The Chang sharp operator

Bill Mitchell
University of Florida

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Outline

1. The Chang sharp operator (a compromise)
2. What is a sharp for a Chang model?
3. Ideas of the proof

~~THE~~ sharp of the Chang model

I was thinking of "the sharp" of the Chang model as the exact analog of Silver's sharp of L; Woodin has criticized this terminology because (among other things) the theory can vary among models with different sets of reals. It is certainly true that it differs in this respect from $0^\sharp$, Silver's sharp for L — which is a unique model.

A proposal

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Certainly the form of my sharp, depending as it does on a mouse over the reals, suggests that it is analogous to $\mathbb{R}^\sharp$, rather than to $0^\sharp$.

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These ideas can be extended to what Woodin has told me he regards as the "true sharp" — given by my construction, but with mice over a Woodin cardinal as well as the reals. This then is the analog of the $M_1^\sharp$ operator: $M_1^{\mathbb{C}\sharp}$.

The Chang model

The Chang model $\mathbb{C}$ is the smallest model of ZF containing all ordinals and all countable sequences of ordinals. Equivalently,

$$\mathbb{C} = L([\Omega]^{<\omega_1}).$$

Alternatively, it can be defined exactly like L, but using the infinitary language $\mathcal{L}_{\omega_1,\omega}$ instead of ordinary first order logic.

The Chang sharp

We assume that there is a $\mathbb{R}$-mouse M with a final extender E such that $\mathrm{len}(E) = (\kappa^{+(\omega+1)})^{M\|E}$ and $\mathrm{cf}(\mathrm{len}(E)) > \omega$.

Credo

M is the sharp $\mathbb{R}^{\mathbb{C}\sharp}$ for the Chang model.

What does this mean?

What does it mean that M is the sharp of $\mathbb{C}$?

$0^\sharp$, the sharp of L, can be characterized as the least mouse $M_L = (J_\alpha, \epsilon, U)$ having an ultrafilter U. Then

$$I_L = \{\, i^U_\alpha(\kappa) \mid \alpha \in \Omega \,\},$$

where $\kappa = \text{crit}(U)$, is the class of Silver indiscernibles for L.

In the same way, we can take the iterated ultrapower

$$i\colon M = M_0 \xrightarrow{i_\alpha = i_{\emptyset,\alpha}} M_\alpha \xrightarrow{i_{\alpha,\alpha+1}} M_{\alpha+1} = \text{ult}(M_\alpha, i_\alpha(E)) \xrightarrow{i_{\alpha+1,\Omega}} M_\Omega$$

giving

$$I = \{\, i^E_\alpha(\kappa) \mid \alpha \in \Omega \,\},$$

where $\kappa = \text{crit}(E)$. This will be the class of indiscernibles for $\mathbb{C}$.

Some more structure

Define a further iterated ultrapower k of M_Ω:

$$N_0 = M_\Omega \xrightarrow{k_\alpha} N_\alpha \xrightarrow{k_{\alpha,\alpha+1}} N_{\alpha+1} = \text{ult}(N_a, F_\alpha) \to N_\Omega$$

where F is the least extender in N_α such that neither of the following is true:

- $\text{cf}(\text{crit}(F)) = \omega$, and $\{\, \alpha' < \alpha \mid F = k_{\alpha',\alpha}(F_{\alpha'}) \,\}$ is unbounded in α.
- $\text{cf}(\text{len}(F)) \neq \omega$

Credo

$N_\Omega \| \Omega$ is the core model $K(\mathbb{R})^{\mathbb{C}}$ of $\mathbb{C}$.

$K(\mathbb{R})^{\mathbb{C}}$ is at least this large

Proposition

- *If there is no mouse such as M, then $K(\mathbb{R})^{\mathbb{C}}$ is an iterated ultrapower (without drops) of $K(\mathbb{R})$.*

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- *If there is no mouse such as* M*, then* $K(\mathbb{R})^{\mathbb{C}}$ *is an iterated ultrapower (without drops) of* $K(\mathbb{R})$*.*
- *If there is such a mouse, then all of the extenders in* $N_\Omega \| \Omega$ *are members of* $\mathbb{C}$*.*

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- For the first case, it is a result of Gitik that if $\mathrm{cf}(\mathrm{crit}(F)) = \omega$, $\{\, \alpha' < \alpha \mid F = k_{\alpha',\alpha}(F_{\alpha'}) \,\}$ is unbounded in α, and $\mathrm{len}(F) < (\kappa^{+(\omega+1)})^{\mathrm{ult}(N_\alpha, F)}$ then $F \in \mathbb{C}$.
- For the second, if $\mathrm{cf}(\mathrm{len}(F)) = \omega$ and every initial segment of F is in $\mathbb{C}$, then the fact that $\mathbb{C}$ is closed under countable sequences implies that $F \in \mathbb{C}$.

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[Question: In the first case, does the iteration described give $K(\mathbb{R})^{\mathbb{C}}$ exactly as in the case when M exists?]

Names for members of $\mathbb{C}$

Note that

- The mouse M is closed under countable sequences.
- The iteration $k\colon M_\Omega \to N_\Omega$ is definable in $M_\Omega \| \Omega$ (using a oracle telling which members of I have codfinality ω).

This means that every member of $\mathbb{C}$ has a name which is a member of M.

More specifically

Every countable sequecnce of ordinals can be written in the form $\hat{T}(b) = k \circ i(T)(b)$ where T is a composition of functions in M and new names for elements of the iteration k.

Indiscernibility

Definition

Call a closed countable increasing sequence b of members of I *suitable* if whenever b_i is a limit member of I of countable cofinality, $b \restriction i$ is cofinal in b_i.

If b and b' are two suitable sequences then we say $b \equiv b'$ if $\text{len}(b) = \text{len}(b')$, and for all but finitely many $i + 1 < \text{len}(b)$,

- $b_{i+1} = \min(I \setminus b_i + 1)$ iff $b'_{i+1} = \min(I \setminus b'_i + 1)$, and
- b_{i+1} is a limit member of I iff b'_{i+1} is a limit member of I. (Note that b_{i+1} and b'_{i+1} necessarily have uncountable cofinality.)

Theorem

Suppose that ϕ is a formula of set theory, $T(b)$ as name as described above, and b and b' are equivalent suitable sequences. Then $\mathbb{C} \models \phi(T(b)) \iff \phi(T(b'))$.

Corollaries

- Any map $\pi\colon I \to I$ such that $\pi(b) \equiv b$ for all suitable b defines an embedding $\pi^*\colon \mathbb{C} \to \mathbb{C}$.
- Any elementary embedding *preserving terms from k* is of this form.
- The mouse M, and hence the class I and set of true sentences, depends only on the set $\mathbb{R}$ of reals.
  (Of course it is possible to change the cofinality of limit points of I, which changes the class of suitable sequences.)
- Every member of $\mathbb{C}$ is definable in $\mathbb{C}$ from a countable subset of I. Hence, for example, every successor cardinal in $\mathbb{C}$ has cofinality ω_1.

Use forcing to define names

[Question: Is this really needed? After all, perfectly good names are already present in $M\|E$.]

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- Problem: You can't force over M, as $\rho_1^M = \mathbb{R}$.
- Idea: Let $\langle\, \delta_\alpha \mid \alpha < \omega_1 \,\rangle$ enumerate the ordinals $\delta < (\kappa^{+(\omega+1)})^M$ such that $M \| (E \restriction \delta) \models \delta = \kappa^{+(\omega+1)}$. Force with the extenders $E \| \delta_\alpha$ (as I did when using the stronger assumption $\text{len}(E) = \kappa^{+\omega_1}$).

- Problem: The ultrafilter $E_{\{\kappa^{+n} \mid n<\omega\} \cup \{\delta_\alpha\}}$ does not appear in $E{\restriction}\delta_\alpha$. Hence forcing over $E{\restriction}\delta_\alpha$ cannot give pointers to indiscernibles for all generators of E.
- Idea: Use Radin forcing over $\langle F \mid \mathrm{crit}(F) = \mathrm{crit}(E) \;\&\; F \lhd E{\restriction}\delta_\alpha \rangle$ instead of $E{\restriction}\delta_\alpha$.
- Problem: $M \| E_{\delta_\alpha}$ is not closed under countable subsets.
- Write $M \| E_{\delta_\alpha}$ as a direct sum of a sequence of transitive, countably closed, models Q_α with $|Q|_\alpha = \kappa^{+(1+2n)}$ and $\kappa^{1+2n} \subset Q_\alpha$, and force simultaneously over all of them.

Thank you for your attention. . .

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And to Menachem:

happy birthday and thank you.