Generic absoluteness for Chang models

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New directions in the higher infinite,
10–14 July 2017

Starting point

Theorem

(Woodin) (ZFC) Suppose there is a supercompact cardinal or a proper class of Woodin cardinals. Then for every set–forcing $\mathcal{P}$ and every $\mathcal{P}$–generic G over V,

$$(\mathcal{C}_\omega^V; \in, r)_{r \in \mathbb{R}^V} \equiv (\mathcal{C}_\omega^{V[G]}; \in, r)_{r \in \mathbb{R}^V}$$

Given a cardinal κ, $\mathcal{C}_\kappa$ denotes the κ–Chang model $L([\mathrm{Ord}]^{\leq\kappa})$ (= the $\subseteq$–minimal inner model of ZF closed under $\leq\kappa$–sequences).

How to extend this?

I: Dropping AC

[This first part is joint work with Asaf Karagila.]

Definition

(Woodin) (ZF) Given an ordinal α, a cardinal κ is V_α–*supercompact* iff there is some β and some elementary embedding $j : V_\beta \longrightarrow M$ with critical point κ and such that ${}^{V_\alpha}M \subseteq M$ and $j(\kappa) > \alpha$.

κ is *supercompact* iff it is V_α–supercompact for all α.

If AC holds, this definition coincides with the usual definition.

Theorem

(ZF + DC)

(1) *Suppose there is, for every Π_2 definable closed and unbounded class of ordinals C, a $V_{\kappa+2}$–supercompact cardinal κ such that $\kappa \in C$. Suppose $\mathcal{P}$ is a set–forcing preserving DC. Then, for every $\mathcal{P}$–generic G over V,*

$$(\mathcal{C}_\omega^V; \in, r)_{r \in \mathbb{R}^V} \equiv_{\Sigma_2} (\mathcal{C}_\omega^{V[G]}; \in, r)_{r \in \mathbb{R}^V}$$

(2) *Suppose there is, for every closed and unbounded class of ordinals C, a $V_{\kappa+2}$–supercompact cardinal κ such that $\kappa \in C$. Suppose $\mathcal{P}$ is a set–forcing preserving DC. Then, for every $\mathcal{P}$–generic G over V,*

$$(\mathcal{C}_\omega^V; \in, r)_{r \in \mathbb{R}^V} \equiv (\mathcal{C}_\omega^{V[G]}; \in, r)_{r \in \mathbb{R}^V}$$

Theorem

(ZF + DC) Suppose there is a supercompact cardinal κ. Suppose $\mathcal{P}$ is a set–forcing preserving DC. Then, for every $\mathcal{P}$–generic G over V,

- $(L(\mathbb{R})^V; \in, r)_{r\in\mathbb{R}^V} \equiv (L(\mathbb{R})^{V[G]}; \in, r)_{r\in\mathbb{R}^V}$, *and*
- *the Axiom of Determinacy holds in* $L(\mathbb{R})^{V[G]}$.

Main ingredient of these proofs:

Theorem

(Woodin) (ZF + DC) Suppose κ is a supercompact cardinal. Then there is a σ–closed forcing $\mathbb{Q} \subseteq V_\kappa$ such that

- $V_\kappa[G] \models \text{ZFC}$
- *For all $\delta \leq \lambda < \kappa$, if δ is V_λ–supercompact in V, then δ remains V_λ–supercompact in $V_\kappa[G]$.*

Selling this

Most of analysis (including descriptive set theory) can be naturally carried out in ZF + DC.

This makes it natural to enquire whether generic absoluteness for the theory of an inner model containing analysis can be derived over the basis theory ZF+DC (Is full AC an overkill, as of course is CH, or $\neg$ CH, etc?)

Limitations

Σ_2–generic absoluteness for the ω–Chang model may fail for set–forcing destroying DC:

Suppose $V \models$ ZFC. Let M be the symmetric submodel of a generic extension of V by the Levy collapse $\mathcal{P} = \mathrm{Coll}(\omega_1, <\aleph_{\omega_1})$ given by the filter of subgroups of permutations of $\mathcal{P}$ generated by $\mathcal{G}_\alpha = \{\pi \in \mathrm{Aut}(\mathcal{P}) \,:\, \pi(p) = p \text{ for all } p \in \mathrm{Coll}(\omega_1, \alpha)\}$ (for $\alpha < \aleph_{\omega_1}$).

As in the classical Feferman–Levy model for ω_1 singular, $\aleph_{\omega_1}^V = \omega_2^M$ and hence M thinks that ω_2 has cofinality ω_1.

M satisfies DC since $\mathcal{P}$ is σ–closed in V.

All supercompact cardinals κ in V remains supercompact in M, essentially since $|\mathcal{P}| < \kappa$ in V.

But then generic absoluteness for the Σ_2–theory of the Chang model has to fail between M and the extension of M by the collapse of ω_1^M $(= \omega_1^V)$ to ω with finite conditions since $\mathcal{C}_\omega^M$ thinks that ω_1 is regular whereas $\mathcal{C}_\omega^{M^{\mathrm{Coll}(\omega,\,\omega_1)}}$ thinks that ω_1 $(= \omega_2^M)$ is singular.

Proof sketch of part (1) of first theorem

Given a set X, $\mathrm{Coll}(\omega, {<}X)$ is the set, ordered by reverse inclusion, of all finite functions p such that $\mathrm{dom}(p) \subseteq \omega \times X$ and $p(i, x) \in x$ for all $(i, x) \in \mathrm{dom}(p)$.

Forcing with $\mathrm{Coll}(\omega, {<}X)$ makes every member of X countable.

Lemma

For every set X, $\mathrm{Coll}(\omega, {<}X)$ is homogeneous in the sense that for all $p, p' \in \mathrm{Coll}(\omega, {<}X)$ there are $q \le p$ and $q' \le p'$ such that $\mathrm{Coll}(\omega, {<}X) \restriction q$ and $\mathrm{Coll}(\omega, {<}X) \restriction q'$ are isomorphic.

Next lemma follows from the fact that $\omega + \omega$ is bijective with ω.

Lemma

For all sets $X \subseteq Y$, $\mathrm{Coll}(\omega, {<}Y)$ and $\mathrm{Coll}(\omega, {<}X) \times \mathrm{Coll}(\omega, {<}Y)$ are isomorphic.

The main lemma:

Main Lemma (ZF + DC) Suppose κ is a $V_{\kappa+2}$–supercompact cardinal, $\mathbb{P} \in V_\kappa$ is a forcing notion preserving DC, and G is a $\mathbb{P}$–generic filter G over V. In $V[G]$ there is then a forcing iteration $(\mathcal{Q}_\xi)_{\xi \leq \kappa}$ such that

(1) $\mathcal{Q}_\kappa \subseteq V_\kappa^{V[G]}$.

(2) $\mathcal{Q}_\kappa$ is σ–closed in $V[G]$,

and for which there are unboundedly many cardinals $\delta < \kappa$ such that $\mathbb{P} \in V_\delta$ and such that the following holds:

(3) $\mathcal{Q}_\delta$ forces that $\mathcal{Q}_\kappa / \dot{G}_{\mathcal{Q}_\delta}$ is δ–closed.

(4) If H is $\mathcal{Q}_\kappa$–generic over $V[G]$, then

(a) $\mathcal{C}_\omega^{V[G]} = \mathcal{C}_\omega^{V[G][H]}$,
(b) $V_\kappa[G][H] \models \mathrm{ZFC}$,
(c) δ is, in $V_\kappa[G][H]$, sufficiently supercompact (say, 2^δ–supercompact), and
(d) in some outer model there is a $\mathrm{Coll}(\omega, {<}V_\delta^V)$–generic filter K over $V[G][H]$ and an elementary embedding $j : \mathcal{C}_\omega^{V_\kappa[G][H]} \longrightarrow \mathcal{C}_\omega^{V_\kappa[G][H][K]}$.

(5) Letting $X = V_\delta^V$, the partial orders $\mathrm{Coll}(\omega, {<}X)$, $\mathcal{P} \times \mathrm{Coll}(\omega, {<}X)$ and $(\mathcal{P} * \dot{\mathcal{Q}}_\delta) \times \mathrm{Coll}(\omega, {<}X)$ are, in V, forcing–equivalent.

Proof sketch: By a standard reflection argument there are unboundedly many $\delta < \kappa$ such that δ is (say) $V_{\delta+1}$–supercompact. Let us fix such a δ and let $\mathbb{P} \in V_\delta$ be a partial order. Let G be $\mathbb{P}$–generic filter over V and let us work in $V[G]$. Since DC holds, by the proof of Theorem 226 in [Woodin, "Suitable extender models I"], there is a forcing iteration $(\mathcal{Q}_\xi)_{\xi \leq \kappa} \subseteq V_\kappa[G]$ and a strictly increasing sequence $(\kappa_\xi)_{\xi < \kappa}$ of cardinals below κ with the following properties in $V[G]$.

- (i) $\mathcal{Q}_\kappa$ is σ–closed.
- (ii) For all $\xi < \kappa$, $\mathcal{Q}_\xi$ forces that $\mathcal{Q}_\kappa / \dot{G}_{\mathcal{Q}_\xi}$ is κ_ξ–closed.
- (iii) For every $\bar{\delta} < \kappa$, if $\bar{\delta}$ is (say) $V_{\bar{\delta}+1}$–supercompact, then
  - (a) $\mathcal{Q}_\xi \in V_{\bar{\delta}}$ for all $\xi < \bar{\delta}$,
  - (b) $\kappa_{\bar{\delta}} = \bar{\delta}$,
  - (c) for every dense $D \subseteq \mathcal{Q}_{\bar{\delta}}$ there is some $\xi < \bar{\delta}$ such that $D \cap \mathcal{Q}_\xi$ is predense in $\mathcal{Q}_{\bar{\delta}}$, and
  - (d) $\bar{\delta}$ remains $V_{\bar{\delta}+1}$–supercompact after forcing with $\mathcal{Q}_\kappa$.
- (iv) If H is $\mathcal{Q}_\kappa$–generic over $V[G]$, then $V_\kappa[G][H] \models$ ZFC.

Let H be a $\mathcal{Q}_\kappa$–generic filter H over $V[G]$. By (iii) (d) for δ, we know that δ remains 2^δ–supercompact in $V_\kappa[G][H]$. Since DC holds in $V[G]$ and $\mathcal{Q}_\kappa$ is σ–closed in $V[G]$, we have that $\mathcal{C}_\omega^{V[G]} = \mathcal{C}_\omega^{V[G][H]}$. Also, by
(ii) together with (iii) (b) for δ, $\mathcal{Q}_\delta$ forces that $\mathcal{Q}_\kappa/\dot{G}_{\mathcal{Q}_\delta}$ is δ–closed.

By a classical ZFC result of Woodin using the 2^δ–supercompactness of δ in $V_\kappa[G][H]$ (this degree of supercompactness is more than enough) and the fact that $V_\kappa[G][H] \models$ ZFC, there is, in some outer model N, a $\mathrm{Coll}(\omega, {<}\delta)$–generic filter K_0 over $V_\kappa[G][H]$ for which there is an elementary embedding $j : \mathcal{C}_\omega^{V_\kappa[G][H]} \longrightarrow \mathcal{C}_\omega^{V_\kappa[G][H][K_0]}$.

Since $\mathrm{Coll}(\omega, {<}\delta)$ and $\mathrm{Coll}(\omega, {<}V_\delta^V)$ are isomorphic in $V_\kappa[G][H]$ as $\delta = |V_\delta|$ holds there, there is in N a $\mathrm{Coll}(\omega, {<}V_\delta^V)$–generic filter K over $V[G][H]$ such that $V_\kappa[G][H][K_0] = V_\kappa[G][H][K]$.

Finally, stepping back into V and letting $X = V_\delta^V$, it is easy to check that all of $\mathrm{Coll}(\omega, {<}X)$, $\mathbb{P} \times \mathrm{Coll}(\omega, {<}X)$ and $(\mathcal{P} * \dot{\mathcal{Q}}_\delta) \times \mathrm{Coll}(\omega, {<}X)$ are forcing–equivalent.

(The forcing–equivalence between $\mathrm{Coll}(\omega, {<}X)$ and $(\mathbb{P} * \dot{\mathcal{Q}}_\delta) \times \mathrm{Coll}(\omega, {<}X)$ uses the fact that $\mathrm{Coll}(\omega, {<}X)$ adds a sequence $(J_\xi)_{\xi<\delta}$ such that each J_ξ is a V–generic filter for $\mathbb{P} * \dot{\mathcal{Q}}_\xi$ and such that $J_\xi \subseteq J_{\xi'}$ for all $\xi < \xi'$, which, by (iii) (c) for δ, implies that $\bigcup_{\xi<\delta} J_\xi$ is a V–generic filter for $\mathbb{P} * \dot{\mathcal{Q}}_\delta$.) $\square$

Recall:

Theorem

($\mathsf{ZF}+\mathsf{DC}$) Suppose there is, for every closed and unbounded class of ordinals C, a $V_{\kappa+2}$–supercompact cardinal κ such that $\kappa \in C$. Then, for every set–forcing $\mathbb{P}$ and every $\mathbb{P}$–generic filter G over V, if $V[G] \models \mathsf{DC}$, then the structures $(\mathcal{C}_\omega^V; \in, r)_{r\in\mathbb{R}^V}$ and $(\mathcal{C}_\omega^{V[G]}; \in, r)_{r\in\mathbb{R}^V}$ have the same Σ_2–theory.

Proof sketch:

Suppose, towards a contradiction, that $\mathbb{P}$ forces that $(\mathcal{C}_\omega^V; \in, r)_{r\in\mathbb{R}^V}$ and $(\mathcal{C}_\omega^{V[\dot{G}]}; \in, r)_{r\in\mathbb{R}^V}$ disagree on the truth value of some sentence $\sigma = (\exists x)(\forall y)\varphi(x,y)$, where φ is a restricted formula.

By our large cardinal assumption we may then fix a $V_{\kappa+2}$–supercompact cardinal κ such that

$$(\mathcal{C}_\omega^{V_\kappa}; \in, r)_{r\in\mathbb{R}^V} \models \sigma$$

iff

$$(\mathcal{C}_\omega^{V_\kappa[G]}; \in, r)_{r\in\mathbb{R}^V} \models \neg\sigma$$

We show that $(\mathcal{C}_\omega^{V_\kappa}; \in, r)_{r\in\mathbb{R}^V}$ and $(\mathcal{C}_\omega^{V_\kappa[G]}; \in, r)_{r\in\mathbb{R}^V}$ are elementarily equivalent, which will be a contradiction.

By the Main Lemma we know that there is some cardinal $\delta < \kappa$ such that $\mathbb{P} \in V_\delta$ and that there are, in some outer model, $\mathrm{Coll}(\omega, {<}V_\delta)$–generic filters K and K' over V and $V[G]$, resp., for which there are elementary embeddings

$$j : \mathcal{C}_\omega^{V_\kappa} \longrightarrow \mathcal{C}_\omega^{V_\kappa[K]}$$

and

$$j' : \mathcal{C}_\omega^{V_\kappa[G]} \longrightarrow \mathcal{C}_\omega^{V_\kappa[K']}$$

But then, by the homogeneity of $\mathrm{Coll}(\omega, {<}V_\delta)$, the theories of $(\mathcal{C}_\omega^{V_\kappa}; \in, r)_{r \in \mathbb{R}^V}$ and $(\mathcal{C}_\omega^{V_\kappa[G]}; \in, r)_{r \in \mathbb{R}^V}$ are the same.

□

Question

Can one get these generic absoluteness results assuming more modest large cardinal hypotheses (in the region of Woodin cardinals)?

Back in ZFC. Larger inner models

[This second part is joint work with Matteo Viale.]

A cardinal κ is super–huge iff for every α there is an elementary embedding $j : V \longrightarrow M$ with critical point κ such that $j(\kappa) > \alpha$ and ${}^{j(\kappa)}M \subseteq M$.

Viale defines a strengthening MM^{+++} of Martin's Maximum, and even of MM^{++}, and he proves:

Theorem

(Viale)

(1) *Suppose there is a super–huge cardinal κ. Then there is an $SSP(\omega_1)$ poset forcing MM^{+++}.*

(2) *(MM^{+++}) Suppose there is a proper class of super–huge cardinals. If $\mathcal{P}$ is an $SSP(\omega_1)$ poset forcing MM^{+++} and G is $\mathcal{P}$–generic over V, then*

$$(\mathcal{C}^{V}_{\omega_1}; \in, x)_{x\in\mathcal{P}(\omega_1)^V} \equiv (\mathcal{C}^{V[G]}_{\omega_1}; \in, x)_{x\in\mathcal{P}(\omega_1)^V}$$

MM^{+++} is $\mathsf{CFA}(\mathsf{SSP}(\omega_1))$, the Category Forcing Axiom for $\mathsf{SSP}(\omega_1)$ = set–forcings preserving stationary subsets of ω_1.

Category forcing axioms

Given a class Γ of complete Boolean algebras and $B, C \in \Gamma$, a complete homomorphism of Boolean algebras $i : B \longrightarrow C$ is Γ–*correct* iff

$$\Vdash_B C/i[\dot{G}_B] \in \Gamma$$

- $C \leq_\Gamma B$ means that there is a Γ–correct complete homomorphism $i : B \longrightarrow C$.
- $\mathbb{U}^\Gamma$ is the class–forcing $(\Gamma, \leq_\Gamma)$.
- Also, given $B, C \in \Gamma$, $C \leq^*_\Gamma B$ means that there is an injective complete homomorphism of Boolean algebras $i : B \longrightarrow C$ such that

$$\Vdash_B C/i[\dot{G}_B] \in \Gamma$$

Given cBa's B, C and a class Δ of cBa's, a complete homomorphism $k : B \longrightarrow C$ Δ–*freezes* C iff for every cBa $D \in \Delta$ and all Δ–correct $i_0 : C \longrightarrow D$ and $i_1 : C \longrightarrow D$ we have that $i_0 \circ k = i_1 \circ k$.

$B \in \Delta$ is Δ–*totally rigid* iff $Id_B : B \longrightarrow B$ Δ–freezes B; equivalently, iff

- for every $C \in \Delta$ there is at most one Δ–correct $i : B \longrightarrow C$,

i.e.,

- for every $C \in \Delta$ there is at most one complete homomorphism $i : B \longrightarrow C$ such that if H is C–generic over V, then $V[i^{-1}(H)] \models C/i[i^{-1}(H)] \in \Delta$.

Given a cardinal κ, a class Γ has the *SSP(κ)–freezability property* iff for every $B \in \Gamma$ there is some $C \in \mathrm{SSP}(\kappa)$ and some injective Γ–correct $k : B \longrightarrow C$ such that k SSP(κ)–freezes B.

$$\mathrm{SSP}(\kappa) = \{C \,:\, C \text{ preserves stationary subsets of } \kappa\}$$

Note: Suppose $\Vdash_B \dot{C} \in \Gamma$ and if G is B–generic, then $C = \dot{C}_G$ forces that there is some $A_G \subseteq \kappa$ coding G in an absolute way mod. $\mathrm{SSP}(\kappa)$, in the sense that there is some restricted formula $\varphi(x, y, z)$ and some $p \in V$ such that, if H is C–generic over $V[G]$, then

- $(H(\kappa^+); \in, \mathrm{NS}_\kappa)^{V[G][H]} \models \varphi(G, A_G, p)$, and
- in every outer model M of $V[G][H]$ such that $\mathcal{P}(\kappa)^{V[G][H]} \cap (\mathrm{NS}_\kappa)^M = (\mathrm{NS}_\kappa)^{V[G][H]}$, if

$$(H(\kappa^+); \in, \mathrm{NS}_\kappa)^M \models \varphi(G_0, A_{G_0}, p)$$

and

$$(H(\kappa^+); \in, \mathrm{NS}_\kappa)^M \models \varphi(G_1, A_{G_1}, p),$$

then $G_0 = G_1$.

Then the natural inclusion

$$i : B \longrightarrow B * \dot{C}$$

$\mathrm{SSP}(\kappa)$–freezes B.

κ–suitable classes

Definition Given a cardinal κ, a class Γ of cBa's is κ–suitable iff the following holds.

(1) There is some $a \in H(\kappa^+)$ and some Σ_2–formula $\varphi(x, y)$ such that for every cBa B, $B \in \Gamma$ if and only if $\varphi(a, B)$.

(2) For every $B \in \Gamma$ and every $b \in B$, $B \restriction b \in \Gamma$.

(3) Γ is closed under two–step iterations (where Γ^{V^B} is of course the class of cBa's defined in V^B by $\varphi(a, y)$), under lottery sums, under forcing–equivalence, and under preimages of complete homomorphisms.

(4) Γ is iterable in the sense that there is a (class–sized) winning strategy for player II in the iteration game $\mathcal{G}_\Gamma$ of length an arbitrary limit ordinal λ in which players I and II take turns in building a $\leq^*_\Gamma$–decreasing sequence $(B_i)_{i\leq\lambda'}$ for some $\lambda' \leq \lambda$, II moves at limit stages, and for all limit $i \leq \lambda'$,

(a) $\varinjlim\{B_\alpha : \alpha < i\} \subseteq B_i \subseteq \varprojlim\{B_\alpha : \alpha < i\}$, and
(b) if i is inaccessible and $|B_j| < i$ for all $j < i$, then $B_i = \varinjlim\{B_\alpha : \alpha < i\}$,

and where of course player II wins if and only if she can play all the way up to stage λ (i.e., if and only if B_λ exists).

(5) For every ordinal $\alpha > \kappa$, Γ contains some B forcing $|\alpha| = \kappa$.

(6) Γ contains all $<\delta$–closed cBa's, where $\delta > \kappa$ is any inaccessible cardinal.

(7) Γ has the SSP(κ)–freezability property.

ω_1–suitable classes

(1) Semiproper $= \{B : B \text{ semiproper}\}$ is ω_1–suitable, assuming the existence of a proper class of supercompact cardinals (Viale).

(2) Proper $= \{B : B \text{ proper}\}$ is ω_1–suitable:

Clauses (1)–(6) in the definition of ω_1–suitable clearly hold for Proper. The SSP(ω_1)–freezability property for Proper follows from:

Theorem

(Moore) Given $A \subseteq \omega_1$, a ladder system $\vec{C} = (C_\alpha \,:\, \alpha \in \mathrm{Lim}(\omega_1))$ and a sequence $\vec{S} = (S_\alpha)_{\alpha<\omega_1}$ of pairwise disjoint stationary subsets of ω_1 there is a proper forcing $\mathcal{P}_A^{\vec{C},\vec{S}}$ adding a $\subseteq$–increasing and continuous sequence $(X_\nu)_{\nu<\omega_1}$ of countable sets such that

- $\bigcup_{\nu<\omega_1} X_\nu = \omega_2^V$, *and*
- *for every limit $\nu < \omega_1$ there is some $\nu_0 < \nu$ such that for all $\xi \in [\nu_0, \nu)$, $w(X_\xi \cap \omega_1, X_\nu \cap \omega_1) < w(X_\xi, X_\nu)$ if and only if $X_\nu \cap \omega_1 \in \bigcup_{\alpha\in A} S_\alpha$, where for countable $X \subseteq Y \subseteq Ord$, $w(X, Y) = |\sup(X \cap \pi_Y^{-1}(C_{\mathrm{ot}(Y)})|$ (π_Y the transitive collapsing function of Y).*

The forcing $\mathcal{P}_A^{\vec{C},\vec{S}}$ in above theorem is the natural forcing for adding a suitable instance of the Mapping Reflection Principle (MRP) by initial segments. Let us call such forcings MRP–forcings.

(3) Proper $\cap\ \omega^\omega$–bounding is ω_1–suitable:

Clauses (1)–(6) in the def. of ω_1–suitability are again easy (iterability of proper ${}^\omega\omega$–bounded forcing under countable support iterations is a result of Shelah).

The SSP(ω_1)–freezability property for Proper $\cap\ \omega^\omega$–bounding follows from the fact that every instance of Moore's $\mathcal{P}^A_{\vec{C},\vec{S}}$ is σ–distributive (so certainly ${}^\omega\omega$–bounding).

(4) Proper $\cap$ Suslin tree preserving is ω_1–suitable:

Clauses (1)–(6) in the def. of ω_1–suitability are again easy (iterability of proper proper Suslin tree preserving forcing under countable support iterations is a result of Miyamoto).

The SSP(ω_1)–freezability property for Proper $\cap$ Suslin tree preserving follows from:

Lemma

(Miyamoto) MRP–forcings preserve Suslin trees.

We immediately have the following now:

(5) Proper $\cap$ ω^ω–bounding $\cap$ Suslin tree preserving is ω_1–suitable.

Also:

Lemma

(Miyamoto?)

- *The limit of every RCS–iteration of semiproper ${}^{\omega}\omega$–bounding partial orders is semiproper and ${}^{\omega}\omega$–bounding.*
- *The limit of every RCS–iteration of semiproper Suslin tree preserving partial orders is semiproper and Suslin tree preserving.*

Hence we immediately have:

(6) Semiproper $\cap\ \omega^{\omega}$–bounding is ω_1–suitable.

(7) Semiproper $\cap$ Suslin tree preserving is ω_1–suitable.

(8) Semiproper $\cap\ \omega^{\omega}$–bounding $\cap$ Suslin tree preserving is ω_1–suitable.

Shelah's PIF §XI S–condition

Shelah isolates in PIF, §XI, a property he calls S–condition such that

1. if CH holds and $\mathcal{P}$ is a forcing notion satisfying the S–condition, then $\mathcal{P}$ adds no new reals, and such that
2. if $\langle \mathcal{P}_\alpha, \dot{\mathcal{Q}}_\beta : \alpha \leq \lambda, \beta < \lambda \rangle$ is an RCS iteration such that for all $\beta < \lambda$, $\Vdash_{\mathcal{P}_\beta} |\mathcal{P}_\beta| = \aleph_1$ and $\dot{\mathcal{Q}}_\beta$ has the S–condition, then $\mathcal{P}_\lambda$ has the S–condition.

Forcing notions satisfying the S–condition:

- Namba forcing (and natural variations thereof),
- all σ–closed forcing notions,
- the natural poset which, for a fixed stationary $S \subseteq \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$, adds an ω_1–club through S with countable conditions.

Given a tree T and a node η of T, let $\mathrm{succ}_T(\eta)$ denote the set of immediate successors of η in T. For a partial order $\mathcal{P}$ and $p \in \mathcal{P}$, let $\mathcal{P}_p^{\mathcal{P}}$ be the following game of length ω between players *I* and *II*, with *I* paying at even stages and *II* playing at odd stages.

Definition

(1) At any given stage n of the game, the corresponding player picks a pair T^n, $(p^n_\eta)_{\eta \in T^n}$, where T^n is a tree consisting of finite sequences of ordinals in $|\mathcal{P}|$ without infinite branches and where $(p^n_\eta)_{\eta \in T^n}$ is a sequence of conditions in $\mathcal{P}$ extending p such that p^n_ν extends p^n_η in $\mathcal{P}$ whenever ν extends η in T^n.

(2) If $n > 0$, then

  (a) T^n and $(p^n_\eta)_{\eta \in T^n}$ end–extend T^{n-1} and $(p^{n-1}_\eta)_{\eta \in T^{n-1}}$, respectively,

  (b) every terminal node in T^{n-1} has a proper extension in T^n, and

  (c) every node in $T^n \setminus T^{n-1}$ extends a unique terminal node in T^{n-1}.

(3) Player I starts by playing $T_0 = \{\emptyset\}$ and $p^0_\emptyset = p$.

(4) At any given even stage $n > 0$ of the game, player I picks, for every terminal node η of T^{n-1}, a finite sequence ν_η of ordinals in $|\mathcal{P}|$ such that ν_η extends η properly. He then builds T^n as
$T^{n-1} \cup \{\nu_\eta \restriction k \, : \, k \leq |\nu_\eta|, \eta \text{ a terminal node of } T^{n-1}\}$.
Player I also has to choose of course $(p^n_\eta)_{\eta \in T^n}$ in such a way that (1) and (2) are satisfied.

(5) At any given odd stage n of the game, player II chooses, for every terminal node η of T^{n-1}, a regular cardinal $\kappa^n_\eta \in [\aleph_2, |\mathcal{P}|]$ and builds T^n from T^{n-1} by adding to T^{n-1} a next level where, for each terminal node η of T^{n-1}, the set of immediate successors of η in T^n is $\{\eta^\frown\langle\alpha\rangle \, : \, \alpha < \kappa^n_\eta\}$.
Player II also has to choose of course $(p^n_\eta)_{\eta \in T^n}$ in such a way that (1) and (2) are satisfied.

After ω moves, the players have naturally built a tree $T = \bigcup_n T^n$ of height ω whose nodes are finite sequences of ordinals in $|\mathcal{P}|$, together with a sequence $(p_\eta)_{\eta \in T} = \bigcup (p^n_\eta)_{\eta \in T^n}$ of $\mathcal{P}$–conditions such that for all nodes η, ν in T, if ν extends η in T, then p_ν extends p_η in $\mathcal{P}$. Finally, player II wins the game iff for every subtree T' of T, if $|\mathrm{succ}_{T'}(\eta)| = |\mathrm{succ}_T(\eta)|$ for every $\eta \in T'$, then there is a condition in $\mathcal{P}$ forcing that there is an ω–branch b through T' such that $p_{b \restriction n} \in \dot{G}$ for all $n < \omega$.

The definition of the S–condition is the following (Shelah's definition is more general, but the present form suffices for our purposes):

Definition

A partial order $\mathcal{P}$ has the S–condition if and only if $|\mathcal{P}| \geq \aleph_2$ and for every $p \in \mathcal{P}$ player II has a winning strategy σ in the game $\mathcal{G}_p^{\mathcal{P}}$ such that for every partial run of the game, the output of σ at any given sequence $\eta \in {}^{<\omega}|\mathcal{P}|$ depends only on η, $(p_{\eta \restriction k})_{k \leq |\eta|}$ and $\{k < |\eta| \,:\, |\mathrm{succ}_T(\eta \restriction k)| > 1\}$, where T denotes the tree built by the players up to that point.

Let S–cond denote the class of B such that B is a complete Boolean subalgebra of some cBa C with the S–condition.

(9) S–cond is ω_1–suitable:

Clauses (1)–(6) in the definition of ω_1–suitability are easy (iterability holds by Shelah's result). The SSP(ω_1)–freezability property for S–cond holds by a variant of Foreman–Magidor–Shelah's coding for showing that MM implies $2^{\aleph_1} = \aleph_2$:

Lemma

For every partial order $\mathcal{P}$, regular cardinals $\lambda > \kappa > |\mathcal{P} * \mathrm{Coll}(\omega_1, |\mathcal{P}|)|$, every partition $(S_\alpha)_{\alpha<\omega_1}$ of $\{\xi < \kappa : \mathrm{cf}(\xi) = \omega\}$ into stationary sets and every stationary $U \subseteq \{\xi < \lambda : \mathrm{cf}(\lambda) = \omega\}$ such that $\{\xi < \lambda : \mathrm{cf}(\lambda) = \omega\} \setminus U$ is also stationary, there is a $(\mathcal{P} * \mathrm{Coll}(\omega_1, |\mathcal{P}|))$–name $\dot{A}$ for a subset of ω_1 such that $\mathcal{Q} = \mathrm{Coll}(\omega_1, |\mathcal{P}|) * \dot{\mathcal{S}}_{\dot{A}}^{\vec{S},U}$ has the S–condition and freezes $\mathcal{P}$, where $\mathcal{S}_A^{\vec{S},U}$ is the partial order, ordered by end–extension, of all strictly $\subseteq$–increasing and $\subseteq$–continuous sequences $(Z_\nu)_{\nu\leq\nu_0}$, for some $\nu_0 < \omega_1$, such that for all $\nu \leq \nu_0$ and all $\alpha < \omega_1$, $Z_\nu \in [\lambda]^{\aleph_0}$ and

- *if $sup(Z_\nu \cap \kappa) \in S_\alpha$, then $sup(Z_\nu) \in U$ if and only if $\alpha \in A$.*

($\mathcal{S}_A^{\vec{S},U}$ forces the existence of strictly increasing and continuous functions $f : \omega_1 \longrightarrow \kappa$, $g : \omega_1 \longrightarrow \lambda$, with range cofinal in κ and λ, respectively, and such that for all ν, $\alpha < \omega_1$, if $f(\nu) \in S_\alpha$, then $g(\nu) \in U$ if and only if $\alpha \in A$.)

Strongly presaturated towers

Given regular cardinals δ and κ, $\mathcal{T}$ is a κ–tower of height δ of normal ideals iff

- each member of $\mathcal{T}$ is a normal ideal on $[X]^{\leq\kappa}$ for some $X \in V_\delta$, and
- for each $I \in \mathcal{T}$ and each $Y \in V_\delta$,
  - if $Y \supseteq \cup I$, then $\{y \in [Y]^{\leq\kappa} : y \cap (\cup I) \in I\} \in \mathcal{T}$, and
  - if $Y \subseteq \cup I$, then $\{x \cap Y : x \in I\} \in \mathcal{T}$.

There is a natural ordering $\leq_{\mathcal{T}}$ on $\{a \in V_\delta : a \text{ is an } I\text{–positive subset of } [\bigcup a]^{\leq\kappa} \text{ for some } I \in \mathcal{T}\}$:

(∘) $b \leq_{\mathcal{T}} a$ iff $\cup a \subseteq \cup b$ and for every $y \in b$, $y \cap (\cup a) \in a$.

We will abuse language and will identify $\{a \in V_\delta : a \text{ is an } I\text{–positive subset of } [\bigcup a]^{\leq\kappa} \text{ for some } I \in \mathcal{T}\}$ with $\mathcal{T}$.

Note: Forcing with $(\mathcal{T}, \leq_{\mathcal{T}})$ over V gives rise to a natural elementary embedding $j : V \longrightarrow M$ with critical point $(\kappa^+)^V$.

We say that $\mathcal{T}$ is *presaturated* iff $j((\kappa^+)^V) = \delta$ and M is forced to be closed under $<\delta$–sequences in $V[\dot{G}]$.

Given a κ–suitable class Γ of cBa's, we say that $\mathcal{T}$ *is* Γ*–strongly presaturated* iff:

(1) $(\mathcal{T}, \leq_{\mathcal{T}}) \in \Gamma$.

(2) $(\mathcal{T}, \leq_{\mathcal{T}})$ is presaturated.

(3) $(\mathcal{T}, \leq_{\mathcal{T}})$ is SSP(κ)–totally rigid.

(4) For every SSP(κ)–totally rigid $B \in \Gamma \cap V_\delta$ and every $I \in \mathcal{T}$ on $H(|B|^+)$, the dual filter of I concentrates on the set of $M \in [H(|B|^+)]^\kappa$ such that M has exactly one SSP(κ)–correct B–generic filter G (i.e., G meets every dense of B in M and for every $S \in (\pi_M``M[\pi_M``G]) \cap \mathcal{P}(\kappa)$, if S is stationary in $\pi_M``M[\pi_M``G]$, then it is stationary in V).

CFA(Γ)

Definition

Suppose Γ is a κ–suitable class of cBa's. The *Category Forcing Axiom for* Γ, CFA(Γ), is the following assertion:

The class of Γ–strongly presaturated normal κ–towers is dense in $(\mathbb{U}^{\Gamma}, \leq^*_{\Gamma})$.

These are strong forcing axioms:

Fact

(Viale) Suppose there is a proper class of Woodin cardinals. Then the fllowing are equivalent.

(1) MM^{++}

(2) *The class of* $SSP(\omega_1)$ *towers is dense in* $(\mathbb{U}^{SSP(\omega_1)}, \leq^*_{SSP(\omega_1)})$

CFA(SSP(ω_1)) is MM^{+++}.

Theorem

Suppose Γ is κ–suitable. Then CFA(Γ) implies FA(Γ)$^{++}$.

Consistency and generic absoluteness

Definition

A cardinal δ is *super–Σ_2–huge* iff for every ordinal α there is an elementary embedding $j : V \longrightarrow M$ with critical point δ such that $j(\delta) > \alpha$ and such that M is closed under λ–sequences for some $\lambda > j(\delta)$ such that V_λ is Σ_2–correct in V.

Theorem

Suppose Γ is κ–suitable and δ is a super–Σ_2–huge cardinal. Let $\mathbb{U}^\Gamma_\delta = (\mathbb{U}^\Gamma \cap V_\delta, \leq_\Gamma)$. Then $\mathbb{U}^\Gamma_\delta \in \Gamma$ and $\mathbb{U}^\Gamma_\delta$ forces $CFA(\Gamma)$ over V.

Theorem

Suppose Γ is κ–suitable and there is a proper class of super–Σ_2–huge cardinals. If $CFA(\Gamma)$ holds, $\mathcal{P}$ is a partial order in Γ forcing $CFA(\Gamma)$, and G is $\mathcal{P}$–generic over V, then

$$(\mathcal{C}_\kappa^V; \in, x)_{x \in \mathcal{P}(\kappa)^V} \equiv (\mathcal{C}_\kappa^{V[G]}; \in, x)_{x \in \mathcal{P}(\kappa)^V}$$

The following summarizes what we have seen so far.

Theorem Let Γ be one of the following classes.

- SSP(ω_1)
- Proper
- Proper $\cap\ \omega^\omega$–bounding
- Proper $\cap$ Suslin tree preserving
- Proper $\cap\ \omega^\omega$–bounding $\cap$ Suslin tree preserving
- Semiproper $\cap\ \omega^\omega$–bounding
- Semiproper $\cap$ Suslin tree preserving
- Semiproper $\cap\ \omega^\omega$–bounding $\cap$ Suslin tree preserving
- S–cond

1. Suppose δ is a super–Σ_2–huge cardinal. Then $\mathbb{U}_\delta^\Gamma$ forces $\mathrm{CFA}(\Gamma)$.
2. Suppose there is a proper class of super–Σ_2–huge cardinals. If $\mathrm{CFA}(\Gamma)$ holds, $\mathcal{P}$ is a partial order in Γ forcing $\mathrm{CFA}(\Gamma)$, and G is $\mathcal{P}$–generic over V, then

$$(\mathcal{C}_\kappa^V; \in, x)_{x \in \mathcal{P}(\kappa)^V} \equiv (\mathcal{C}_\kappa^{V[G]}; \in, x)_{x \in \mathcal{P}(\kappa)^V}$$

Incompatible category forcing axioms

These category forcing axioms are pairwise incompatible:

- $\mathsf{MM}^{+++} \implies \mathsf{MM}$, and $\mathsf{MM} \implies \delta^1_2 = \omega_2$ by well–known result of Woodin.
- If there is a super–huge cardinal δ and CFA(Proper) ($=\mathsf{PFA}^{+++}$) holds, then $\delta^1_2 < \omega_2$: If δ is super–huge, then $\mathbb{U}^{\text{Proper}}_\delta$ is proper and collapses ω_2^V to $\aleph_1$. Also, by a result of Neeman-Zapletal, under this large cardinal hypothesis (a proper class of Woodin cardinals suffices), if $\mathcal{P}$ is a proper poset and G is $\mathcal{P}$–generic over V, then there is an elementary embedding between $L(\mathbb{R})^V$ and $L(\mathbb{R})^{V[G]}$ which is the identity on the ordinals. Hence $V^{\mathbb{U}^{\text{Proper}}_\delta} \models \delta^1_2 < \omega_2$. Since "$\delta^1_2 < \omega_2$" is expressible over $H(\omega_2)$ and PFA^{+++} holds in V, it follows from (the proof) of the generic absoluteness theorem that $V \models \delta^1_2 < \omega_2$.

Similarly the following holds in the presence of large cardinals.

- If CFA(Proper $\cap\, \omega^\omega$–bounding) ($=$ PFA^{+++}(ω^ω–bounding)) holds, then $2^{\aleph_0} = \aleph_2$, $\delta^1_2 < \omega_2$, $\mathfrak{d} = \omega_1$, and there are no Suslin trees.
- If CFA(Proper $\cap$ Suslin tree preserving) ($=$ PFA^{+++}(Suslin tree preserving)) holds, then $\delta^1_2 < \omega_2$, $2^{\aleph_0} = \mathfrak{d} = \omega_2$, and there are Suslin trees.
- If CFA(Proper $\cap\, \omega^\omega$–bounding $\cap$ Suslin tree preserving) ($=$ PFA^{+++}(ω^ω–bounding + Suslin tree preserving)) holds, then $2^{\aleph_0} = \aleph_2$, $\delta^1_2 < \omega_2$, $\mathfrak{d} = \omega_1$, and there are Suslin trees.

- If CFA(Semiproper $\cap\, \omega^\omega$–bounding) ($=$ MM^{+++}(ω^ω–bounding)) holds, then $2^{\aleph_0} = \aleph_2$, $\delta^1_2 = \omega_2$, $\mathfrak{d} = \omega_1$, and there are no Suslin trees.
- If CFA(Semiproper $\cap$ Suslin tree preserving) ($=$ MM^{+++}(Suslin tree preserving)) holds, then $2^{\aleph_0} = \aleph_2$, $\delta^1_2 = \omega_2$, $\mathfrak{d} = \omega_2$, and there are Suslin trees.
- If CFA(Semiproper $\cap\, \omega^\omega$–bounding $\cap$ Suslin tree preserving) ($=$ MM^{+++}(ω^ω–bounding + Suslin tree preserving)) holds, then $2^{\aleph_0} = \aleph_2$, $\delta^1_2 = \omega_2$, $\mathfrak{d} = \omega_1$, and there are Suslin trees.

- If CFA(S–cond) holds, then $\diamondsuit$ holds.

This uses the fact that if $\diamondsuit$ holds, then there is a $\diamondsuit$–sequence indestructible under S–condition forcing (this has been recently proved by Magidor).

Question

Are there κ–suitable classes for $\kappa > \omega_1$?

One conclusion of 2nd part:

Completeness modulo forcing is not by itself a good criterion for choosing new (true) axioms if we want a univocal description of the universe. Maximality is better. Completeness modulo forcing can be a pleasant added value of maximality axioms, though.