Mutual Stationarity Revisited

P.D.Welch, U. of Bristol, MM^{+70} Meeting

Mutual Stationarity

• A kind of stationarity in some $\mathcal{P}(\kappa)$ with constraints imposed on the stationary set.

• Or: as a property intermediate between Chang like properties and the Jónsson Property.

MS with constant cofinalities

Definition

• Let $\langle \kappa_\alpha | \alpha < \delta \rangle$ for $\omega \leq \delta < \kappa_0$ be an ascending sequence of regular cardinals. Let $\kappa_\delta =_{df} \sup \kappa_\alpha$.
• Let $\mathcal{S} = \langle S_\alpha | \alpha < \delta \rangle$ be a sequence of stationary sets, with $S_\alpha \subseteq \kappa_\alpha \cap cof(\lambda)$ $(\lambda < \kappa_0)$.

Then $\mathcal{S}$ is *mutually stationary* if for all algebras $\mathfrak{A}$ on κ_δ, there exists a sub-algebra $\mathfrak{B} \prec \mathfrak{A}$ with $\forall \alpha < \delta : \quad \kappa_\alpha \in \mathfrak{B} \to \sup(|\mathfrak{B}| \cap \kappa_\alpha) \in S_\alpha$.

(More generally . . .)

In general we don't need to stick to fixed cofinalities, and we can say with $\mathcal{S}$ any sequence of stationary sets, with $S_\alpha \subseteq \kappa_\alpha$ and $sup_\alpha \kappa_\alpha = \kappa_\delta$, if for any set X with $\kappa_\delta \subseteq X$ then $\mathcal{S}$ is *mutually stationary* iff

$$\{Y \subseteq X \mid Y \text{ `meets' } \mathcal{S}\}$$

is stationary in $\mathcal{P}(X)$.

[Here, as above, Y 'meets' $\mathcal{S}$ iff $\forall \alpha (\kappa_\alpha \in Y \rightarrow sup Y \cap \kappa_\alpha \in S_\alpha)$

Definition

Let $\lambda < \kappa_0$, $MS(\langle \kappa_\alpha \rangle_{\alpha<\delta}, \lambda)$ mean:
For *all* such sequences $\mathcal{S} = \langle S_\alpha | \alpha < \delta \rangle, S_\alpha \subseteq \kappa_\alpha \cap cof(\lambda), S_\alpha$ stationary, $\mathcal{S}$ is mutually stationary.

Theorem (Foreman-Magidor)

ZFC $\vdash MS(\langle \kappa_\alpha \rangle_{\alpha<\delta}, \omega)$ *for any sequence* $\langle \kappa_\alpha \rangle_{\alpha<\delta}$.

Theorem (Foreman-Magidor)

Assume $V = L$. *Then* $\forall k > 0 \ \neg MS(\langle \aleph_n \rangle_{n<\omega}, \omega_k)$

Their latter argument can be seen to be valid for fine-structural inner model of the form $L[\vec{E}]$.

With cardinals far apart

Theorem (Cummings-Foreman-Magidor)

If each κ_α is measurable, then any such $\mathcal{S}$ with $\delta < \kappa_0$ is mutually stationary. Indeed this holds even for $\mathcal{S}$ with possible varying cofinalities all less than some $\lambda < \kappa_0$

Theorem (Cummings-Foreman-Magidor)

Con(ZFC+"there exists a measurable cardinal"
$\Longrightarrow Con(ZFC + \exists\langle\kappa_n | n < \omega\rangle(\forall\lambda < \kappa_0\ MS(\langle\kappa_n\rangle_{n<\omega}, \lambda))$.
Again, the same is true with $\mathcal{S}$ of varying cofinalities in the conclusion.

Theorem (Koepke-W)

Con(ZFC+ "there exists a measurable cardinal")
$\Longleftarrow Con(ZFC + \exists\langle\kappa_n | n < \omega\rangle(\ MS(\langle\kappa_n\rangle_{n<\omega}, \omega_1))$.

Theorem (Koepke-W)

$ZFC + MS(\langle\aleph_n\rangle_{n<\omega}, \omega_1) \vdash$ *"There exists an inner model so that* $\forall k \exists r > k \; \{\kappa \mid o(\kappa) \geq \omega_k\}$ *is stationary below* ω_r*"*.

Theorem (Koepke)

$Con(ZFC+$ *"there exists a measurable cardinal"* $)$
$\Longrightarrow Con(ZFC + MS(\langle\aleph_{2n+1}\rangle_{n<\omega}, \omega_1)$.

More particular cofinality variation

Theorem (Liu-Shelah '97)

Let $A \subseteq \omega$ be infinite/coinfinite, with the property that $m \in A \rightarrow m+1 \notin A$. Let $1 \leq p, q < \omega$. Then:
$Con(ZFC + \exists^{\infty}$ measurable cardinals of $o(\kappa) = \omega_p + 1) \Rightarrow Con(ZFC+$ "$S =_{df}$

$$\left\{ X \in [\aleph_\omega]^{<\aleph_\omega} \mid \exists m_0 \forall m > m_0 \; cf(\sup(X \cap \omega_m) = \begin{array}{l} \omega_p : \; if \; m \in A \\ \omega_q : \; if \; m \notin A \end{array} \right\}$$

is stationary.")

Note:

- A is chosen first, and then assumptions are fitted up accordingly.
- The method of proof requires gaps are required in the patterns of cofinalities.

Theorem (Shelah)

$Con(ZFC + \exists^\infty \textit{ supercompacts}) \Rightarrow$
$\Rightarrow Con(ZFC + \forall f : \omega \to \{0, p\}\ (0 < p))$

$$\left\{X \in [\aleph_\omega]^{<\aleph_\omega} \;\middle|\; \forall m > 1\ cf(\sup(X \cap \omega_m) = \omega_{f(m)}\right\}$$

is stationary.")

Theorem (Magidor)

The following are equiconsistent:
(1) ZFC+ "If $A \subseteq \omega$ is infinite/coinfinite and $S =_{df}$

$$\left\{ X \in [\aleph_\omega]^{<\aleph_\omega} \;\middle|\; \forall m > 1 \; cf(\sup(X \cap \omega_m)) = \begin{array}{lcl} \omega & : & \textit{if } m \in A \\ \omega_1 & : & \textit{if } m \notin A \end{array} \right\}$$

is stationary";
(2) ZFC+ "there exists infinitely many measurable cardinals."

If we raise the cofinalities in Magidor's earlier Theorem:

Theorem (W)

ZFC $\vdash$ "If $A \subseteq \omega$ is infinite/coinfinite, $1 \leq p < q$ and

$$\left\{ X \in [\aleph_\omega]^{\omega_q} \;\middle|\; \forall m > max\{p,q\} \;\; cf(\sup(X \cap \omega_m)) = \begin{array}{l} \omega_p : \; if \; m \in A \\ \omega_q : \; if \; m \notin A \end{array} \right\}$$

is stationary, then there exists an inner model with infinitely many inaccessible cardinals below each of which the class of measurable cardinals κ with $o(\kappa) \geq \omega_p$ is Mahlo."

- This is close to an equiconsistency with the Liu-Shelah result.

Theorem (W)

Let $A \subseteq \omega$ and there are both infinitely many n with $n, n+1 \in A$, and infinitely many m with $m, m+1 \notin A$. Let $1 \leq p < q$, $S =$

$$\left\{ X \in [\aleph_\omega]^{\omega_2} \;\middle|\; \forall m > 2 \;\; cf(\sup(X \cap \omega_m)) = \begin{array}{l} \omega_p : \text{ if } m \in A \\ \omega_q : \text{ if } m \notin A \end{array} \right\}$$

*If S is stationary, then 0^{pistol} exists (*i.e.* a $\sharp$ for an inner model with a strong cardinal).*

- It suffices to have arbitrarily large successive pairs of suprema the same cofinality, as long as those cofinalities do not themselves settle down.

Still Open Question

Q. Is $ZFC + MS(\langle \aleph_n \rangle_{n<\omega}, \omega_1)$ consistent relative to large cardinals?

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