subscript $_\omega$ is self-explanatory.

Fact

For every ordinal α the following holds:

$$\mathcal{P}(J_\alpha) \cap J_{\alpha+1} = \boldsymbol{\Sigma}_\omega(J_\alpha) \tag{1}$$

So the notion of acceptability can be reformulated as follows:

- If there is a set $a \subseteq \tau$ definable over J_α with parameters such that $a \notin J_\alpha$ then there is a surjection $f : \tau \to J_\alpha$ definable over J_α with parameters.

How to strengthen this? We can demand that there is a match between the degree of definability of the set a and the degree of definability of the surjection f.
Also note:

- A little inspection of the situation reveals that it suffices to consider τ least possible.
- Every J_α satisfies the Σ_0-separation axiom. So every subset x of $a \in J_\alpha$ which is Σ_0-definable (over) J_α with parameters. So the sets a of interest are of complexity $\boldsymbol{\Sigma}_1$ and higher.
- Without loss of generality the parameter can be taken to be a finite sets of ordinals. From now on, we will always consider this is the case, unless explicitly stated otherwise.

This motivates the following definitions.

Definition

An ordinal $\varrho \leq \alpha$ is the Σ_1-**projectum**, or the **first projectum** of J_α iff

- there is a $\mathbf{\Sigma}_1(J_\alpha)$ set $a \subseteq \omega\varrho$ such that $a \notin J_\alpha$, and
- ϱ is the least ordinal with this property.

We denote the first projectum of J_α by $\varrho_1(J_\alpha)$.

Fact

If $\varrho_1(J_\alpha) < \alpha$ then $\omega\varrho = \varrho$ and ϱ is a (Σ_1-) cardinal in J_α.

Definition

Assume $\varrho_1(J_\alpha) < \alpha$. A parameter $p \in [\omega\alpha]^{<\omega} = [\mathbf{On}]^{<\omega} \cap J_\alpha$ is:

- A 1-**good** parameter of J_α iff there is a set $a \subseteq \varrho_1(J_\alpha)$ which is $\Sigma_1(J_\alpha)$-definable in the parameter p and $a \notin J_\alpha$.
- A 1-**very good** parameter of J_α iff there is a partial surjection $f : \varrho_1(J_\alpha) \xrightarrow{\text{onto}} J_\alpha$.

- $P_1(J_\alpha)$ = the set of all 1-good parameters of J_α.
- $R_1(J_\alpha)$ = the set of all 1-very good parameters of J_α.

By Cantor diagonalization mentioned above, $R_1(J_\alpha) \subseteq P_1(J_\alpha)$.

Definition

The structure J_α is 1-sound iff $P_1(J_\alpha) = R_1(J_\alpha)$.

We can view $[\mathbf{On}]^{<\omega}$ as the set of all finite descending sequences of ordinals. Let $<^*$ be the lexicographical ordering of this set. Then $<^*$ is a set-like well-ordering.

Definition

The first standard parameter $p_1(J_\alpha)$ is the $<^*$-least element of $P_1(J_\alpha)$.

Fact

J_α is 1-sound iff $p_1(J_\alpha) \in R_1(J_\alpha)$.

Generalizations

- Everything we did so far goes through for amenable structure of the form $M = (J_\alpha^A, B)$ where J_α^A is acceptable. We will call such structure M **acceptable**.
- One can now generalize the above notions from $n = 1$ to arbitrary $n \in \omega$. Instead of the notion of Σ_1-definability one uses here a higher analogue, which we call n-definability. In the case of **L** one could work with Σ_n-definability instead, but loses the uniformity of definitions that the $n = 1$ case enjoys. The notion of n-definability is somewhat technical, but for the purpose of this talk one does not need to know what it is. All one needs to know that the notion of n-definability behaves the same way as the notion of Σ_1-definablity.

- We thus obtain the notions of the n-th projectum $\varrho_n(M)$, n-good/very good parameter, $P^n(M)$, $R^n(M)$, the n-th standard parameter $p_n(M)$ and the notion of n-soundness.

We also obtain the ultimate versions. Clearly

$$\varrho^1 \geq \varrho^2 \geq \cdots \geq \varrho^n \ldots,$$

so this sequence eventually stabilizes.

- $\varrho(M) =$ the ultimate value of $\varrho_n(M)$; this ordinal is called the **ultimate projectum** of M.
- The sets $P(M), R(M)$ are defined in the obvious way.
- In all important cases, including the case $M = J_\alpha$ we will have that $p_n(M)$ is an initial segment of $p_{n+1}(M)$ (when viewed as descending sequences). In that case we take

$$p(M) = \bigcup_n p_n(M) = \text{ the ultimate value of } p_n(M)$$

and call $p(M)$ the **standard parameter** of M.

Definition

M is **sound** iff $p_n(M) \in R_n(M)$ for all n.

For M where $p(M)$ is defined we have:

Fact

M is sound iff $p(M) \in R(M)$.

Finally we have:

Theorem

Each J_α is sound.

The proof of this theorem is by induction on α, and simultaneously proves the acceptability of J_α. Note: acceptability of $J_{\alpha+1}$ trivially follows from soundness of J_α. On the other hand, the acceptability of J_α is used to prove the soundness of J_α.

The notion of Σ_1-preserving map generalizes also in he obvious way. For general n we talk about n-**embeddings**; these are embeddings that preserve n-definability, or more precisely statements whose complexity is at the level n in the n-definability hierarchy.

Covering Theorem: Proof of an instance

Assume λ is singular. We show that either $0^{\#}$ exists or $\lambda^{+\mathbf{L}} = \lambda^{+}$. So set $\tau = \lambda^{+\mathbf{L}}$ and assume $\tau < \lambda^{+}$.

- Pick a set $X \subseteq \tau$ of cofinality $< \lambda$ cofinal in τ.
- By forming an elementary substructure $Y \supseteq X$ of size $< \lambda$ and collapsing, we obtain a cofinal elementary map

$$\sigma : J_{\bar{\tau}} \to J_{\tau}.$$

- **Key point.** Y can be chosen so that for every $\beta \geq \bar{\tau}$: If $\bar{\tau}$ is a cardinal in J_β then σ can be extended to a Σ_0-preserving map $\tilde{\sigma} : J_\beta \to J_{\tilde{\beta}}$ such that
  - If $J_\beta \models \mathrm{ZFC}^{-}$ then $\tilde{\sigma}$ is fully elementary.
  - If n is least such that $\varrho_{n+1}(J_\beta) < \tau$ then $\tilde{\sigma}$ is an n-embedding.

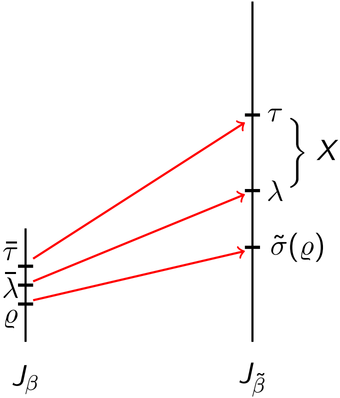
- If $\bar{\tau}$ is a cardinal in $\mathbf{L}$ then $\tilde{\sigma} : \mathbf{L} \to \mathbf{L}$. Also, $\tilde{\sigma} \neq \mathrm{id}$ as $\sigma \neq \mathrm{id}$.
- If $\bar{\tau}$ is not cardinal in $\mathbf{L}$ let β be largest such that $\bar{\tau}$ is a cardinal in J_β. Then $\bar{\tau}$ is not a cardinal in $J_{\beta+1}$, so there is a set $a \subseteq \bar{\lambda} < \bar{\tau}$ coding a well-ordering of order type $\bar{\tau}$. Hence $a \in J_{\beta+1} \setminus J_\beta$.
- Hence $\varrho(J_\beta) < \bar{\tau}$. Write ϱ for $\varrho(J_\beta)$.
- Let n be least such that $\varrho_{n+1}(J_\beta) = \varrho$.
- Then $\tilde{\sigma} : J_\beta \to J_{\tilde{\beta}}$ is an n-embedding, so it preserves the n-Skolem function.
- $J_\beta = h^n_{J_\beta}(\rho \cup \{p_{J_\beta}\})$ as J_β is sound.
- We thus have:

$$X \subseteq \mathrm{rng}(\tilde{\sigma}) = \tilde{\sigma}[h^n_{J_\beta}(\rho \cup \{p_{J_\beta}\})] = h^n_{J_{\tilde{\beta}}}(\tilde{\sigma}[\varrho] \cup \{\tilde{\sigma}(p_{J_\beta})\})$$

The right side is contained as a subset in the set

$$h^n_{J_{\tilde\beta}}(\tilde\sigma(\varrho) \cup \{\tilde\sigma(p_{J_\beta})\}) \in \mathbf{L}.$$

So it suffices to take $Y = h^n_{J_{\tilde\beta}}(\tilde\sigma(\varrho) \cup \{\tilde\sigma(p_{J_\beta})\}) \cap \tau$. □



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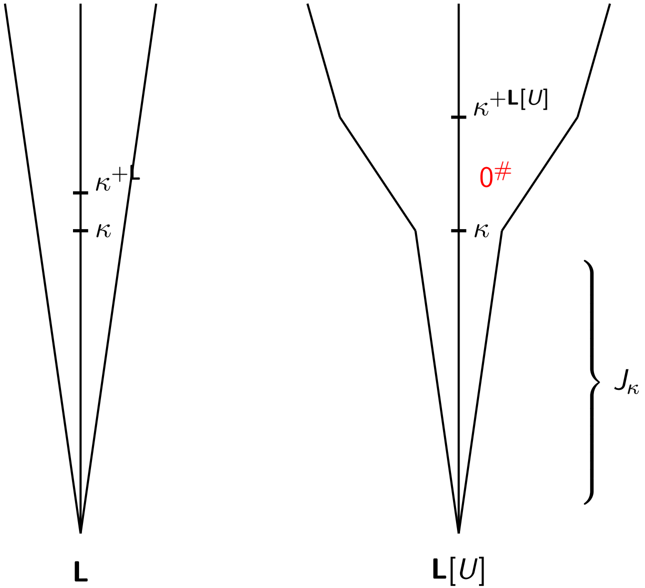
What Next?

Toward a model with a measurable cardinal, we can try $\mathbf{L}[U]$ where U is a normal measure on some κ in $\mathbf{V}$. This model is nice: $\mathbf{L}[U]$ has combinatorics similar to that $\mathbf{L}$ has, but has some deficiencies.

- The construction of $\mathbf{L}[U]$ requires a measurable cardinal in $\mathbf{V}$. (This is not too much of a problem.)
- $\mathbf{L}[U]$ is not acceptable. For instance, $0^{\#}$ is a subset of ω, but enters the J^U-hierarchy at some stage between κ and $\kappa^{+\mathbf{L}[U]}$.

The former issue tells us that $\mathbf{L}[U]$ may not be as helpful as $\mathbf{L}$ for calculating lower bounds for consistency strengths, as for consistency strengths we do not want to assume large cardinals in $\mathbf{V}$; the second issue is a source of difficulties if one wants to construct useful models for large cardinal axioms significantly stronger than measurability. Also, it gives rise to many problems in the study of internal structure of the model.



$\kappa^{+\mathbf{L}}$
κ
$\mathbf{L}$
$\kappa^{+\mathbf{L}[U]}$
$0^{\#}$
κ
J_κ
$\mathbf{L}[U]$

- One now can prove analogous results for $\mathbf{L}[0^{\#}]$, $\mathbf{L}[0^{\#\#}]$ ...
- In early 80's, Dodd and Jensen found a way of iterating this procedure and built a model which picks all large cardinal properties of $\mathbf{V}$, under the assumption that there is no inner model with a measurable cardinal.
- The model they constracted was named $\mathbf{K}$, the core model.
- $\mathbf{K}$ has the following crucial properties:
  - **Covering.** Every set of ordinals X can be covered by a set $Y \in \mathbf{K}$ such that $\mathrm{card}^{\mathbf{V}}(Y) = \mathrm{card}^{\mathbf{V}}(X) + \aleph_1$.
  - **Rigidity.** There is no nontrivial elementary embedding $j : \mathbf{K} \to \mathbf{K}$.
  - **Generic absoluteness.** If $\mathbb{P}$ is a poset in $\mathbf{V}$ and G is a filter generic for $\mathbb{P}$ over $\mathbf{V}$ then $\mathbf{K}^{\mathbf{V}[G]} = \mathbf{K}^{\mathbf{V}}$.

(Of course, all assuming there is no inner model with a measurable cardinal.)

In particular,

- SCH holds, similarly as in the case of **L**.
- If λ is a singular cardinal in **V** then $\lambda^{+\mathbf{K}} = \lambda^+$.
- In particular, since $\mathbf{K} \models \square_\kappa$, any $\square_\kappa$-sequence of **K** is also a $\square_\kappa$-sequence in the sense of **V**.
- If λ is a singular cardinal in **V** then it is singular in **K**.

So this gives a way of calculating lower bounds for consistency strength up to a measurable cardinal.

How about going further upwards? Dodd and Jensen also proved a version of covering theorem for $\mathbf{L}[U]$, but at this point new features come in: It may happen that a singular **V**-cardinal can be regular in the inner model (Prikry forcing).

The next substantial step upward is by increasing the Mitchell order.

Definition

Given two normal measures U_1, U_2 on κ, we write $U_1 \lhd U_2$ iff $U_1 \in \mathrm{Ult}(\mathbf{V}, U_2)$.

- $\lhd$ is called the **Mitchell order** on measures.
- $\lhd$ is a well-founded partial ordering.
- The Mitchell order of U is the rank of U under $\lhd$.
- The Mitchell order $o(\kappa)$ of κ is the rank of $\lhd$ on all measures on κ. Clearly, $o(\kappa) \leq 2^{2^\kappa}$.

Can we get a model of the form $\mathbf{L}[U]$ for a measure U of Mitchell order larger than 0? Note that if U is on κ then this requires that

$$\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in U.$$

But $\mathbf{L}[U]$ agrees with $\mathbf{L}$ below κ!

Recall that $\mathbf{L}[U]$ is iterable.

- $\mathbf{L}[U]$ is unique in the following sense. Given two iterable models $\mathbf{L}[U_1]$, $\mathbf{L}[U_2]$, if both U_1, U_2 are on the same ordinal κ then $U_1 = U_2$.
- In particular, if $U_1 \lhd U_2$ then still $\mathbf{L}[U_1] = \mathbf{L}[U_2]$!

Solution. In order to construct a model for higher Mitchell order one has to construct a model relative to a **sequence** of measures, i.e. a model

$$\mathbf{L}[\vec{U}]$$

where $\vec{U}$ is a sequence of measures. It will be crucial that this sequence is **coherent**.

Mitchell constructed such models in early 70's, but was not able to prove the GCH as he did not have fine structure theory for these models. The problem of constructing fine structural models, in particular the core model analogous to that of Dodd-Jensen was open for a while, as it was not clear how to arrange acceptability.

Here is one of the central issues one encounters when building models for sequences of measures:

- Say at stage α we build a structure $J_\alpha^{\vec{U}}$ and there is a measure $U_\alpha \subseteq J_\alpha^{\vec{U}}$ such that $\mathcal{M}_\alpha = (J_\alpha^{\vec{U}}, U_\alpha)$ is a premouse, i.e. we are at a stage where we are putting a new measure on the sequence $\vec{U}$. We let $\vec{U}_{\alpha+1} = \vec{U}_\alpha{}^\frown\langle U_\alpha \rangle$. Say U_α is a measure on κ_α.
- At some later stage β. in the construction we may unintentionally add a set $a \subseteq \kappa$, so $a \notin J_\alpha^{\vec{U}}$. But then U_α is not an ultrafilter on the resulting structure $J_\beta^{\vec{U}}$, so it is not possible to form $\mathrm{Ult}(J_\beta^{\vec{U}}, U_\alpha)$!
- In the situation above we way that U_α is a **partial measure** on the $\mathcal{M}_\beta$-sequence.

Follwing an idea of Stuart Baldwin, Mitchell worked out the fine structure theory for models below $o(\kappa) = \kappa^{++}$ and built the core model at this level.

Roughly speaking, the idea was

- Not to worry about creating partial measures during the process of building the model; just keep them on the sequence.
- Whenever we need to form the ultrapower using a partial measure, first truncate the model to the longest initial segment for which the measure is still total.

About the same time Jensen developed a similar theory up to one strong cardinal. These core model enjoyed the similar properties as the Dodd-Jensen's, in particular rigidity and generic absoluteness, but the Covering Theorem must have been weakened, as the model now has total measures, so the following is possible.

- Measurable cardinals can be singularized by forcing, and
- if the model contains a measurable cardinal κ with $o(\kappa) = \kappa^{++}$ then one can force the failure of SCH.

The following theorem is called **Weak Covering Theorem**.

Theorem (Appropriate Anti-Large Cardinal Assumption)

Assume κ be a **K***-cardinal such that $\kappa \geq \omega_2^{\mathbf{V}}$. Then*

$$\mathrm{cf}^{\mathbf{V}}(\kappa^{+\mathbf{K}}) \geq \mathrm{card}^{\mathbf{V}}(\kappa)$$

In particular, if κ is a singular cardinal in **V** then

$$\kappa^{+\mathbf{K}} = \kappa^{+}.$$

Mitchell also proved a strong form of covering theorem for his core model up to $o(\kappa) = \kappa^{++}$. Later, Gitik and Mitchell worked out a strong form of covering theorem for **K** up to a strong cardinal and use it to characterize the failure of SCH in terms of Mitchell order (for extenders).

Extenders

If U is a normal measue in $\mathbf{V}$ then and $i_U : \mathbf{V} \to M$ is the corresponding ultrapower embedding then

- Every element of M is of the form $i_U(f)(\kappa)$ for some f; equivalently, M is the Σ_0-hull of $\mathrm{rng}(i_U) \cup \{\kappa\}$.
- More precisely, $[f]_U = i_U(f)(\kappa)$.
- $V^M_{\kappa+1} = V_{\kappa+1}$ but $V^M_{\kappa+2} \subsetneq V_{\kappa+1}$ as $U \notin M$.

One can increase the large cardinal strength of the embedding by

- increasing the agreement between $\mathbf{V}$ and the target model in terms of rank;
- making the set from which the target model is generated larger. (In the case of a single measure this set is just $\{\kappa\}$.)

It turns out that the latter also enables to arrange the former.

We define:

- Given an ordinal β we say that the embedding $j : \mathbf{V} \to M$ is β-**strong** iff $V_\beta \subseteq M$.
- A cardinal κ is β-**strong** iff there is a β-strong embedding with critical point κ.
- A cardinal κ is **strong** iff κ is β-strong for every ordinal β.
- Given a set/class A, the embedding j is (β, A)-strong (or (β, A)-reflecting) iff j is β-strong and

$$j(A) \cap V_\beta = A \cap V_\beta.$$

- κ is (β, A)-strong (or (β, A)-reflecting) iff there is (β, A)-reflecting j with critical point κ.
- κ is A-reflecting iff κ is (β, A)-reflecting for every ordinal β.

For our purpose, we always consider $\beta < j(\kappa)$ where $\kappa = \mathrm{cr}(j)$. So κ is measurable iff κ is $(\kappa + 1)$-strong.

Definition

Assume $j : \mathbf{V} \to M$ is an elementary embedding with critical point κ and $\beta < \kappa$. Given a set $a \in [\beta]^{<\omega}$ let

$$\begin{aligned} E_a &= \text{the measure on } [\kappa]^{|a|} \text{ derived from } j \\ &= \{x \subseteq [\kappa]^{|a|} \mid a \in j(x)\} \end{aligned}$$

Then the system

$$E = \langle E_a \mid a \in [\beta]^{<\omega} \rangle$$

is called the (κ, β)-**extender derived from** j. Ordinals below β are called **coordinates** of E.

If β has good closure properties, e.g. if β is primitive recursively closed, we also have the so-called **map representation** of extender: $E : \mathcal{P}(\kappa) \to \mathcal{P}(\beta)$ is such that

$$E(x) = j(x) \cap \beta.$$

An extender is a **set** which, similarly as an ultrafilter, codes an elementary embedding, which may be a proper class. Given a (κ, β)-extender E we can form an ultrapower. In order to establish properties of the ultrapower and prove the Łoś theorem we need the following properties of $E = \langle E_a \mid a \in [\beta]^{<\omega} \rangle$ which can be easily derived from the above definition, and which can be taken as an alternative definition.

If $a \subseteq b$ are two sets in $[\beta]^{<\omega}$ then the canonical projection $p^b_a : [\kappa]^{|\,b\,|} \to [\kappa]^{|\,a\,|}$ so that

$p^b_a(u)$ "sits" inside u the same way the set a "sits" inside b.

We write "u^b_a" instead of $p^b_a(u)$. Similarly, if $\alpha \in a$ then for $u \in [\kappa]^{|\,a\,|}$,

$(u)^a_\alpha =$ the ordinal which "sits" in u the same way α "sits" in a.

A system $E = \langle E_a \mid a \in [\beta]^{<\omega} \rangle$ of measures is a (κ, β)-extender iff the following are satisfied.

- Each E_a is a κ-complete ultrafilter on $[\kappa]^{|a|}$, and at least one E_a is non-principal.
- If $\xi < \kappa$ then $E_{\{\xi\}}$ is principal with $\{\xi\} \in E_{\{\xi\}}$.
- (Coherence) If $a \subseteq b$ then E_a is the canonical projection of E_b.
- (Normality) If $\gamma < \beta$ and

$$\{u \in [\kappa]^{|a \cup \{\gamma\}|} \mid f(z_a^{a \cup \{\gamma\}}) < (z)_\gamma^{a \cup \{\gamma\}}\} \in E_{a \cup \{\gamma\}}$$

  then there is some $\gamma' < \gamma$ such that

$$\{u \in [\kappa]^{|a \cup \{\gamma'\}|} \mid f(z_a^{a \cup \{\gamma'\}}) = (z)_{\gamma'}^{a \cup \{\gamma'\}}\} \in E_{a \cup \{\gamma'\}}.$$

Many authors also include a clause that would guarantee well-foundedness of the ultrapower, but we leave it out. This is useful, as we will later talk about M-extenders, and in that case the well-foundedness of the ultrapower cannot be expressed internally.

Ultrapower Construction with Extenders

Objects in the ultrapower by E are represented by pairs (a, f) where

- $a \in [\beta]^{<\omega}$,
- $f : [\kappa]^{|a|} \to \mathbf{V}$.

We let

$$(a, f) \sim (b, g) \iff \{u \in [\kappa]^{|a \cup b|} \mid f(u_a^{a\cup b}) = g(u_b^{a\cup b})\} \in E_{a\cup b},$$

show that $\sim$ is an equivalence relation, and denote the equivalence class of (a, f) by $[a, f]_E$ or briefly $[a, f]$ if E is clear from the context. We then define the membership relation of the ultrapower in the obvious way.

Łoś theorem is the equivalence of the following two statements.

- $\mathrm{Ult}(V, E) \models \varphi([a_1, f_1], \ldots, [a_\ell, f_\ell])$.
- $\{u \in [\kappa]^{|a_1 \cup \cdots \cup a_\ell|} \mid \mathbf{V} \models \varphi(f_1(u^{a_1 \cup \cdots \cup a_\ell}_{a_1}), \ldots, f_\ell(u^{a_1 \cup \cdots \cup a_\ell}_{a_\ell}))\} \in E_{a_1 \cup \cdots \cup a_\ell}$.

We define the ultrapower embedding $i_E : \mathbf{V} \to \mathrm{Ult}(\mathbf{V}, E)$ by

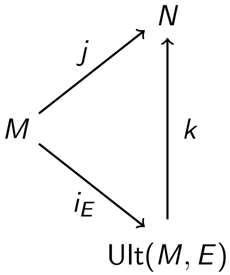
$$i_E(a) = [\{\varnothing\}, c_a]$$

where $c_a : \{\varnothing\} \to \mathbf{V}$ is the constant function with value a.

Properties of i_E:

- $\mathrm{cr}(i_E) = \kappa$ and $i_E(\kappa) \geq \beta$.
- $[a, f] = i_E(f)(a)$, so $\mathrm{Ult}(\mathbf{V}, E) =$ the Σ_0-hull of $\mathrm{rng}(i_E) \cup \beta$.
- $\alpha = [\{\alpha\}, p]$ for $\alpha < \beta$ where $p(\{\xi\}) = \xi$.
- $\alpha = [\{\varnothing\}, c_\alpha]$ if $\alpha < \kappa$.

If $j : \mathbf{V} \to M$ has critical point κ and E is the (κ, β)-extender derived from j where $\beta \leq j(\kappa)$ then we get the following diagram, so in particular $\mathrm{Ult}(\mathbf{V}, E)$ is well-founded.



Again using the Łoś theorem and the above fact on how objects in $\mathrm{Ult}(\mathbf{V}, E)$ are represented:

- $k : \mathrm{Ult}(\mathbf{V}, E) \to M$ is elementary.
- $k \restriction \beta = \mathrm{id} \restriction \beta$.

In particular, if $M =$ the Σ_0-hull of $\mathrm{rng}(i_E) \cup \beta$ then $\mathrm{Ult}(M, E) = M$ and $k = \mathrm{id}$.

Generators

We saw the ultrapower is generated from $\mathrm{rng}(i_E)$ and ordinals $< \beta$, but not all ordinals $< \beta$ are needed. Ordinals which are necessary to include in the generating set are called **generators** of E. Precisely:

- ζ is a generator of E iff ζ is not of the form $[\xi, f] = i_E(f)(\xi)$ for any $f : \kappa \to \kappa$ and any $\xi < \zeta$.

The least generator of any non-trivial extender is its critical point. Extenders with only one generator are essentially measures, so genuine extenders have at least two generators.

- An extender with two generators has consistency strength $o(\kappa) = \kappa^{++}$.

We define ν_E, the **natural length** of E to be the strict supremum of all generators of E, that is,

- $\nu_E = \sup\{\zeta + 1 \mid \zeta \text{ is a generator of } E\}$.

M-EXTENDERS

Given a transitive structure M satisfying a reasonable fragment of ZFC, we define an M-**extender** or an **extender over** M in an analogous fasion as M-ultrafilter. We demand that all clauses in the definition of extender hold with the following modifications.

- We only demand that E measures all sets in $\mathcal{P}(\kappa) \cap M$, in other words, each E_α is an M-ultrafilter.
- The normality condition applies only to function in M.

All other relevant notions also refer only to functions in M.

- When forming $\mathrm{Ult}(M, E)$ we use only functions $f \in M$.
- The notion of a generator is defined using only functions from M; the natural length ν_E is again the strict sup of the genearators.

Weak amenability of E is defined in the obvious way.

- E is weakly amenable iff $\mathrm{Ult}(M, E)$ agrees with M on subsets of κ. (This is equivalent to demanding that every E_a is a weakly amenable M-ultrafilter.)

Some notation:

- If $\gamma < \beta$ then $E \,|\, \gamma$ denotes the restriction of E to coordinates below γ, i.e. $E \,|\, \gamma = (E_a \mid a \in [\gamma]^{<\omega})$.
- The **Mitchell-Steel completion** or **trivial completion** of E is the extender $E \,|\, \nu_E^{+\mathrm{Ult}(\mathbf{M},E)}$.
- The **Jensen completion** of E is the map $i_E \restriction (\mathcal{P}(\kappa) \cap M)$.

It this talk we focus primarily on Mitchell-Steel (trivial) completions.

We will consider structures relatively constructed from a sequence of objects indexed by ordinals. In general, such a sequence has the form

$$A = \{(\beta, A_\beta) \mid \beta \leq \alpha \text{ is a limit ordinal.}\}$$

Given a structure $M = (J^A_\alpha, A_{\omega\alpha})$ where A is as above, the **initial segment** of length $\beta \leq \alpha$ of M is the structure

$$M \,||\, \beta \stackrel{\text{def}}{=} (J^A_\beta, A_{\omega\beta}).$$

If A is as above then the structure J^A_α satisfies the crucial property

$$\mathcal{P}(J^A_\beta) \cap J^A_{\beta+1} = \mathbf{\Sigma}_\omega(J^A_\beta),$$

which makes it possible to develop the fine structure theory for M, as long as M is amenable.

Premice

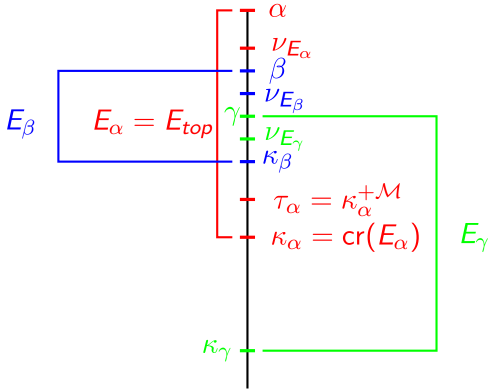
A **premouse** is an acceptable structure of the form $\mathcal{M} = (J^E_\alpha, E(\omega\alpha))$ satisfying the following requirements.

- E is a sequence indexed by limit ordinals such that for each limit $\beta \leq \alpha$ either $E(\beta) = \varnothing$ or $E(\beta)$ codes an extender.
- If $E(\beta)$ codes a J^E_β-extender (which we also denote by $E(\beta)$) then
$$\beta = \nu_{E(\beta)}^{+\mathrm{Ult}(J^E_\beta, E(\beta))}.$$
We say that β is the **Mitchell-Steel index** of $E(\beta)$.
- (Coherence) If $J^{E'}_{\beta'} = \mathrm{Ult}(J^E_\beta, E(\beta))$ then $E' \restriction \beta = E \restriction \beta$ and $E'(\beta) = \varnothing$. (The latter is automatic, as β is a cardinal successor in the ultrapower.)
- Every proper initial segment of $\mathcal{M}$ is sound.

- (Initial segment condition, ISC). For every limit ordinal $\beta \leq \omega\alpha$ and every generator ζ of $E(\beta)$:

  $E(\beta) \restriction \zeta$ is on the sequence E or on the sequence of $\mathrm{Ult}(J^E_\zeta, E(\zeta))$

  as long as $E(\beta) \restriction \zeta$ is not pathological.



Some Explanations.

- $\mathcal{M} \,||\, \beta$ is typically not amenable. However, we can translate $\mathcal{M}$ into an amenable structure $\hat{\mathcal{M}}$ which carries the same information. What we literally mean in the clause on soundness is that $\hat{\mathcal{M}}$ is sound.
- One can show that extender sequences which reach a measurable cardinal above a strong cardinal always contain pathological extenders which cannot be on any extender sequence. So the ISC tells us that initial segments of the extender are always on the sequence (of $\mathcal{M}$ or of a suitable ultrapower) as long as they have a chance to be extenders of some premice. Actually, the weaker condition $E_\beta \,|\, \zeta \in \mathcal{M} \,||\, \beta$ woud suffice.
- If $\mathcal{M} = (J^E_\alpha, E(\omega\alpha))$ then $E(\omega\alpha)$ is called the **top extender** of $\mathcal{M}$. All extenders on the $\mathcal{M}$-sequence except for the top one are elements of $\mathcal{M}$.

Fine Ultrapowers

- We saw that if M is an amenable structure and U is an M-ultrafilter then the ultrapower map $i_U : M \to \text{Ult}(M, U)$ is Σ_0-preserving (and in fact Σ_1-preserving), but not necessarily more. The same applies to ultrapower maps by extenders.
- To increase the preservation degree of ultrapower maps we need to use the so-called **fine ultrapowers**.
- As a premouse $\mathcal{M}$ is typically not amenable, we apply the extender E to its amenable translate $\hat{\mathcal{M}}$ to get

$$i_E : \hat{\mathcal{M}} \to N.$$

  One can show that $N = \hat{\mathcal{M}}'$ for some premouse $\mathcal{M}'$. This $\mathcal{M}'$ will be consider the premouse which corresponds to the desired ultrapower. By abuse of notation we will write $i_E : \mathcal{M} \to \mathcal{M}'$, although in reality we have

$$i_E : \hat{\mathcal{M}} \to \hat{\mathcal{M}}'.$$

- Each fine ultrapower is assigned a degree $n \in \omega$. A fine ultrapower of degree n is called an n-ultrapower.
- To form an n-ultrapower of a structure M (which needs to be both acceptable and amenable), we consider not only functions which are elements of M, but also certain functions that are n-definable over M (with parameters).
- A 0-ultrapower is the same as a usual ultrapower. We can form the n-ultrapower of M whenever $\rho_n(M) > \mathrm{cr}(E)$. If n is maximal possible we talk about a **maximal** ultrapower.
- For the purpose of this talk it is not important how a fine ultrapower is defined. What is important is the following:
  - All what we said about M-ultrapowers by extenders above applies to fine ultrapowers.
  - An n-ultrapower map is an n-embedding, so in particular it preserves the canonical n-Skolem functions.

Iterating Ultrapowers II: Linear Iterations of Total Extenders

These iterations are obvious generalizations of iterations of a single M-ultrafilter. Given a premouse $\mathcal{M}$, an n-iteration satisfies the following.

- $\mathcal{M}_0 = \mathcal{M}$
- $\mathcal{M}_{\alpha+1} = \mathrm{Ult}^n(\mathcal{M}_\alpha, E_\alpha)$ where E_α is a total extender on the $\mathcal{M}_\alpha$-sequnence (so we do not necessarily have $E_\alpha \in \mathcal{M}_\alpha$), and $i_{\alpha,\alpha+1} = i_{E_\alpha}$.
- The remaining embeddings are obtained by composing the ultrapower embeddings of adjacent models.
- At limit steps we take direct limits.

We say that a linear iteration as above is **normal** iff

- $\kappa_\alpha \geq \nu_{E_\beta}$ whenever $\beta < \alpha$.

The extenders used in a normal iteration are **non-overlapping**.

- The normality condition is crucial in reconstructing the extenders used in the iteration from the iteration maps. If $\alpha < \beta$ and $i_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is the iteration map then by the normality condition $i_{\alpha+1,\beta} \upharpoonright \nu_{E_\alpha} = \mathrm{id} \upharpoonright \nu_{E_\alpha}$. For $a \in [\nu_{E_\alpha}]^{<\omega}$ and $x \in \mathcal{P}([\kappa]^{|a|})$:

$$\begin{aligned} x \in (E_\alpha)_a &\iff a \in i_{E_\alpha}(x) = i_{\alpha,\alpha+1} \\ &\iff a = i_{\alpha+1,\beta}(a) \in i_{\alpha+1,\beta}(i_{\alpha,\alpha+1}(x)) = i_{\alpha,\beta}(x) \end{aligned}$$

So

$$\begin{aligned} E_\alpha &= \text{the extender derived from } i_{\alpha,\alpha+1} \\ &= \text{the extender derived from } i_{\alpha,\beta}. \end{aligned}$$

- The index of E_α is equal to $(\nu_{E_\alpha})^{+\mathcal{M}_\gamma}$ for every $\gamma > \alpha$.

The Comparison Argument

- Given two linearly iterable premice $\mathcal{M}, \mathcal{N}$ whose extender sequences are non-overlapping, we can form a pair of linear normal **padded** iterations $\mathcal{I} = (\mathcal{M}_\alpha, i^{\mathcal{I}}_{\alpha,\beta})_{\alpha,\beta}$ and $\mathcal{J} = (\mathcal{N}_\alpha, i^{\mathcal{J}}_{\alpha,\beta})_{\alpha,\beta}$ by iterating the least disagreement between models $\mathcal{M}_\alpha, \mathcal{N}_\alpha$ for every α.
- If for instance $E^{\mathcal{I}}_\alpha = \varnothing$ then at the α-th step in $\mathcal{I}$ we "do nothing", that is, se set $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha$ and $i_{\alpha,\alpha+1} = \mathrm{id}$. Iterations which contain such steps are called padded.

Theorem

If λ is a regular cardinal larger than $\max(\mathrm{card}(\mathcal{M}), \mathrm{card}(\mathcal{N}))$ then the comparison stops at some $\alpha < \lambda$. This means: The extender sequence of $\mathcal{M}_\alpha$ is an initial segment of the extender sequence of $\mathcal{N}_\alpha$ or vice versa.

- Assuming $\mathcal{I}, \mathcal{J}$ run for at least λ-steps, consider their initial segments of length $\lambda + 1$ which we also denote by $\mathcal{I}, \mathcal{J}$.
- Let X be an elementary substructure of H_λ, where $\lambda > \theta$, such that $\mathcal{I}, \mathcal{J} \in X$ and $\bar{\lambda} = X \cap \lambda$ is an ordinal.
- Collapse X; let $\sigma : H \to H_\theta$ be the inverse of the collapsing map and $\sigma(\bar{\mathcal{I}}, \bar{\mathcal{J}}) = (\mathcal{I}, \mathcal{J})$. Notice $\bar{\lambda} = \mathrm{cr}(\sigma)$ and $\sigma(\bar{\lambda}) = \lambda$.
- By inspecting how iterations are built, one proves:
  - $\bar{\mathcal{I}} \restriction (\bar{\lambda} + 1) = \mathcal{I} \restriction (\bar{\lambda} + 1)$
  - $\bar{\mathcal{J}} \restriction (\bar{\lambda} + 1) = \mathcal{J} \restriction (\bar{\lambda} + 1)$

  so
  - $\mathcal{M}^{\bar{\mathcal{I}}}_{\bar{\lambda}} = \mathcal{M}^{\mathcal{I}}_{\bar{\lambda}}$
  - $\mathcal{M}^{\bar{\mathcal{I}}}_{\bar{\lambda}} = \mathcal{M}^{\mathcal{I}}_{\bar{\lambda}}$
- Additionally,
  - $i^{\mathcal{I}}_{\bar{\lambda},\lambda} \restriction \mathcal{M}_{\bar{\lambda}} = \sigma \restriction \mathcal{M}_{\bar{\lambda}}$
  - $i^{\mathcal{J}}_{\bar{\lambda},\lambda} \restriction \mathcal{N}_{\bar{\lambda}} = \sigma \restriction \mathcal{N}_{\bar{\lambda}}$
- $\mathcal{P}(\bar{\lambda}) \cap \mathcal{M}_{\bar{\lambda}} = \mathcal{P}(\bar{\lambda}) \cap \mathcal{N}_{\bar{\lambda}}$.

From the agreement of the iteration maps we get

- The extenders $E^{\mathcal{I}}_{\bar{\lambda}}$ and $E^{\mathcal{J}}_{\bar{\lambda}}$ are **compatible**, that is, either

$$E^{\mathcal{I}}_{\bar{\lambda}} = E^{\mathcal{J}}_{\bar{\lambda}} \mid \alpha^{\mathcal{I}}_{\bar{\lambda}}$$

  where $\alpha^{\mathcal{I}}_{\bar{\lambda}}$ is the index of $E^{\mathcal{I}}_{\bar{\lambda}}$ in $\mathcal{M}^{\mathcal{I}}_{\bar{\lambda}}$ or vice versa.
- If one of the extenders is longer than the other we get a contradiction with the fact that the index of the shorter extender is a cardinal successor of the natural length of the other extender in the model and the Initial Segment Condition. This is because an extender induces a cofinal map from the successor of its critical point into its index.
- If the two extenders are equal we get a contradiction with the way we form the iterations: iterating by the least disagreement.

Iterating Ultrapowers III: Iteration Trees

Why iteration trees?

- If our premice have overlapping extenders then the comparison iterations may not be normal if we iterate linearly. This will de-rail the comparison argument: Assuming
  - $\kappa_{E_1} < \nu_{E_0}$ and (for simplicity)
  - all remaining critical points are above ν_{E_1},

  for $a \in [\nu_{E_1}]^{<\omega}$ and $x \in \mathcal{P}([\kappa]^{|a|}) \cap \mathcal{M}_0$ we have:

$$
\begin{aligned}
a \in i_{0,\infty}(x) &\iff a \in i_{1,\infty} \circ i_{12} \circ i_{01}(x) \\
&\iff a \in i_{1,\infty} \circ i_{12}(i_{01}(x) \cap [\kappa_{E_1}]^{|a|}) \\
&\iff a \in i_{12}(i_{01}(x \cap [\kappa_{E_1}]^{|a|}) \\
&\iff i_{01}(x \cap [\kappa_{E_1}]^{|a|}) \in (E_2)_a
\end{aligned}
$$

- More importantly, linear iterability for countable premice is a $\Pi_1(H_{\omega_1})$ property and can be internalized. It folllows that if the comparison argument works just with linear iterations, then every linearly iterable premouse has a Δ^1_3-well-ordering of reals. Such premice cannot have a Woodin cardnal and a measurable above, by Martin-Steel.
- Even a little bit past one strong cardinal, an argument of Woodin shows that **K** must be more complex than one can achieve with linear iterations.

- We consider trees $T = (\tau, <_T)$ of particular form, namely
  - 0 is the root of T,
  - $<_T$ is consistent with the usual ordering of ordinals, precisely $\alpha <_T \beta \implies \alpha < \beta$, and
  - if α is a limit ordinal then the branch in T below α

$$[0, \alpha)_T \stackrel{\text{def}}{=} \{\xi < \tau \mid \xi <_T \alpha\}$$

    is a closed unbounded subset of α.

Trees of this kind will constitute the base combinatorial structure of iteration trees.

Definition

Let $\mathcal{M}$ be a premouse. A normal maximal iteration tree on $\mathcal{M}$ of length λ is a structure

$$\mathcal{T} = (T, \mathcal{M}_\alpha^\mathcal{T}, E_\alpha^\mathcal{T}, i_{\alpha\beta}^\mathcal{T} \mid \alpha \leq \beta < \lambda)$$

such that

- $\mathcal{M}_0^\mathcal{T} = \mathcal{M}$.
- $E_\alpha^\mathcal{T}$ is an extender on the $\mathcal{M}_\alpha^\mathcal{T}$-sequence.
- Successor case. Letting
  - α^* be the least ordinal ξ such that $\kappa_\alpha^\mathcal{T} \stackrel{\text{def}}{=} \text{cr}(E_\alpha^\mathcal{T}) < \nu_{E_\xi^\mathcal{T}}$,
  - $\zeta_\alpha^\mathcal{T}$ be the largest ζ such that E_α is total on $\mathcal{M}_{\alpha^*} \,||\, \zeta$, and
  - $\mathcal{M}_{\alpha+1}^* = \mathcal{M}_{\alpha^*}^\mathcal{T} \,||\, \zeta_\alpha$,

  we have:
  - α^* is the immediate $<_T$-predecessor of $\alpha + 1$,
  - $\mathcal{M}_\alpha^\mathcal{T} = \text{Ult}^*(\mathcal{M}_{\alpha+1}^*, E_\alpha)$ and

Definition (Continued)

  - the map
    $$i_{\alpha^*,\alpha+1} : \mathcal{M}^*_\alpha \to \mathcal{M}^{\mathcal{T}}_{\alpha+1}$$
    is the corresponding fine ultrapower embedding.
- The maps along the branches commute, as long as they are total.
- Each branch b has only finitely many **truncation points**, that is, ordinals α such that $\zeta_\alpha \in \mathcal{M}^{\mathcal{T}}_{\alpha^*}$.
- If α is a limit ordinal then $\mathcal{M}^{\mathcal{T}}_\alpha$ is the direct limit along the tail-end of the branch below α which has not truncation points. The maps $i^{\mathcal{T}}_{\gamma,\alpha}$ are the direct limit maps.

Some explanations.

- The first condition for the successor case guarantees that $E^{\mathcal{T}}_{\alpha}$ is an $\mathcal{M}^{\mathcal{T}}_{\alpha^*} \,||\, \nu_{E^{\mathcal{T}}_{\xi}}$-extender.
- The second condition for the successor case guarantees that $E^{\mathcal{T}}_{\alpha}$ is an $\mathcal{M}^*_{\alpha}$-extender.
- The definition would make sense even if we did not demand α^* to be the least possible.

Properties of Normal Iteration Trees

- All models $\mathcal{M}_\beta^\mathcal{T}$, $\beta \geq \alpha$ agree below

$$\lambda_\alpha^\mathcal{T} = \text{ the index of } E_\alpha^\mathcal{T} \text{ in } \mathcal{M}_\alpha^\mathcal{T}.$$

- $\lambda_\alpha = (\nu_{E_\alpha^\mathcal{T}})^{+\mathcal{M}_\beta^\mathcal{T}}$ whenever $\beta > \alpha$.
- Critical points along the branches of T are ascending and the extenders used along the branches are non-overlapping.
- If there is no truncation point in the interval $(\alpha, \beta]_T$ and $\mathrm{cr}(i_{\alpha,\beta}) > \varrho(\mathcal{M}_\alpha)$ then all $\varrho(\mathcal{M}_\gamma)$, $\gamma \in [\alpha, \beta]_T$ are the same and the iteration maps send good parameters to good parameters.
- None of $\mathcal{M}_\gamma$, $\gamma > \alpha$ is sound.

Comparison Argument Again.

The comparison argument goes through for iteration trees in the following form. We say that a pair of iteration trees $(\mathcal{T},\mathcal{U})$ where $\mathcal{T}$ is on $\mathcal{M}$ and $\mathcal{U}$ is on $\mathcal{N}$ is a **pair of comparison trees** iff

- $\mathcal{T},\mathcal{U}$ are **padded** normal iteration trees of the same length.
- Extenders $E^{\mathcal{T}}_\alpha, E^{\mathcal{U}}_\alpha$ are picked according to the least disagreement.

A pair of comparison trees is **terminal** iff the last models on these trees are compatible, i.e. one of them is an initial segment of the other.

Theorem (Comparison Theorem)

If λ is a regular cardinal larger than $\max(\mathrm{card}(\mathcal{M}), \mathrm{card}(\mathcal{N}))$ and $(\mathcal{T},\mathcal{U})$ is a pair of non-terminal comparison trees of length λ. Then at least one of these trees does not have a cofinal branch.

- The only new element is that we have to work on a given branch of each tree $\mathcal{T},\mathcal{U}$. Assuming the comparison pair $(\mathcal{T},\mathcal{U})$ has length $\lambda+1$, we consider branches $[0,\lambda]_{\mathcal{T}}$ and $[0,\lambda]_{\mathcal{U}}$.
- As before X is an elementary substructure of H_θ with everything of our interet in X such that $X\cap\lambda=\bar{\lambda}<\lambda$, and $\sigma : H \to H_\theta$ is the inverse of the collapsing map.
- Focus on $\mathcal{T}$. Say $\sigma(\bar{T})=T$. Then
  - $\sigma([0,\bar{\lambda}]_{\bar{T}})=[0,\lambda]_T$ and
  - $\alpha=\sigma(\alpha)\in[0,\lambda]_T$ whenever $\alpha\in[0,\bar{\lambda}]_{\bar{T}}$.

  Thus, $\bar{\lambda}$ is a limit point of $[0,\lambda]_T$ and hence $\bar{\lambda}\in[0,\lambda]_T$ as branches are closed. Similarly for $\mathcal{U}$.
- Now repeat the comparison argument from the linear case on branches $[\bar{\lambda},\lambda]_T$, $[\bar{\lambda},\lambda]_U$.

ITERABILITY

Definition

A λ-**normal iteration strategy** for a premouse $\mathcal{M}$ is a function Σ such that for every normal itertion tree $\mathcal{T}$ on $\mathcal{M}$ of length $< \lambda$,

$$\Sigma(\mathcal{T}) = \text{ a cofinal well-founded branch through } \mathcal{T}.$$

Thus, such a branch b is a cofinal branch of the underlying tree T and the direct limit along b is well-founded.

Definition

A premouse $\mathcal{M}$ is λ-**normally iterable** iff there is a λ-normal iteration strategy for $\mathcal{M}$.

Normal iterability yields comparison.

Theorem

Assume $\mathcal{M}, \mathcal{N}$ are $(\lambda + 1)$-iterable premice where λ is regular larger than $\max(\mathrm{card}(\mathcal{M}), \mathrm{card}(\mathcal{N}))$. Then the comparison of $\mathcal{M}$ with $\mathcal{N}$ terminates strictly below λ.

Comparison is used to establish fine structural properties of premice. Two crucial such properties are:

- **Solidity** of the standard parameter.
- **Universality** of the standard parameter.

The notion of solidity is technical. For us the following consequence of solidity is relevant.

- If $\mathcal{M}$ is solid then the iteration maps send standard parameters to standard parameters. Precisely: if $i_{\alpha\beta}$ is an iteration map then $i_{\alpha\beta}(p_{\mathcal{M}_\alpha}) = p_{\mathcal{M}_\beta}$.

Solidity yields the following crucial properties of comparison.

Theorem

Assume

- $\mathcal{M}, \mathcal{N}$ *are premice such that every proper initial segment of* $\mathcal{M}, \mathcal{N}$ *is solid.*
- $(\mathcal{T}, \mathcal{U})$ *is a pair of terminal comparison trees where* $\mathcal{T}$ *is on* $\mathcal{M}$ *and* $\mathcal{U}$ *is on* $\mathcal{M}$*. Say* $\mathcal{T}, \mathcal{U}$ *are of length* λ*.*

Then at most one of the **main branches** $[0, \lambda]_{\mathcal{T}}$, $[0, \lambda]_{\mathcal{U}}$ *involves a truncation.*

- Assuming there are truncation points on both branches, say α is the last truncation point on $[0, \lambda]_{\mathcal{T}}$ and β is the last truncation point on $[0, \lambda]_{\mathcal{U}}$.
- Then both models $\mathcal{M}_{\lambda}^{\mathcal{T}}, \mathcal{M}_{\lambda}^{\mathcal{U}}$ fail to be sound.
- Thus $\mathcal{M}_{\lambda}^{\mathcal{T}} = \mathcal{M}_{\lambda}^{\mathcal{U}}$. Denote this model by $\mathcal{M}'$.

- Since
  - $\mathcal{M}_\alpha^{\mathcal{T},*}$ is sound,
  - the critical points along the branch $[\alpha_{\mathcal{T}}^-, \lambda]_{\mathcal{T}}$ (α^- is the immediate $<_{\mathcal{T}}$-predecessor of α) are $\geq \varrho(\mathcal{M}_\alpha^{\mathcal{T},*}) \stackrel{\text{def}}{=} \varrho$ and
  - $i_{\alpha^-,\lambda}^{\mathcal{T}}(p_{\mathcal{M}_\alpha^{\mathcal{T},*}}) = p_{\mathcal{M}'}$,

$$\mathrm{rng}(i_{\alpha^-,\lambda}^{\mathcal{T}}) = h_{\mathcal{M}'}^*(\varrho \cup \{p_{\mathcal{M}'}\}).$$

Similarly

$$\mathrm{rng}(i_{\beta^-,\lambda}^{\mathcal{U}}) = h_{\mathcal{M}'}^*(\varrho \cup \{p_{\mathcal{M}'}\})$$

- Thus $\mathcal{M}_\alpha^{\mathcal{T},*} = \mathcal{M}_\beta^{\mathcal{U},*}$ and $i_{\alpha^-,\lambda}^{\mathcal{T}} = i_{\beta^-,\lambda}^{\mathcal{U}}$.
- Letting $\alpha = \gamma + 1$ and $\beta = \delta + 1$, we see that $E_\gamma^{\mathcal{T}}, E_\delta^{\mathcal{T}}$ is a compatible pair of extenders. □

Definition

Let $\mathcal{M}$ be a premouse and $\bar{\mathcal{M}}$ be the transitive collapse of $h^*_{\mathcal{M}}(\varrho(\mathcal{M}))$. Then $\varrho(\bar{\mathcal{M}}) = \varrho(\mathcal{M})$; denote this value by ϱ. We say that $p_{\mathcal{M}}$ is **universal** iff

$$\mathcal{P}(\varrho) \cap \bar{\mathcal{M}} = \mathcal{P}(\varrho) \cap \mathcal{M}.$$

Fine structural properties are **internal** to premice. So it suffices to establish such properties to a countable transitive isomorph to a premouse $\mathcal{M}$; by elementarity the same property is true for $\mathcal{M}$ as well. This is important, as one needs smaller amount of iterability to deal with countable premice. On the other hand, to run the arguments one seems to need more than normal iterability.

Definition

Given a premouse $\mathcal{M}$, a (λ,μ)-**iteration strategy** is a function assigning a cofinal well-founded branch to each linear composition of lenth $< \mu$ of normal iteration trees of length $< \lambda$. Such compositions are also called **stacks** of normal iterations.

Definition

A premouse $\mathcal{M}$ is (λ,μ)-**iterable** iff there is a (λ,μ)-iteration strategy for $\mathcal{M}$.

Definition

A premouse $\mathcal{M}$ is **countably iterable** iff every countable $\bar{\mathcal{M}}$ elementarily embeddable into $\mathcal{M}$ is (ω_1+1,ω_1)-iterable.

Theorem

Let $\mathcal{M}$ be a countably iterable premouse. Then the standard parameter of $\mathcal{M}$ is solid and universal.

Definition

Let $\mathcal{M}$ be a countably iterable premouse. Let $\bar{\mathcal{M}}$ be the transitive collapse of $h^*_{\mathcal{M}}(\varrho(\mathcal{M}) \cup \{p_{\mathcal{M}}\})$. Then $\bar{\mathcal{M}}$ is called the **core** of $\mathcal{M}$ and denoted by $\text{core}(\mathcal{M})$. The associated embedding $\sigma : \text{core}(\mathcal{M}) \to \mathcal{M}$ is called the **core map**.

Theorem

Let $\mathcal{M}$ be a countably iterable premouse. Then

- $\mathcal{M}$ *and* $\text{core}(\mathcal{M})$ *have the same ultimate projectum* ϱ.
- $\mathcal{P}(\varrho) \cap \text{core}(\mathcal{M}) = \mathcal{P}(\varrho) \cap \mathcal{M}$.
- $\varrho^{+\text{core}(\mathcal{M})} = \varrho^{+\mathcal{M}}$.
- $\text{core}(\mathcal{M})$ *is sound and solid. (Its standard parameter is trivially uniersal.)*

Iterability also yields condensation. For our immediate purposes a very basic variant of the condensation lemma suffices.

Theorem

Let $\sigma : \bar{\mathcal{M}} \to \mathcal{M}$ be an n-embedding of two premice such that $\sigma \upharpoonright \varrho_{n+1}(\bar{\mathcal{M}}) = \mathrm{id}$. Assume that $\mathcal{M}$ is countably iterable and $\bar{\mathcal{M}}$ is sound. Then $\bar{\mathcal{M}}$ is an initial segment of $\mathcal{M}$.

Corollary

Let $\mathcal{M}$ be a countably iterable premouse and ϱ be the common ultimate projectum of $\mathcal{M}$ and $\mathrm{core}(\mathcal{M})$ and τ be the common cardinal successor of ϱ. Then $\mathrm{core}(\mathcal{M}) \,||\, \tau = \mathcal{M} \,||\, \tau$.

The Model Construction

We describe the **general background certified construction**

$$\mathbb{C} = \langle \mathcal{N}_\xi, \mathcal{M}_\xi \mid \xi < \Omega \rangle$$

due to Mitchell and Steel.

We will refer a general/vague notion of an extender being **certified**. There are various conditions, chosen according to the purpose of the construction, which can be chosen as certification. Some of them will be discussed later. For the moment the important point is that under suitable circumstances, certified extenders guarantee countable iterability of premice $\mathcal{N}_\xi$ in the construction below.

- $\mathcal{M}_\xi = \text{core}(\mathcal{N}_\xi)$.
- $\mathcal{N}_0 = (\varnothing, \varnothing)$.
- (Successor stage) Assume $\mathcal{M}_\xi = (J^{E^\xi}_{\beta\xi}, E^\xi_{\omega\beta_\xi})$.
  - If $E^\xi_{\omega\beta_\xi} = \varnothing$ and there is a certified extender F such that $(J^{E^\xi}_{\beta_\xi}, F)$ is a premouse let $\tilde{E}^{\xi+1}_{\omega\beta_\xi}$ be such an extender with the least possible natural length. We then let

    $$\tilde{E}^{\xi+1} = \tilde{E}^\xi \restriction \omega\beta_\xi {}^\frown \tilde{E}^{\xi+1}_{\omega\beta_\xi}$$

    and

    $$\mathcal{N}_{\xi+1} = (J^{\tilde{E}^{\xi+1}}_{\beta_\xi}, \tilde{E}^{\xi+1}_{\omega\beta_\xi}) = (J^{E^{\xi+1}}_{\beta_\xi}, \tilde{E}^{\xi+1}_{\omega\beta_\xi})$$

  - in all other cases we let $\tilde{E}^{\xi+1} = E^\xi \cup \langle \omega(\beta_\xi + 1), \varnothing \rangle$ and

    $$\mathcal{N}_{\xi+1} = (J^{\tilde{E}^{\xi+1}}_{\beta_\xi+1}, \tilde{E}^{\xi+1}_{\omega(\beta_\xi+1)}) = (J^{\tilde{E}^{\xi+1}}_{\beta_\xi+1}, \varnothing)$$

- (Limit Case) If ξ is a limit ordinal then

$$\tilde{E}^{\xi} \upharpoonright \beta_{\xi} = \liminf_{\eta<\xi} \tilde{E}^{\eta} = \liminf_{\eta<\xi} E^{\eta}.$$

  This also determines the height of β_{ξ} of $\tilde{E}^{\xi}$. We then let $\tilde{E}^{\xi}_{\omega\beta_{\xi}} = \varnothing$; hence

$$\mathcal{N}_{\xi} = (J^{\tilde{E}^{\xi}}_{\beta_{\xi}}, \tilde{E}^{\xi}_{\omega\beta_{\xi}}) = (J^{\tilde{E}^{\xi}}_{\beta_{\xi}}, \varnothing)$$

If the construction never breaks down or ends up in an infinite cycle we say that the construction **converges**. In this case our model of interest is

$$\mathcal{M}_{\infty} = \mathcal{N}_{\infty}$$

Reasons that may cause the construction break down.

- The next extender in the successor case is not unique.
- The fine structure breaks down.
- We lose too many sets during the collapsing when forming $\mathcal{M}_\xi$.

By choosing the notion of "certified" suitably and making an anti-large cardinal assumption, we can prove countable iterability of models $\mathcal{N}_\xi$. This will guarantee that the above issues do not occur.

Immediate properties of $\mathcal{N}_\infty$.

- $\mathcal{N}_\infty$ is a model with fine structure similar to that of **L**.
- In particular, $\mathcal{N}_\infty$ satisfies GCH.

Definition

A cardinal δ is a **Woodin cardinal** iff for every $A \subseteq V_\delta$,

$$(V_\delta, A) \models \text{"There is an } A\text{-reflecting cardinal"}.$$

Definition

A premouse $\mathcal{M}$ is ***n*-small** iff whenever κ is the critical point of an extender on the $\mathcal{M}$-sequence,

$$\mathcal{M} \,||\, \kappa \models \text{"There are no Woodin cardinals"}.$$

Definition

A premouse $\mathcal{M}$ is **tame** iff there are no δ and α such that

- $\mathcal{M} \,||\, \alpha \models$ "δ is a Woodin cardinal as witnessed by extenders on the $\mathcal{M} \,||\, \alpha$-sequence".
- $\kappa_E \leq \delta < \alpha$.

Thus, $\mathcal{M}$ is tame iff $\mathcal{M}$ "has no Woodin cardinals overlapped by extenders".

Definition

A premouse $\mathcal{M}$ is **bland** iff for every $\delta \in \mathcal{M}$ the set of all $\kappa < \delta$ for which there is $\alpha \in \mathcal{M}$ such that

- δ is Woodin in $\mathcal{M} \,||\, \alpha$ as witnessed by extenders on the $\mathcal{M} \,||\, \alpha$-sequence and
- $\kappa = \mathrm{cr}(E_\alpha)$

is bounded in δ.

Non-bland premice contain Woodin limit of Woodin cardinals.

Fully Background Certified Construction

In this case we say that, given a premouse of the form (J^E_α, F)m an extender F is certified iff there is a **V**-extender F^* such that F is the restriction of F^* to $\mathcal{M}$, more precisely,

$$F_a = F^*_a \cap \mathcal{M}$$

whenever $a \in [\alpha]^{<\omega}$.

The best result concerning fully background certified models is:

Theorem (Neeman 1996)

Assume in **V** *there is a Woodin cardinal which is a limit of Woodin cardinals. If all models $\mathcal{N}_\xi$ are bland then they are all countably iterable. Moreover, every Woodin cardinal in* **V** *is also Woodin in the fully background certified model* $\mathbf{L}[E] = \mathcal{N}_\infty$.

$\mathbf{K}^c$-CONSTRUCTION

Given a premouse $\mathcal{M} = (J^E_\alpha, F)$ where $\kappa = \mathrm{cr}(F)$ and a set $\mathcal{A} \subseteq \mathcal{P}(\bigcup_n [\kappa]^n) \cap \mathcal{M}$ we say that a structure (N, F^*) is an $\mathcal{A}$-certificate for $\mathcal{M}$ iff the following hold.

- N is a transitive model of a reasonable fragment of ZFC such that $V_\kappa \subseteq N$ and $N^\omega \subseteq N$.
- F^* is an N-extender with $\mathrm{cr}(F^*) = \kappa$.
- $V_{\nu_F} + 1 \subseteq \mathrm{Ult}(N, F^*)$ and $\mathrm{Ult}(N, F^*)^\omega \subseteq \mathrm{Ult}(N, F^*)$.
- F^* and F are compatible when restricted to $\mathcal{A}$, that is, $F^*_a \cap \mathcal{A} = F_a \cap \mathcal{A}$ whenever $a \in [\nu_F]^{<\omega}$.
- $E \restriction \alpha = i_{F^*} \restriction \alpha$.

$\mathcal{M}$ is countably certified iff for every countable $\mathcal{A} \subseteq \mathcal{P}(\bigcup_n [\kappa]^n) \cap \mathcal{M}$ there is an $\mathcal{A}$-certificate.

The model $\mathcal{N}_\infty$ built using countably certified $\mathcal{N}_\xi$ is called **countably certified $\mathbf{K}^c$**, or briefly $\mathbf{K}^c$. Models with higher degree of certification are also considered.

Theorem (Schimmerling-Steel)

If all models $\mathcal{N}_\xi$ of the $\mathbf{K}^c$-construction are tame then they are countably iterable, so the $\mathbf{K}^c$-construction converges.

Theorem (Steel, Cheap Covering)

Assume Ω is measurable. Let $\mathbf{K}^c = N_\Omega$. If $\mathbf{K}^c$ has no Woodin cardinals then $\kappa^{+\mathbf{K}^c} = \kappa^+$ for stationarily many regular (in fact inaccessible) $\kappa < \Omega$.

Theorem (Steel)

Assume there is no inner model with a Woodin cardinal. Then $\mathbf{K}^c$ is fully iterable.

Cheap covering combined with full iterability of $\mathbf{K}^c$ yield the true core model $\mathbf{K}$ build in the universe V_Ω where Ω is a measurable cardinal. Much less than a measurable is needed for this construction.

Definition

Let $S \subseteq \Omega$ be stationary. A class $\Gamma \subseteq \Omega$ is S-**thick** iff for all but nonstationarily many $\alpha \in S$:

- α is inaccessible and $\alpha^{+\mathbf{K}^c} = \alpha^+$.
- $\Gamma \cap (\alpha, \alpha^+)$ contains an α-club and $\alpha \in \Gamma$.
- α is not the critical point of an extender on the $\mathbf{K}^c$-sequence which is total.

Let

$$H(S) = \bigcap \{\mathrm{Hull}^{\mathbf{K}^c}(\Gamma) \mid \Gamma \text{ is } S\text{-thick}\}$$

and

$$\mathbf{K}(S) = \text{ the transitive collapse of } H(S).$$

Theorem

$K(S)$ does not depend on S.

Letting $\mathbf{K}$ be the common value of all models $K(S)$, we call $\mathbf{K}$ the **true core model**.

Theorem (Steel)

Assume Ω is measurable. Work in the universe V_Ω. Assuming there is no proper class inner model with a Woodin cardinal,

- $\mathbf{K}$ *is rigid, i.e. there is no nontrivial $j : \mathbf{K} \to \mathbf{K}$.*
- $\mathbf{K}$ *is generically absolute.*

Theorem (Mitchell,Schimmerling,Steel)

*Under the assumptions of the previous theorem, **K** satisfies weak covering. That is,*

$$\mathrm{cf}^{\mathbf{V}}(\kappa^{+\mathbf{K}}) \geq \mathrm{card}^{\mathbf{V}}(\kappa)$$

*whenever $\kappa \geq \omega_2^{\mathbf{V}}$ is a **K**-cardinal. In particular,*

$$\kappa^{+\mathbf{K}} = \kappa^{+}$$

*whenever κ is a singular cardinal in **V**.*

Theorem (Schimmerling,Z.)

In any fine structural extender model we have

$$\square_\kappa \iff \kappa \text{ is not subcompact.}$$

Corollary

Assume $\square_{\aleph_\omega}$ *fails. Then there is a proper class inner model with a Woodin cardinal.*

- Jensen and Steel found a construction of **K** that avoids the use of a measurable cardinal in **V**.
- The above corollary follows from the Jensen-Steel construction directly, but can be obtained also using the original Steel's construction which uses an outer measure on **V** instead of a measurable. This requires some additional work.
- Methods of the **core model induction** push this result much higher.

More on Iterability

Given a normal iteratin tree $\mathcal{T}$, we let

- $\delta(\mathcal{T}) = \sup\{\nu_{E_\alpha^\mathcal{T}} \mid \alpha < \mathrm{lh}(\mathcal{T})\}$
- $E(\mathcal{T}) = \bigcup\{E_\alpha^\mathcal{T} \restriction \nu_{E_\alpha^\mathcal{T}} \mid \alpha < \mathrm{lh}(\mathcal{T})\}$
- $\mathcal{M}(\mathcal{T}) = J_{\delta(\mathcal{T})}^{E(\mathcal{T})}$.

Theorem (Branch Uniqueness, Martin-Steel)

Assume $\mathcal{T}$ is a normal iteration tree with two cofinal well-founded branches $b \neq c$. Let $\mathcal{M}_b^\mathcal{T}, \mathcal{M}_c^\mathcal{T}$ be the two branch models. Then $\delta(\mathcal{T})$ is Woodin in $\mathcal{M}_b^\mathcal{T} \cap \mathcal{M}_c^\mathcal{T}$.

Theorem (Branch Existence, Martin,Steel,...)

Assume $\mathcal{M}$ is a countable mouse and $\sigma : \mathcal{M} \to \mathcal{N}_\xi$ is elementary. Let $\mathcal{T}$ be a countable iteration tree on $\mathcal{M}$. Then there is a **maximal** *branch b through $\mathcal{T}$ and and n-embedding σ' for appropriate n such that one of the following holds:*

- *There is no truncation point on b and then $\sigma' : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{N}_\xi$ is such that $\sigma = \sigma' \circ i_b$ where $\mathcal{M}_b^{\mathcal{T}}$ is the direct limit along b and $i_b : \mathcal{M} \to \mathcal{M}_b^{\mathcal{T}}$ is the branch embedding.*
- *There is a truncation point on b and then $\sigma' : \mathcal{M}_b^{\mathcal{T}} \to \mathcal{N}_\eta$ for some $\eta < \xi$.*

Iterability of $\mathbf{K}^c$

We assume $\mathbf{K}^c$ does not reach a Woodin cardinal and every set $a \subseteq \mathbf{On}$ has a sharp, that is, there is an iterable premouse $a^\# = (J^a_\tau, U)$ with properties analogous to those of $0^\#$.

Assuming $\mathcal{T}$ is a normal iteration tree on $\mathbf{K}^c$ of limit length, we may w.l.o.g. assume tht $\mathcal{T}$ is on $\mathbf{K}^c \,||\, \tau$ where τ is a successor of regular cardinal in $\mathbf{K}^c$.

- Let θ be regular large and X be a countable substructure of H_θ, H be the transitive collapse of X, $\sigma : H \to H_\theta$ be the inverse of the collapsing map, and $\sigma(\bar{\mathcal{T}}, \bar{\mathcal{M}}) = (\mathcal{T}, \mathcal{M})$.
- Let $b \in \mathbf{V}$ be a maximal well-founded branch of $\bar{\mathcal{T}}$ given by branch existence theorem.
- Using the branch uniqueness theorem plus the assumption that $\mathbf{K}^c$ does not reach a Woodin cardinal conclude that b must be cofinal.

- Let $g \in \mathbf{V}$ be generic for $\mathrm{coll}(\omega, \mathrm{trcl}(\bar{\mathcal{T}}))$ over H.
- $\bar{\mathcal{T}}^{\#} \in H$, so by Shoenfield absoluteness in $H[g]$ there is a cofinabranch $b \in H[g]$.
- Again using the branch uniqueness theorem and the the assumption that $\mathbf{K}^c$ does not reach a Woodin cardinal conclude that b must be unique.
- Then b is definable in $H[g]$ from the parameter $\bar{\mathcal{T}} \in H$.
- By the homogeneity of the collapse, $b \in H$.

Then $\sigma(b)$ is a cofinal well-founded branch through $\mathcal{T}$ by the elementarity of σ.