High and low forcing

organized by
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Workshop Summary

Two separate classes of applications of forcing saw substantial progress in recent years: forcing in connection with infinitary combinatorics, and forcing with side conditions in connection with axioms which affect the real line. The former, which we call *high forcing*, addresses the long standing theme in set theory of

how much compactness can there be in universe. The latter, referred to as *low forcing*, provides methods for finally extending to $\aleph_2$

the rich structure theory that has been developed for objects of size $\aleph_1$ using

PFA and properness.

The goal of this workshop was to bring

together researchers involved with these aspects, so that people working on each may learn

about the progress and new techniques of the other. We invited a mix of participants working on the two areas, and also a

mix of young researchers and more established experts.

1. **The structure of the workshop**

Every morning there were two talks, on each of the two main themes. The talks varied from tutorial style to discussion of key new developments.

During the first two mornings Sinapova and then Unger went over a recent result that starting from large cardinals, one can obtain the tree property at $\aleph_{\omega^2+1}$ and $\aleph_{\omega^2+2}$ simultaneously, where $\aleph_{\omega^2}$ is strong limit. This result led to a discussion of open problems related to the long standing project of obtaining the tree property at every regular cardinal greater than $\aleph_1$. For example, can one get the tree property at $\aleph_{\omega+1}$ and failure of SCH at $\aleph_\omega$? Or, can the above result for $\aleph_{\omega^2}$ be extended to also getting the tree property below $\aleph_{\omega^2}$?

On Wednesday, Magidor discussed branch preservation lemmas, which constitute a central ingredient in proving the tree property in forcing extensions. On Thursday Fontanella talked about strengthenings of the tree property: *the strong tree property* and *the super tree property*, and what is currently known and what is open. These stronger properties capture the combinatorial nature of strongly compact and super compact cardinals, in the same way that the tree property does for weakly compact cardinals. Some interesting open problems here include whether one can have the super tree property at successor of a singular, and also whether the strong (or super) tree property implies SCH above.

On the side conditions side, on the first day Neeman talked about his forcing with side conditions of two types and higher analogues of PFA. Then on Tuesday Todorcevic discussed potential applications of higher analogues of PFA to set theoretic topology and infinitary combinatorics. He presented open problems, explained how they relate to analogues of PFA, and mentioned possible strategies for tackling them. He covered a very wide spectrum of topics, with questions on free sequences in set theoretic topology, cofinal types of directed sets of size $\aleph_2$, partition relations at ω_2, and gaps spectrum in $\mathcal{P}(\omega)/\text{fin}$. Continuing with the same purpose of exploring potential applications for higher analogues of PFA, Moore on Wednesday talked about a problem in strong homology, namely whether it is consistent that strong homology is additive for certain classes of metric spaces. The question has connections to set theory and to PFA. Moore explained what the connections are, and how analogues of PFA may help settling the question. He also talked on potential applications of countably closed analogues of PFA to questions on Aronszajn and Countryman lines, and explained the forcing used for some of the existing applications of PFA in this area.

On Thursday, Gitik spoke on higher analogues of semi-proper forcing axioms. He presented an outline of his poset of finite structures with pistes using models of size $\leq \aleph_2$, and an application obtaining the consistency of $\text{Fr}(\aleph_3, \aleph_1)$. On Friday, Velickovic spoke on his setup for iterating using finite side conditions. The setup is similar to Neeman's, but applies more easily in several contexts. Also on Friday, Aspero talked about adapting symmetric systems of side conditions to the two-type context, with a tentative application in set theoretic topology. Also, on one of the afternoons Neeman gave an impromptu presentation of his new theorem that it is consistent to have Baumgartner's principle at $\aleph_2$, namely that any two $\aleph_2$-dense subsets of $\mathbb{R}$ are order isomorphic.

The talks were followed by group activities in the

afternoon. During the afternoon of the first day, we had a problem session. We compiled a list of problems, that fell roughly into three categories: questions related to the low forcing, the high forcing and to both. Then people divided into groups according to

interest in specific problems among these.

1. **Group discussions** Below is a brief description of the work of each group.

Is it consistent that $lim^p \mathbb{A} = 0$ *for all* p*?*

This was one of the questions raised in Moore's lecture. It looks possible that higher analogues of PFA can provide a positive answer. The group spent some time working through the definitions and internalizing the problem, as the notions involved are outside the expertise of most set theorists. Moore presented what he knew about the problem, including results by his student Jeff Bergalf showing connections to forcing axioms, and listed some test questions.

Can we prove from large cardinals the bounded forcing axiom for Namba type forcings?

Bounded forcing axioms are roughly equivalent to Σ_1 absoluteness over $H(\omega_2)$, and under CH, Σ_1 over $H(\omega_2)$ is Σ^2_2. So the motivation behind this question is roughly whether under CH, we can get Σ^2_2 absoluteness for Namba forcing.

(The broader issue of full Σ^2_2 absoluteness from large cardinals is a prominent open question, under combinatorial assumptions, for example $\diamond_{\omega_1}$. A combinatorial assumption is necessary since the existence of a

Suslin tree is Σ^2_2 statement.)

The group observed the following. A result of Ketchersid, Larson and Zapletal showed that a "Namba style" forcing can change the value of δ^1_2 and thus change a weakly homogeneously Suslin equivalence relation. Here the notion of "Namba style" is a tree forcing, but still far from classical Namba. On the other hand, in the paper "Definable Counterexamples" by Foreman and Magidor it is shown that Namba forcing changes a weakly homogeneously Suslin equivalence relation if and only if that equivalence relation already has ω_2 many classes.

This means that anything that can be said with weakly homogeneously Suslin equivalence relations is absolute for Namba forcing under CH. While the original questions remains open, these results provide partial answers.

Can one prove in ZFC that there is a proper $\aleph_2$-c.c. poset, with $\aleph_2$ dense subsets which cannot be simultaneously met by a filter? In other words, is the forcing axiom $\mathrm{FA}_{\aleph_2}$*(proper $\aleph_2$-c.c.) inconsistent?*

The group looked at various principles to check if they can be derived from the forcing axiom and lead to a contradiction. Perhaps the most promising approach involved uniformization principles on ω_2. For now the group showed that several candidate principles, including a weak uniformization principle, follow from a consistent strengthening of the Aspero-Mota axiom $\mathrm{MA}^{1.5}_{\aleph_2}$. It is not yet clear whether any of these can be adjusted to an inconsistent consequence of $\mathrm{FA}_{\aleph_2}$(proper $\aleph_2$-c.c.).

Can we obtain the super tree property at the successor of a singular cardinal? The natural strategy is to take infinitely many super compact cardinals and look at the successor of their limit. By a result of Fontanella, the strong tree property holds in this case. The group looked at possible ways to strengthen the argument and a number of related open problems.

Does the strong tree property at κ imply SCH above κ? This question tests how much of the large cardinal properties does the strong tree property capture. The motivation for a positive answer is that having a strongly compact cardinal κ implies that SCH holds above κ. On the other hand, a positive answer will mean that one cannot obtain the strong tree property at every regular cardinal, since that would require many violations of SCH. A strategy for a negative answer is to look at constructions where the strong tree property holds at $\aleph_2$ in possible combinations with Prikry forcing to violate SCH. In particular, this question lies at the intersection of the high and the low forcing.

Two of the workshop participants, Hayut and Unger, made key partial progress in the direction of a possible negative answer to this question. They obtained the consistency of the strong tree property at each $\aleph_n$, for $1 < n < \omega$ together with weak square at $\aleph_\omega$. Weak square is an incompactness type principle and usually holds in the presence of failure of SCH. Their preprint is available on arXiv.

Is it consistent to have the tree property at $\aleph_{\omega+1}$ together with stationary reflection at $\aleph_\omega$? The group working on this problem analyzed the known models for obtaining the tree property at $\aleph_{\omega+1}$, and discovered that reflection fails in all of them in a way that seems central to each construction. The analogous statement at $\aleph_{\omega^2}$ is indeed consistent, but at this time it is not known how to obtain similar constructions at $\aleph_\omega$. While hard, the problem is very interesting and some of the participants in this group will continue to work on it.

Tree property at κ^{++} for a singular κ with side conditions.

This problem lies at the intersection of both areas of the workshop. The group solved it by combining using properness of side conditions together with branch preservation lemmas for Prikry forcing. Currently there is a draft with the proof. The group also began talking about forcing the tree property at successive cardinals using side conditions.

Is it consistent that the tree property holds at $\aleph_2$ and there is a saturated ideal on $\aleph_2$?

This question was raised by Eskew. Gitik and Magidor suggested using a variation of Mitchell forcing together with an almost huge cardinal. It is still not clear if this would work, but it presents a promising strategy.

There were some other problems that groups briefly discussed, which were too ambitious to attack in one week: can one turn $\aleph_{\omega+1}$ into $\aleph_2$? Is it consistent that every poset either adds a real or collapses a cardinal? Although no progress was made on these questions, bringing them up highlighted the overall motivation of the workshop themes.

2. **Conclusion**

The workshop was a success. Participants were quickly brought up to speed on recent new developments in forcing related to infinitary combinatorics and

side conditions. A lot of open problems were discussed and strategies were outlined for problems that were previously intractable. The workshop provided opportunities for new collaborations and also for young researchers

to interact with experts from both of the main areas.