Classical infinitary logics

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1. $\bigwedge_{\alpha < \gamma} \phi_\alpha$, $\bigvee_{\alpha < \gamma} \phi_\alpha$

2. $\forall x_\gamma \phi$, $\exists x_\gamma \phi$

(Where $x_\gamma = \{x_\alpha : \alpha < \gamma\}$)
Hilbert-style system enough to derive a completeness theorem for $\text{Set}$-valued models. Featuring the following axiom schemata, for each $\gamma < \kappa$: 

1. **Classical distributivity:**
$$\bigwedge_{i < \gamma} \bigvee_{j < \gamma} \psi_{ij} \rightarrow \bigwedge_{f \in \gamma} \bigvee_{i < \gamma} \psi_{if}(i)$$

2. **Classical dependent choice up to $\gamma$** ($\text{DC}_\gamma$):
$$\bigwedge_{\alpha < \gamma} \forall \beta < \alpha \ x_\beta \exists x_\alpha \psi_\alpha \rightarrow \exists \alpha < \gamma x_\alpha \bigwedge_{\alpha < \gamma} \psi_\alpha$$

(for disjoint $x_\alpha$ and such that no variable in $x_\alpha$ is free in $\psi_\beta$ for $\beta < \alpha$).

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$$\bigwedge_{\alpha<\gamma} \forall \beta<\alpha \exists \mathbf{x}_\beta \exists \mathbf{x}_\alpha \psi_\alpha \rightarrow \exists \alpha<\gamma \mathbf{x}_\alpha \bigwedge_{\alpha<\gamma} \psi_\alpha$$

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As an example, it follows by results of de Jongh (1980) that the set of non-equivalent propositional formulas of $\kappa$-propositional intuitionistic logic with two atoms has cardinality $\kappa$. 
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The most useful way of axiomatizing $\kappa$-first-order intuitionistic logic is through the use of sequent calculus. We denote by the sequent $\phi \vdash x \psi$ in context $x$ the sentence $\forall x (\phi(x) \rightarrow \psi(x))$. The axioms and rules of $\kappa$-first-order intuitionistic systems consist of the following:

- **Structural rules:**
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- **Structural rules:**
  - Identity axiom:
    $$\phi \vdash_\kappa \phi$$
  - Substitution rule:
    $$\phi \vdash_\kappa \psi$$
    $$\frac{\phi[\kappa] \vdash_\kappa \psi[\kappa]}{\phi[\kappa] \vdash_\kappa \psi[\kappa]}$$
    where $\psi$ is a string of variables including all variables occurring in the string of terms $\kappa$. 
Cut rule:

\[ \phi \vdash x \psi \quad \psi \vdash x \theta \]

\[ \phi \vdash x \theta \]
Equality axioms:

\[ T \vdash_{x} x = x \]
\(\kappa\)-first-order systems

- Equality axioms:
  \[ (x = y) \land \phi \vdash_{z} \phi[y/x] \]

where \(x, y\) are contexts of the same length and type and \(z\) is any context containing \(x, y\) and the free variables of \(\phi\).
Conjunction axioms and rules:

\[ \bigwedge_{i < \gamma} \phi_i \vdash x \phi_j \]

\[ \{ \phi \vdash x \psi_i \}_{i < \gamma} \]

\[ \phi \vdash x \bigwedge_{i < \gamma} \phi_i \]
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Disjunction axioms and rules:

\[ \phi_j \vdash x \bigvee_{i<\gamma} \phi_i \]

\[ \{ \phi_i \vdash x \theta \}_{i<\gamma} \]

\[ \bigvee_{i<\gamma} \phi_i \vdash x \theta \]
$\kappa$-first-order systems

- Implication rule:

\[
\frac{\phi \land \psi \vdash_{x} \theta}{\phi \vdash_{x} \psi \rightarrow \theta}
\]

Existential rule:

\[
\phi \vdash_{x, y} \psi \quad \exists y \phi \vdash_{x} \psi
\]

where no variable in $y$ is free in $\psi$.

Universal rule:

\[
\phi \vdash_{x, y} \psi \quad \phi \vdash_{x} \forall y \psi
\]

where no variable in $y$ is free in $\phi$. 

Christian Espíndola (Stockholm University) Completeness of infinitary intuitionistic logics September 27th, 2016
κ-first-order systems

- Implication rule:

\[
\phi \land \psi \vdash_x \theta \\
\phi \vdash_x \psi \to \theta
\]

- Existential rule:

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\phi \vdash_{\mathbf{x},\mathbf{y}} \psi \\
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where no variable in $\mathbf{y}$ is free in $\phi$. 
Transfinite transitivity:

\[
\phi_i \vdash y_i \bigvee_{j \in \gamma^{\beta+1}, j \upharpoonright \beta = i} \exists x_j \phi_j \quad \beta < \gamma, \ i \in \gamma^\beta
\]

\[
\phi_i \not\vdash y_i \bigwedge_{\alpha < \beta} \phi_i \upharpoonright \alpha \quad \beta < \gamma, \ \text{limit } \beta, \ i \in \gamma^\beta
\]

\[
\phi_\emptyset \vdash y_\emptyset \bigvee_{i \in \gamma^\gamma} \exists_{\beta < \gamma} x_i \upharpoonright_{\beta+1} \bigwedge_{\beta < \gamma} \phi_i \upharpoonright \beta
\]

for each cardinal \( \gamma < \kappa \), where \( y_i \) is the canonical context of \( \phi_i \), provided that, for every \( i \in \gamma^{\beta+1} \), \( FV(\phi_i) = FV(\phi_i \upharpoonright \beta) \cup x_i \) and \( x_i \upharpoonright_{\beta+1} \cap FV(\phi_i \upharpoonright \beta) = \emptyset \) for any \( \beta < \gamma \), as well as \( FV(\phi_i) = \bigcup_{\alpha < \beta} FV(\phi_i \upharpoonright \alpha) \) for limit \( \beta \). Note that we assume that there is a fixed well-ordering of \( \gamma^\gamma \) for each \( \gamma < \kappa \).
A Kripke model for pure first-order logic over $\Sigma$ is a quadruple $\mathcal{B} = (K, \leq, D, \Vdash)$, where $(K, \leq)$ is a tree, $D$ is a set-valued functor on $K$ and the forcing relation $\Vdash$ is a binary relation between elements of $K$ and sentences of the language with constants from $\bigcup_{k \in K} D(k)$, defined for atomic formulas $\phi$ with the conditions that $k \not\Vdash \perp$ and that $k \Vdash \phi(d) \iff l \Vdash \phi(D_{kl}(d)))$ for $d \subseteq D(k)$, and recursively extended to arbitrary formulas as follows:
A Kripke model for pure first-order logic over $\Sigma$ is a quadruple $\mathcal{B} = (K, \leq, D, \models)$, where $(K, \leq)$ is a tree, $D$ is a set-valued functor on $K$ and the forcing relation $\models$ is a binary relation between elements of $K$ and sentences of the language with constants from $\bigcup_{k \in K} D(k)$, defined for atomic formulas $\phi$ with the conditions that $k \not\models \perp$ and that

$k \models \phi(d) \iff l \models \phi(D_{kl}(d)))$ for $d \subseteq D(k)$, and recursively extended to arbitrary formulas as follows:

- $k \models \bigwedge_{i < \gamma} \phi_i(d) \iff k \models \phi_i(d)$ for every $i < \gamma$
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- $k \Vdash \bigwedge_{i < \gamma} \phi_i(d) \iff k \Vdash \phi_i(d)$ for every $i < \gamma$
- $k \Vdash \bigvee_{i < \gamma} \phi_i(d) \iff k \Vdash \phi_i(d)$ for some $i < \gamma$
\( \kappa \)-Kripke semantics

\[ k \models \phi(d) \rightarrow \psi(d') \iff \forall k' \geq k (k' \models \phi(D_{kk'}(d)) \Rightarrow k' \models \psi(D_{kk'}(d')) \)
\( \kappa \)-Kripke semantics

- \( k \models \phi(d) \rightarrow \psi(d') \iff \forall k' \geq k (k' \models \phi(D_{kk'}(d)) \implies k' \models \psi(D_{kk'}(d'))) \)
- \( k \models \exists x \phi(x, d) \iff \exists e \subseteq D(k) (k \models \phi(e, d)) \)
\( k \models \phi(d) \rightarrow \psi(d') \iff \forall k' \geq k (k' \models \phi(D_{kk'}(d)) \implies k' \models \psi(D_{kk'}(d'))) \)

\( k \models \exists x \phi(x, d) \iff \exists e \subseteq D(k) (k \models \phi(e, d)) \)

\( k \models \forall x \phi(x, d) \iff \forall k' \geq k \forall e \subseteq D_{k'} (k' \models \phi(e, D_{kk'}(d))) \)
A Kripke model for a theory $C$ is a Kripke model forcing all the axioms of the theory.
Connections with large cardinal axioms:

1. The distributivity property suggests to study the case of inaccessible $\kappa$. 
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Consider a language containing one propositional variable $P_a$ for every node $a$ in a tree of height $\kappa$ and levels of size less than $\kappa$. 
Large cardinals

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- $P_a \land P_b \vdash \bot$ for each pair $a \neq b \in L_\alpha$ and each $\alpha < \kappa$
Large cardinals

Connections with large cardinal axioms:

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- \( P_a \land P_b \vdash \bot \) for each pair \( a \neq b \in L_\alpha \) and each \( \alpha < \kappa \)
- \( P_a \vdash P_b \) for each pair \( a, b \) such that \( a \) is a successor of \( b \)
Then:

- Under the assumption of completeness for Kripke semantics, every such tree has a cofinal branch: $B = \{ a : p \vDash P_a \}$. 
Then:

- Under the assumption of completeness for Kripke semantics, every such tree has a cofinal branch: \( B = \{ a : p \models P_a \} \).

We can show, on the other hand, that the hypothesis of weak compactness is enough to derive a completeness theorem for \( \kappa \)-first-order theories of cardinality at most \( \kappa \) with respect to \( \kappa \)-Kripke semantics.
Then:

- Under the assumption of completeness for Kripke semantics, every such tree has a cofinal branch: \[ B = \{ a : p \models P_a \}. \]

We can show, on the other hand, that the hypothesis of weak compactness is enough to derive a completeness theorem for \( \kappa \)-first-order theories of cardinality at most \( \kappa \) with respect to \( \kappa \)-Kripke semantics. We have:

**Theorem (E., 2016)**

Let \( \kappa \) be a weakly (resp. strongly) compact cardinal. Then \( \kappa \)-first-order theories of cardinality at most \( \kappa \) (resp. of arbitrary cardinality) are semantically complete with respect to \( \kappa \)-Kripke models.
If we remove the restriction that the models are Kripke models and allow for models in more general categories, it is possible to avoid the use of weakly (resp. strongly) compact cardinals.
If we remove the restriction that the models are Kripke models and allow for models in more general categories, it is possible to avoid the use of weakly (resp. strongly) compact cardinals. The correct property that we have to look at is an exactness property of the category of sets which reflects the transfinite transitivity rule and that we also call transfinite transitivity: the transfinite composites of jointly covering $\kappa$-families of morphisms are jointly covering.
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**Theorem (E., 2016)**

Let $\kappa$ be an inaccessible cardinal. Then $\kappa$-first-order logic is sound and complete with respect to models in Grothendieck toposes with the transfinite transitivity property.
Applications

It is possible to improve the completeness theorem restricting ourselves to the case of Kripke models over trees. This semantics allows to prove the following:

Corollary

Let $\kappa$ be a weakly compact cardinal. Then $\kappa$-first-order logic over a language without function symbols has the disjunction property: if $\bigvee_{i<\gamma} \phi_i$ is provable (in the empty theory) then, for some $i$, $\phi_i$ is already provable.

Corollary

Let $\kappa$ be a weakly compact cardinal. Then $\kappa$-first-order logic over a language without function symbols and at least one constant symbol has the existence property: if $\exists x \phi(x)$ is provable (in the empty theory) then, for some constants $c$, $\phi(c)$ is already provable.
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**Corollary**

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Another application is a characterization of weakly (resp. strongly) compact cardinals in terms of the existence of certain $\kappa$-complete $\kappa$-prime filters in $\kappa$-complete, $\kappa$-distributive lattices. The correct notion of distributivity is the given by the transfinite transitivity rule applied in the propositional case:
Another application is a characterization of weakly (resp. strongly) compact cardinals in terms of the existence of certain $\kappa$-complete $\kappa$-prime filters in $\kappa$-complete, $\kappa$-distributive lattices. The correct notion of distributivity is the given by the transfinite transitivity rule applied in the propositional case:
Given a cardinal $\kappa$, $\kappa$-complete lattices are lattices that have joins and meets of less than $\kappa$ elements (in particular they are bounded).
Another application is a characterization of weakly (resp. strongly) compact cardinals in terms of the existence of certain $\kappa$-complete $\kappa$-prime filters in $\kappa$-complete, $\kappa$-distributive lattices. The correct notion of distributivity is the given by the transfinite transitivity rule applied in the propositional case:

Given a cardinal $\kappa$, $\kappa$-complete lattices are lattices that have joins and meets of less than $\kappa$ elements (in particular they are bounded). We say that a lattice is $\kappa$-distributive if for every $\gamma < \kappa$ and all elements $\{a_f : f \in \gamma^\beta, \beta < \gamma\}$ such that

$$a_f \leq \bigvee_{g \in \gamma^{\beta+1}, g|_\beta = f} a_g$$

for all $f \in \gamma^\beta, \beta < \gamma$, and
Applications

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for all limit \( \beta \), \( f \in \gamma^\beta \), \( \beta < \gamma \), we have that

\[ a_\emptyset \leq \bigvee_{f \in \gamma^\beta} \bigwedge_{\beta < \gamma} a_f|_{\beta}. \]
\[ a_f = \bigwedge_{\alpha < \beta} a_{f|\alpha} \]

for all limit $\beta$, $f \in \gamma^\beta$, $\beta < \gamma$, we have that

\[ a_\emptyset \leq \bigvee_{f \in \gamma^\beta} \bigwedge_{\beta < \gamma} a_{f|\beta}. \]

In words, if for each $\gamma < \kappa$ we have in the lattice a tree of type $\gamma^\gamma$ (whose partial order is the reverse order of the lattice) in which every element is below the join of its immediate successors and where at limits levels every element is the meet of its predecessors, then the root element is below the join over all cofinal branches of the elements that are intersections of elements in each cofinal branch of the tree.
$\kappa$-distributivity implies the distributivity property:

$$\bigwedge_i \bigvee_j a_{ij} = \bigvee_f \bigwedge_{i<\gamma} a_{if(i)} \quad (1)$$

and in fact one can see that in a $\kappa$-complete Boolean algebra they are equivalent properties.
$\kappa$-distributivity implies the distributivity property:

$$\bigwedge \bigvee a_{ij} = \bigvee \bigwedge a_{if(i)} \quad (1)$$

and in fact one can see that in a $\kappa$-complete Boolean algebra they are equivalent properties.

However $\kappa$-distributivity is stronger in general; for example the interval $[0, 1]$ satisfies (1) but is not $\kappa$-distributive.
A \( \kappa \)-complete filter in the lattice is a filter such that whenever \( a_i \in F \) for every \( i \in I \), \( |I| < \kappa \), then \( \bigwedge_{i \in I} a_i \in F \). A \( \kappa \)-prime filter in the lattice is a filter \( F \) such that whenever \( \bigvee_{i \in I} a_i \) is in \( F \) for \( |I| < \kappa \) then \( a_i \in F \) for some \( i \in I \).
A \( \kappa \)-complete filter in the lattice is a filter such that whenever \( a_i \in \mathcal{F} \) for every \( i \in I \), \( |I| < \kappa \), then \( \bigwedge_{i \in I} a_i \in \mathcal{F} \). A \( \kappa \)-prime filter in the lattice is a filter \( \mathcal{F} \) such that whenever \( \bigvee_{i \in I} a_i \) is in \( \mathcal{F} \) for \( |I| < \kappa \) then \( a_i \in \mathcal{F} \) for some \( i \in I \). We have now:

**Theorem (E., 2016)**

Let \( \kappa \) be an inaccessible cardinal. Then \( \kappa \) is weakly (resp. strongly) compact if and only if every \( \kappa \)-complete, \( \kappa \)-distributive lattice of cardinality at most \( \kappa \) (resp. of arbitrary cardinality) has a \( \kappa \)-complete, \( \kappa \)-prime filter.
Future work

The following are further lines of work to pursue:

Establish conceptual completeness theorems, or to what extent the category of models determines the theory.

Study the case of finite-quantifier theories over $L_{\kappa,\omega}$.

Call $\kappa$ a Heyting cardinal if $\kappa$-first-order theories of cardinality strictly less than $\kappa$ are complete for $\kappa$-Kripke semantics. Determine its strength within the large cardinal hierarchy.
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