PFA implies a class of hereditarily Lindelöf spaces are D spaces

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Question

Is every hereditarily Lindelöf (HL) space a D space?

Connection with basis problem

Fact

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- *X has HL inner topology;*
- *X has a "nice" outer topology.*

Open set mapping axiom

OSM_X asserts that for any open set mapping N, there is a partition of X into countably many pieces such that for each x, y in the same piece, either $x \in N(y)$ or $y \in N(x)$.

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Theorem (PFA)

If there is a counterexample to the basis problem, there is a sub-Sorgenfrey one.

OSM

Some HL spaces.

	Sorgenfrey line	Moore's L space	P-Wu's L group
Menger	No	Yes	Yes/No
D space	Yes	Yes	Yes
OSM	Yes	No	No

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Theorem (PFA)

If X has a HL inner topology, then OSM_X holds.

σ-ideal and reflection induced by ONA

Fix HL space X and ONA $N, N' : X \to \tau$ such that for any $x \in X$, $\overline{N'(x)} \subset N(x)$. There is a natural σ-ideal

$$\mathcal{I} = \langle Y \subset X : \text{ for any } x, y \in Y, x \in N(y) \vee y \in N(x) \rangle_\sigma .$$

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Let $\mathcal{M}$ be a countable elementary submodel and

$$T_x = \{y \in A : x \notin N(y) \wedge y \notin N(x)\}.$$

(N, N') reflects in $\mathcal{M}$ if for any $A \notin \mathcal{I}$, there are $x \in A$ and $B \in \mathcal{M}$ such that $A \cap B \neq \emptyset$ and $B \subset T_x$.

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Theorem (PFA)

If for any ONA N, there are corresponding $N', \mathcal{M}$ such that (N, N') reflects in $\mathcal{M}$, then OSM_X.

Inner topology

For a space (X, τ) and a countable collection $\mathcal{C} \subset P(X)$, the inner topology $(X, \tau^{I,\mathcal{C}})$ induced by $\mathcal{C}$ is the topology with base $\{\{x\} \cup O^{I,\mathcal{C}} : x \in O, O \text{ is open}\}$ where $O^{I,\mathcal{C}} = \bigcup\{C \in \mathcal{C} : C \subset O\}$.

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Sketch of proof. Fix $A \notin \mathcal{I}$. If we can find $x, y \in A$ such that $x \in C \subset (\overline{N'(y)}^c)^{I,\mathcal{C}}$ and $y \in C' \subset (\overline{N'(x)}^c)^{I,\mathcal{C}}$, then take $B = \{z \in C' : C \subset (\overline{N'(z)}^c)^{I,\mathcal{C}}\} \in \mathcal{M}$ and we are done.

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Fix a countable elementary submodel $\mathcal{N}$ containing A.

Recall $T_x = \{y \in A : x \notin N(y) \wedge y \notin N(x)\}$.

Fix $z \in A \setminus \mathcal{N}$ such that T_z is uncountable.

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$z \in \overline{N'(x)}^c$ for any $x \in C \cap T_z \subset C \cap A$. By HL, we can find $x \in C \cap A \cap \mathcal{N}$ such that $z \in \overline{N'(x)}^c$.

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Then $Y = \{y \in A : C \subset (\overline{N'(y)}^c)^{I,\mathcal{C}}$ and $y \in \overline{N'(x)}^c\}$ is uncountable by elementarity.

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By HL again, we can find $y \in Y$ such that $y \in (\overline{N'(x)}^c)^{I,\mathcal{C}}$

Questions

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Note OSM_X implies HL.

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Thank you!