# Lecture 4: Univalent foundations

Nicola Gambino

School of Mathematics
University of Leeds

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# Univalent foundations

The simplicial model of type theory suggest to:

1. use type theory as a language for speaking about homotopy types,
2. develop mathematics using this language; in particular

$$\text{sets} \ =_{\mathrm{def}}\ \text{discrete homotopy types},$$

3. add axioms to type theory motivated by homotopy theory

# Overview

Part I: Homotopy theory in type theory

Part II: The univalence axiom in simplicial sets

Part III: Univalent foundations

## Part I: Homotopy theory in type theory

# Contractibility

**Definition.** We say that a type $A$ is **contractible** if the type

$$\mathsf{iscontr}(A) =_{\mathrm{def}} (\Sigma x : A)(\Pi y : A)\mathsf{Id}_A(x, y)$$

is inhabited.

**Examples**

- The singleton type 1.
- For all $a : A$, the type
$$(\Sigma x : A)\mathsf{Id}_A(a, x)$$

# Weak equivalences in type theory

Let $f : A \to B$.

- The **homotopy fiber** of $f$ at $y : B$ is the type

$$\mathrm{hfiber}(f, y) =_{\mathrm{def}} (\Sigma x \in A)\, \mathrm{Id}_B(fx, y)\,.$$

- We say that $f : A \to B$ is a **weak equivalence** if the type

$$\mathrm{isweq}(f) =_{\mathrm{def}} (\Pi y : B)\, \mathrm{iscontr}\,(\mathrm{hfiber}(f, y))$$

  is inhabited.

**Note.** For $A, B : \mathrm{type}$, there is a type

$$\mathrm{Weq}(A, B) = (\Sigma f : A \to B)\, \mathrm{isweq}(f)$$

of weak equivalences from $A$ to $B$.

**Note.** The identity $1_A : A \to A$ is a weak equivalence.

## Homotopies

**Definition.** Let $f, g : A \to B$. A **homotopy** $\alpha : f \sim g$ is an element

$$\alpha : (\Pi x : A)\, \mathrm{Id}_B(fx, gx)$$

**Proposition.** Weak equivalences are homotopy equivalences, i.e. if $f : A \to B$ is a weak equivalence then there is $g : B \to A$ and homotopies

$$\alpha : g \circ f \sim 1_A\,, \quad \beta : f \circ g \sim 1_B\,.$$

## Part II: The univalence axiom

# Univalent types

Let

$$x : A \vdash B(x) : \mathrm{type}$$

be a dependent type.

For $x, y \in A$, we have:

- the type of paths from $x$ to $y$, $\mathrm{Id}_A(x, y)$
- the type $\mathrm{Weq}(B(x), B(y))$ of weak equivalences from $B(x)$ to $B(y)$.

**Note.** We have

$$j_{x,y} : \mathrm{Id}_A(x, y) \to \mathrm{Weq}(B(x), B(y))$$

**Definition.** We say $x : A \vdash B(x) : \mathrm{type}$ is **univalent** if $j_{x,y}$ is an equivalence for all $x, y : A$.

# The univalence axiom

Recall

$$\frac{a : \mathsf{U}}{\mathsf{El}(a) : \mathsf{type}}$$

**Univalence Axiom.** The dependent type

$$x : \mathsf{U} \vdash \mathsf{El}(x) : \mathsf{type}$$

is univalent.

Explicitly, we have equivalences

$$j_{x,y} : \mathsf{Id}_{\mathsf{U}}(x, y) \to \mathsf{Weq}(\mathsf{El}(x), \mathsf{El}(y))$$

for all $x, y : \mathsf{U}$.

**Slogan**

- Isomorphism is equality

# Univalence in simplicial sets (I)

The rich structure of **SSet** allows us to ‘internalize’ a lot of constructions.

Let $p : B \to A$ a fibration. There exists a fibration

$$(s, t) : \mathrm{Weq}(A) \to A \times A$$

such that the fiber over $(x, y)$ is

$$\mathrm{Weq}(A)_{x,y} = \{ w : B_x \to B_y \mid w \in \mathbf{Weq} \}$$

**Note.** Given $p : B \to A$, we have

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Weq}(B) \\ {\scriptstyle i}\downarrow & \overset{j_p}{\nearrow} & \downarrow{\scriptstyle (s,t)} \\ \mathrm{Path}(A) & \longrightarrow & A \times A \end{array}$$

## Univalent fibrations

**Definition.** A fibration $p\colon B \to A$ is said to be **univalent** if

$$j_p\colon \mathrm{Path}(A) \to \mathrm{Weq}(B)$$

is a weak equivalence.

**Idea.** Weak equivalences between fibers are 'witnessed' by paths in the base.

**Proposition.** A fibration $p\colon B \to A$ is **univalent** if and only if for every fibration $q\colon D \to C$, the space of squares

$$\begin{array}{ccc} D & \xrightarrow{s} & B \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ C & \xrightarrow[t]{} & A \end{array}$$

such that

$$D \to C \times_A B$$

is a weak equivalence, is either empty or contractible.

**Idea.** Essential uniqueness of $s$, $t$ (if they exist).

# Univalence in simplicial sets (II)

**Theorem.** The fibration $\pi : \tilde{U} \to U$ is univalent.

We consider the diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad w \quad} & \mathrm{Weq}(U) \\ & {\scriptstyle \Delta} \searrow \quad \swarrow {\scriptstyle (s,t)} & \\ & U \times U & \end{array}$$

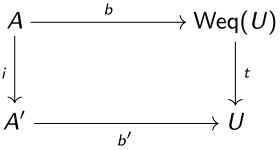
and show $w \in \mathbf{Weq}$. By composing with $\pi_2 : U \times U \to U$, we get

$$\begin{array}{ccc} U & \xrightarrow{\quad w \quad} & \mathrm{Weq}(U) \\ & {\scriptstyle 1_U} \searrow \quad \swarrow {\scriptstyle t} & \\ & U & \end{array}$$

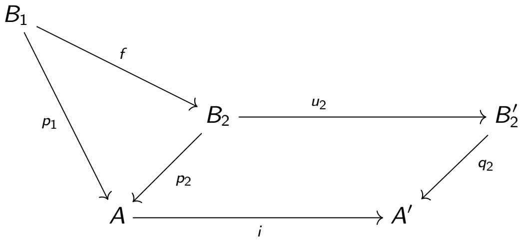
Hence, it suffices to show that $t \in \mathbf{Weq}$.

Since $t \in \mathbf{Fib}$, we show that $t \in \mathbf{Weq} \cap \mathbf{Fib}$. Suffices $i \pitchfork t$, for all $i \in \mathbf{Cof}$.

So, we need to prove the existence of a diagonal filler in a diagram of the form

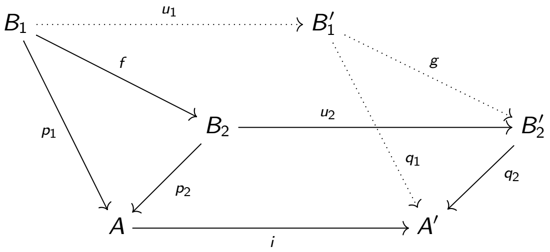



where $i \in \mathbf{Cof}$. By the definition of $t$, such a square amounts to a diagram




where $p_1, p_2, q_2 \in \mathbf{Fib}$, $f \in \mathbf{Weq}$ and $i \in \mathbf{Cof}$.

The required diagonal filler amounts to a diagram of the form




where $q_1$ is a fibration, $g$ is weak equivalence and all squares are pullbacks.

This is also established via the theory of minimal fibrations.

# A relative consistency result

The definition of the simplicial model is carried out within

- ZFC + 2 inaccessible cardinals

**Theorem.** The extension of Martin-Löf type theory with the univalence axiom is consistent relatively to ZFC + 2 inaccessible cardinals.

**Questions**

- Can this result be improved?
- Can the simplicial model be redeveloped constructively, so as to give relative consistency with respect to Martin-Löf type theory?

**Note** Recent work on models in cubical sets.

## Part III: Mathematics in univalent type theories

# Homotopy levels in type theory

**Definition**

- A type $A$ has **homotopy level** 0 if it is contractible.

**Definition**

- A type $A$ has **homotopy level** 1 if for all $x, y : A$, the type $\mathrm{Id}_A(x, y)$ has h-level 0, i.e. it is contractible.

**Note:**

$$
\begin{aligned}
A \text{ has h-level } 1 \quad &\Leftrightarrow \quad \text{for all } x, y : A,\ \mathrm{Id}_A(x, y) \text{ is contractible} \\
&\Leftrightarrow \quad \text{if } A \text{ is inhabited, then it is contractible} \\
&\Leftrightarrow \quad \text{either } 0 \text{ or } 1
\end{aligned}
$$

Types of h-level 1 will be called **h-propositions**.

**Example:** $\mathrm{isweq}(f)$ is a h-proposition.

# Sets

**Definition**

- A type $A$ has **homotopy level** 2 if for all $x, y : A$, the type $\mathsf{Id}_A(x, y)$ has h-level 1, i.e. it is an h-proposition.

**Note:**

$$
\begin{aligned}
A \text{ has h-level } 2 \quad &\Leftrightarrow \quad \text{for all } x, y : A,\ \mathsf{Id}_A(x, y) \text{ is a h-proposition} \\
&\Leftrightarrow \quad A \text{ is discrete}
\end{aligned}
$$

Types of h-level 2 will be called **h-sets**.

**Idea.** This hierarchy can be extended inductively:

| | | | | |
|---|---|---|---|---|
| Level | 0 | 1 | 2 | 3 |
| Types | $*$ | 0 , 1 | sets | groupoids |
| Mathematics | – | “logic” | “algebra” | “category theory” |

# Remarks on the univalence axiom

**Theorem.** The univalence axiom implies Function Extensionality, i.e.

$$(\Pi x : A)\mathsf{Id}_B(fx, gx) \to \mathsf{Id}_{A\to B}(f, g)$$

**Theorem.** Assuming the univalence axiom, the type universe U is not an h-set.

**Proof.** Assume U is a h-set.

Then $\mathsf{Id}_{\mathsf{U}}(a, b)$ would be a h-proposition, i.e. either empty or contractible.

By univalence, so would $\mathsf{Weq}\big(\mathsf{El}(a), \mathsf{El}(b)\big)$.

But, for example, $\mathsf{Weq}\big(\mathsf{Bool}, \mathsf{Bool}\big)$ is neither empty nor contractible.

**Corollary.** Univalence is not valid in the types-as-sets model.

# Further aspects

**Other topics**

- Synthetic homotopy theory
- Higher inductive types
- Homotopy-initial algebras in type theory
- Models of type theory in cubical sets
- Models of type theory with uniform fibrations
- Relation with the theory of $(\infty, 1)$-categories

**Open problems**

- Constructivity of the simplicial model
- Direct definition of $(\infty, 1)$-category in type theory

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**Type theory and set theory**

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**Homotopical algebra**

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# References (II)


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**Univalent Foundations**

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