The semantics of algebraic quantum mechanics and the role of model theory.

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B. Zilber, *The semantics of the canonical commutation relations*

arxiv.org/abs/1604.07745
Geometric dualities

Affine commutative $\mathbb{C}$-algebra

$R = \mathbb{C}[X_1, \ldots, X_n]/I$

Commutative unital $C^*$-algebra

$A$

Affine reduced $k$-algebra

$R = k[X_1, \ldots, X_n]/I$

Complex algebraic variety

$\mathbf{V}_R$

Compact topological space

$\mathbf{V}_A$

The geometry of $k$-definable points, curves etc of an algebraic variety $\mathbf{V}_R$

$\ldots$

$\ldots$
Why model theory?

These are syntax – semantics dualities.
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In general the syntax may come with a topology!
Why model theory?

These are syntax – semantics dualities.

In general the syntax may come with a topology! (as in $C^*$-algebras).
Zariski geometries as geometric semantics

The structure $V = (V, L)$ with a topology on its cartesian powers is said to be (Noetherian) Zariski if it satisfies

- Closed subsets of $V^n$ are exactly those which are $L$-positive-quantifier-free definable.
- The projection of a closed set is quantifier-free definable (positive quantifier-elimination).
- A *good* dimension notion on closed subsets is given.
- ...
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- \( \ldots \)

**Theorem.** Noetherian Zariski geometries allow elimination of quantifiers and are stable of finite Morley rank.
Further geometric dualities

Affine commutative \( \mathbb{C} \)-algebra \( R \)
Commutative \( C^* \)-algebra \( A \)
Affine reduced \( k \)-algebra \( R \)
\(*\)-algebra \( A \) at roots of unity
Weyl-Heisenberg algebra
\( \langle Q, P : \ QP - PQ = i\hbar \rangle \)

Complex algebraic variety \( V_R \)
Compact topological space \( V_A \)
The \( k \)-definable structure on an algebraic variety \( V_R \)
Zariski geometry \( V_A \)
Further geometric dualities

Affine commutative $\mathbb{C}$-algebra $R$

Commutative $C^*$-algebra $A$

Affine reduced $k$-algebra $R$

*-algebra $A$ at roots of unity

Weyl-Heisenberg algebra $\langle Q, P : QP - PQ = i\hbar \rangle$

Complex algebraic variety $V_R$

Compact topological space $V_A$

The $k$-definable structure on an algebraic variety $V_R$

Zariski geometry $V_A$

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The semantics of algebraic quantum mechanics and the role of model theory.
A noncommutative duality Theorem

For the category of algebras “at roots of unity” there is an equivalence of categories

\[ A_V \leftrightarrow V_A. \]

\( A_V \) – co-ordinate algebra, \( V_A \) – Zariski geometry.
A non-commutative example “at root of unity”

Non-commutative 2-torus $V_A$ at $\epsilon = e^{2\pi i \frac{m}{N}}$
has co-ordinate ring $A = \langle U, V : U^* = U^{-1}, V^* = V^{-1}, UV = \epsilon VU \rangle$
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Points $\alpha$ on the torus have structure of an $N$-dim Hilbert space $V_\alpha$ with a distinguished system of canonical orthonormal bases

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This does not allow a $C^*$-algebra (Banach algebra) setting. Also does not fit a model-theoretic construction.

On suggestion of Weyl and following Stone – von Neumann Theorem replace the Weyl-Heisenberg algebra by the category of Weyl $*$-algebras

$$A_{a,b} = \left\langle U^a, V^b : U^a V^b = e^{2\pi i ab} V^b U^a \right\rangle, \quad a, b \in \mathbb{R}.$$ 

Think:

$$U^a = e^{iaQ}, \quad V^b = e^{\frac{2\pi}{\hbar} ibP}.$$
\[ QP - PQ = i\hbar \]

and physics assumes that \( Q \) and \( P \) are self-adjoint.

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Ignore the non-rational ones. Replace “the algebra given by \(QP - PQ = i\hbar\)” by the category \(\mathcal{A}_{\text{fin}}\) of rational Weyl algebras

\[
A_{a,b} = \left\langle U^a, V^b : U^a V^b = e^{2\pi iab} V^b U^a \right\rangle, \quad a, b \in \mathbb{Q}
\]

with morphisms = embeddings.
Categories $A_{\text{fin}}$ and $V_{\text{fin}}$

Note:

$$A_{a,b} \hookrightarrow A_{c,d} \text{ iff } \exists n, m \in \mathbb{Z} \; cn = a \& dm = b$$
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In the dual category $\mathcal{V}_{\text{fin}}$ morphisms of Zariski geometries

$$\mathcal{V}_{A_{a,b}} \rightarrow \mathcal{V}_{A_{c,d}}$$

are certain relations that make each such pair a Zariski geometry again.
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Note: $\mathcal{V}_{A_{a,b}}$ is interpretable in $\mathcal{V}_{A_{c,d}}$ but not the other way round.
The duality functor $A \mapsto V_A$ can be interpreted as defining a sheaf of Zariski geometries over the lattice $A_{\text{fin}}$.
How noncommutative $\mathbf{V}_{A_{\frac{1}{m}, \frac{1}{n}}}$ deforms into $\mathbf{V}_{A_{\frac{1}{\mu}, \frac{1}{\mu}}}$
Not all elements of the non-standard algebra $A_{\frac{1}{\mu}, \frac{1}{\mu}}$ can be given a limit meaning!
Not all elements of the non-standard algebra $A_{1, \mu, \mu}$ can be given a limit meaning!

Not all elements of the non-standard $V_{A_{1, \mu, \mu}}$ can be given a limit meaning!
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The subalgebra of operators which survive the limit

$$A_* \subset A_{\frac{1}{\mu}, \frac{1}{\mu}}$$

acts on the substructure

$$V_* \subset V_{A_{\frac{1}{\mu}, \frac{1}{\mu}}}$$

which survive the limit.
The space of states $\mathcal{S}$. 

The structure $\mathcal{S}$ is a homomorphic image of $V_*$ under a homomorphism called $\lim$, 

$$\lim : V_* \rightarrow \mathcal{S}, \quad *Q \rightarrow \mathbb{R}.$$ 

This can also be classified as a generalisation of the

- standard part map,
- specialisation,
- residue map.
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Can be explained in terms of *positive model theory.*
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Can be explained in terms of *positive model theory*.

$\mathcal{S}$ is a symplectic space with a vector field and Fourier transforms on it.

See e.g. G. Lion and M. Vergne, *The Weil Representation, Maslov Index, and Theta Series* Birkhauser 1980
Operators acting on $S$

Remark. Operators $U_\mu^1$ and $V_\mu^1$ “do not survive” $\lim$.
Operators acting on $S$

Remark. Operators $U^1_\mu$ and $V^1_\mu$ “do not survive” $\lim$. We define (interdefinably) in each member $V_{a,b}$ of the ultraproduct:

$$Q := \frac{U^a - U^{-a}}{2ia}, \quad P := \frac{V^b - V^{-b}}{2ib}$$

in accordance with

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Then for any vector $e$ of norm 1,

$$(QP - PQ)e = i\hbar e + (s_1 - s_2)$$

where $s_1, s_2$ are vectors of norm 1 which depend on $a, b$ and $e$. 
Remark. Operators $U_1^\mu$ and $V_1^\mu$ “do not survive” $\lim$. We define (interdefinably) in each member $V_{a,b}$ of the ultraproduct:

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Then for any vector $e$ of norm 1,

$$(QP - PQ)e = i\hbar e + (s_1 - s_2)$$

where $s_1, s_2$ are vectors of norm 1 which depend on $a, b$ and $e$. Under the $\lim s_1 - s_2$ vanishes!

So, in the space of states: $QP - PQ = i\hbar I$. 

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Observables

A relation, a function or an operator which is defined on the multisorted structure $\mathcal{V}_{\text{fin}}$ is said to be observable if it is respected by $\lim$ and the image in $\mathcal{S}$ is non-trivial. In particular, an observable relation is Zariski closed.
Observables

A relation, a function or an operator which is defined on the multisorted structure $\mathcal{V}_{\text{fin}}$ is said to be **observable** if it is respected by $\lim$ and the image in $S$ is non-trivial. In particular, **an observable relation is Zariski closed**.

Examples.

- Operators $P$ and $Q$.
- $|\langle w_1|w_2 \rangle|_{\text{Dir}} := \mu \cdot |\langle w_1|w_2 \rangle|$, renormalised probability.
- ...
Gauss quadratic sums survive the limit

\[ \sum_{n=0}^{N-1} e^{2\pi i \frac{n^2}{N}} = e^{-i \frac{\pi}{4} \sqrt{N}} \]

if \( N \) is even, e.g. \( N = \mu^2 \).
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This allows us to calculate (approximate) oscillating Gaussian integrals, for \(a \in \mathbb{Q}\),

\[\int_{\mathbb{R}} e^{iax^2} \, dx\]

and eventually for \(a \in \mathbb{R}\).
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Here, for \( a = \frac{k}{m} \) it is crucial that \( \mu \) is divisible by \( k \).
Example of calculation. Quantum harmonic oscillator.

The Hamiltonian:

\[ H = \frac{1}{2} (P^2 + Q^2) \]
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The time evolution operator:

\[ K^t = K^t_H := e^{-\frac{iHt}{\hbar}} , \quad t \in \mathbb{R}. \]

This "induces" the automorphism of the category of algebras

\[ U^a \mapsto e^{-\frac{2\pi a^2 \sin t \cos t}{2}} U^a \sin t \ V^a \cos t \]
\[ V^a \mapsto e^{\frac{2\pi a^2 \sin t \cos t}{2}} U^{-a \cos t} \ V^a \sin t \]

(in \( V_* \) we only consider \( t \) such that \( \sin t, \cos t \in \mathbb{Q} - \{0\} \)).

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Example. Quantum harmonic oscillator.

Write $|x\rangle$ for eigenvectors of $Q$ with eigenvalues $x \in \mathbb{R}$.

Then the kernel of the Feynman propagator is calculated in \( \lim V_* \) as

$$\langle x_1 | K^t x_2 \rangle_{\text{Dir}} = \sqrt{\frac{1}{2\pi i \hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2) \cos t - 2x_1 x_2}{2\hbar \sin t}.$$
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The trace of $K^t$,

$$\text{Tr}(K^t) = \int_{\mathbb{R}} \langle x | K^t x \rangle = \frac{1}{\sin \frac{t}{2}}.$$
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Note that in terms of conventional mathematical physics we have calculated \( \text{Tr}(K^t) = \sum_{n=0}^{\infty} e^{-it(n+\frac{1}{2})} \), a non-convergent infinite sum.
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a non-convergent infinite sum.
An analogy: p-adic and motivic integration

\[ \int_{A(\mathbb{P})} |f(z)|^t \, dz = g(q, t) \]

where \( \mathbb{P} \) is a locally compact non-archimedean field, \( q = p^n \) is the cardinality of the residue field of \( \mathbb{P} \), \( t \in \mathbb{R} \) and \( g \) is a nice function which does not depend on \( \mathbb{P} \).
An analogy: $p$-adic and motivic integration

\[ \int_{A(\mathcal{F})} |f(z)|^t dz = g(q, t) \]

where $\mathcal{F}$ is a locally compact non-archimedean field, $q = p^n$ is the cardinality of the residue field of $\mathcal{F}$, $t \in \mathbb{R}$ and $g$ is a nice function which does not depend on $\mathcal{F}$.

In the formulae above $x$ appears at any high enough level of $V_{\frac{1}{m}, \frac{1}{m}}$ of the category as

\[ q = p^{n^2} = e^{ix^2} ; \quad p = e^{\frac{2\pi i}{m^2}} \]
\[ \langle x_1 | K^t x_2 \rangle_{\text{Dir}} = \int_{\mathbb{R}} f(y)^t \, dy \]

\[ g(q, t) = \sqrt{\frac{1}{2\pi i\hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2) \cos t - 2x_1x_2}{2\hbar \sin t}. \]
Conclusions

- The resulting semantics of the canonical commutation relation $QP - PQ = i\hbar$ suggests that the universe of quantum mechanics is a huge finite space of states.
- The known list of observables can be explained by the semantics.
- The calculations of key integrals can be reduced to calculations of finite sums without invoking continuous limits.