A Minimal Predicative Framework for Formalizing Mathematics

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Joint work with Arnon Avron

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Outline

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1. Motivation
2. Adjusting the Ideal Calculus
3. The Basic Framework
4. A Better Way to Handle Infinity

Formalized Mathematics

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Informal Math

- Type-free set theory is viewed by most mathematicians as the foundation of the mathematics they practice.
- Thus it seems to be the most natural framework for MKM.

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Computer-aided Math

- Employs sophisticated type theories, with different notions of constructibility and computation (Coq, Nuprl, Agda).
- Uses fragments of higher-order logic (Isabelle/HOL).

Goals

- Present a set-theoretical formal framework with the properties:
  - Able to handle set theories of different strength
  - Has computational content
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- Present a set-theoretical formal framework with the properties:
  - Able to handle set theories of different strength
  - Has computational content
  - Uses abstractions (static language)
  - Suitable for mechanization
- Identify the minimal ontological commitments which are indispensable to the formalization of applicable mathematics.
- Demonstrate how large portions of mathematics can be developed within the framework.

Outline

The Ideal Calculus via Safety

Extensionality

$$\forall z(z \in x \leftrightarrow z \in y) \to x = y$$

$$(t = \{x \mid x \in t\})$$

The Comprehension Scheme

$$\forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi)$$

$$(t \in \{x \mid \varphi\} \leftrightarrow \varphi[t/x])$$

The Regularity Schema ($\in$-induction)

$$(\forall x(\forall y(y \in x \to \varphi[y/x]) \to \varphi)) \to \forall x\varphi$$

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$$\forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi), \quad \textit{when } \varphi \textit{ is safe w.r.t. } \{x\}$$

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What is Safety?

- Safety is a relation between formulas and sets of variables.
- Differences between set theories is mainly due to differences in their notions of safety.
- The safety relation should be decidable, and defined syntactically, in a static way.

Predicativity

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Principle I

Higher-order constructs, such as sets or functions, are acceptable only when introduced through non-circular definitions.

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Principle II

The natural numbers sequence is a basic well understood mathematical concept, and as a totality it constitutes a set.

- Predicativism does not sanction power-sets of infinite sets. Thus, in predicative mathematics apparently all sets are countable.

Principles and Examples

The intuitive meaning of a safety relation

$$\varphi\,(x_1, ..., x_n, y_1, ..., y_k) \succ \{x_1, ..., x_n\}$$

- The collection $\{\langle x_1, ..., x_n\rangle\,|\varphi\}$ is an "acceptable" set for all "acceptable" values of $y_1, ..., y_k$.
- It can be constructed from these values.

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$$\varphi(y_1, \ldots, y_k) \succ \emptyset$$

- The collection $\{\langle y_1, \ldots, y_k\rangle \in z|\varphi\}$ is an "acceptable" set for all "acceptable" values of z.

Outline

Language

Terms:

- Every variable is a term (HF is a term)
- $\{ x \mid \varphi \}$ is a term if φ is a formula s.t. $\varphi \succ \{x\}$

Formulas:

- If t and s are terms then $t = s$ and $t \in s$ are formulas.
- If φ and ψ are formulas, and x is a variable, then $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $\exists x \varphi$ ($(\varphi \rightarrow \psi)$, $\forall x \varphi$) are formulas.

Safety Relation $\succ$:

- $\varphi \succ \emptyset$ if φ is atomic.
- $\varphi \succ \{x\}$ if $\varphi \in \{x = t, t = x, x \neq x, x \in t\}$, and $x \notin Fv(t)$.
- $\neg\varphi \succ \emptyset$ if $\varphi \succ \emptyset$.
- $\varphi \vee \psi \succ X$ if $\varphi \succ X$ and $\psi \succ X$.
- $\varphi \wedge \psi \succ X \cup Y$ if $\varphi \succ X$, $\psi \succ Y$ and $Y \cap Fv(\varphi) = \emptyset$.
- $\exists y \varphi \succ X - \{y\}$ if $y \in X$ and $\varphi \succ X$.

The System(s) *RST*

Extensionality

$$\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y$$

The Comprehension Scheme

$$\forall x(x \in \{x \mid \varphi\} \leftrightarrow \varphi)$$

The Restricted Regularity Schema ($\in$-induction)

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In the intuitionistic case we add:

Restricted Excluded Middle

$$\varphi \vee \neg\varphi, \ \text{ for } \varphi \succ \emptyset$$

Static Extensions by Definitions

Predicates If $Fv(\varphi) \subseteq \{x_1, ..., x_n\}$ and $\varphi \succ \emptyset$, then one may add a new n-ary relation symbol P, and the axiom:

$$\forall x_1 ... \forall x_n . P(x_1, ..., x_n) \leftrightarrow \varphi$$

- Instead of $P(t)$, it is more convenient to use $t \in \{\, x \mid \varphi \,\}$.
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Functions If t is a term, the one may introduce a new n-ary function symbol F together with the axiom:

$$\forall x_1 ... \forall x_n . F(x_1, ..., x_n) = t$$

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All extensions are static

Handling Infinity

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The System(s) RST_{HF}

$$\emptyset \in HF$$
$$\forall x \forall y \, (x \in HF \wedge y \in HF \to x \cup \{y\} \in HF)$$
$$\forall y \, (\emptyset \in y \wedge \forall v, w \in y . v \cup \{w\} \in y \to HF \subseteq y)$$

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$$\mathbb{N} := \{x \in HF \mid Ord(x)\}$$

- The minimal model of RST_{HF} is J_2.
- Definitional theory – $a \in J_2$ iff it is definable by a closed term of RST_{HF}.

Mathematics within the Minimal Framework

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Cons

- The reals form a proper class
- Involves a lot of coding
- Schematic theory

Working with Classes - Coding

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Example

$U \subseteq \mathbb{R}$ is *open* if there is a set $u \subseteq \mathbb{Q} \times \mathbb{Q}^{+}$ s.t.

$$U = \bigcup_{\langle p,q\rangle \in u} B_q(p) = \{r \in \mathbb{R} \mid \exists p, q\,(\langle p,q\rangle \in u \wedge |r-p| < q)\}$$

We call u a code for U.

Working with Classes — Schemes

- The basic language is extended by incorporating free class variables, $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z} \ldots$
- These variables stand for arbitrary class terms, and they may only appear as *free* variables.
- Intuitively, $\psi(\boldsymbol{X})$ is interpreted as "for any given class X, $\psi(X)$ holds".
- $\psi(\boldsymbol{X})$ is a schema, where $\boldsymbol{X}$ is a "place holder" whose substitution instances are the official formulas of the language.

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Example — LUB

$$(\boldsymbol{U} \subseteq \mathbb{R} \land \boldsymbol{U} \neq \emptyset \land \mathit{separ}(\boldsymbol{U}) \land \exists w \in \bar{\mathbb{R}}.\mathit{bound}_{\boldsymbol{U}}(w)) \rightarrow$$
$$\exists v \in \mathbb{R}\,(\mathit{bound}_{\boldsymbol{U}}(v) \land \forall w \in \bar{\mathbb{R}}\,(\mathit{bound}_{\boldsymbol{U}}(w) \rightarrow v \leq w))$$

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Ancestral Logic (AL)

Ancestral Logic $=$ FOL $+$ a transitive closure operator.

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Ancestral Logic = FOL + a transitive closure operator.

The Language

The language $\mathcal{L}_{AL}$ is defined as $\mathcal{L}_{FOL}$, with the following additional clause:

- $(TC_{x,y}\varphi)(s,t)$ is a formula, when φ is a formula, x, y are distinct variables, and s, t are terms.

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- $(TC_{x,y}\varphi)(s,t)$ is a formula, when φ is a formula, x, y are distinct variables, and s, t are terms.

The intended meaning of $(TC_{x,y}\varphi)(s,t)$

$$\varphi(s,t) \vee \exists w_1 . \varphi(s,w_1) \wedge \varphi(w_1,t)$$
$$\vee \exists w_1 \exists w_2 . \varphi(s,w_1) \wedge \varphi(w_1,w_2) \wedge \varphi(w_2,t) \vee \ldots$$

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- AL does have natural sound finitary proof systems.

Induction rule for TC

$$\frac{\Gamma, \varphi(x,y) \Rightarrow \psi(x,y), \Delta \quad \Gamma, \psi(u,v), \psi(v,w) \Rightarrow \psi(u,w), \Delta}{\Gamma, (TC_{x,y}\varphi)(s,t) \Rightarrow \psi(s,t), \Delta}$$

provided $x, y \notin FV(\Gamma \cup \Delta)$ and $u, v, w \notin FV(\Gamma \cup \Delta \cup \{\varphi, \psi\})$

Ancestral Logic — Properties

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provided $x, y \notin FV(\Gamma \cup \Delta)$ and $u, v, w \notin FV(\Gamma \cup \Delta \cup \{\varphi, \psi\})$

- The Gentzen-type system for classical *AL* is complete for an appropriate Henkin-type semantics.

Using *AL* in the Current Framework

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- RST^{AL} is obtained from RST by the addition of the appropriate TC inference rules.

Properties of RST^{AL}

- Expressive Power:

$$\mathbb{N} := \{x \mid x = \emptyset \vee \exists y. y = \emptyset \wedge (TC_{x,y} x = y \cup \{y\})(x, y)\}$$

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- The minimal model of RST^{AL} is $J_{\omega^\omega} = L_{\omega^\omega}$.

- Definitional theory – $a \in J_{\omega^\omega}$ iff it is definable by a closed term of RST^{AL}.

  In particular, J_2, J_3, ... , J_ω, J_{ω^2}, J_{ω^3}, ... are definable in RST^{AL}.

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  - This set does not include, of course, all the "real" real numbers.
- Involves no coding.
- No need to use class terms or class variables, thus all claims are fully provable in the system.

Further Work

- Develop larger portions of predicative mathematics
  - More analysis, topology, algebra.
- Going beyond predicativity
  - Practice reverse mathematics to determine what set theoretical assumptions are essential for various fragments of mathematics.
- The dynamic approach
  - Replace the static approach with a dynamic one, in which both being a legal term and equality of terms are major judgments.
- Implementation
  - Implement and test the formal systems developed, and then use them for concrete applications.

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Thank You