Approximate Ramsey theory

Dana Bartošová

Carnegie Mellon University

Ultrafilters, Ramsey theory and dynamics
Université de Lyon
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Ramsey's theorem

For every $k \leq m$, $r \geq 2$, and every colouring of k-element subsets of $\mathbb{N}$ with r-many colours there is an infinite subset X of $\mathbb{N}$ such that all k-element subsets of X have the same colour.

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Finite Ramsey's theorem

For every $k \leq m$ and $r \geq 2$, there exists n such that for every colouring of k-element subsets of n with r-many colours there is a subset X of n of size m such that all k-element subsets of X have the same colour.

Reformulation of the finite Ramsey theorem

THE CLASS OF FINITE LINEAR ORDERS IS RAMSEY

Given A and B finite linear orders, $|A| \leq |B|$ and $r \geq 2$, there exists a finite linear order C such that whenever we colour copies of A in C by r colours, there is a copy B' of B in C such that all copies of A in B' have the same colour.

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A topological group G is extremely amenable if it has a fixed point under any continuous action on a compact Hausdorff space. Equivalently, every minimal G-flow is a singleton.

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Finite linear orders are a Ramsey class $\longleftrightarrow$ every partition $G = \bigcup_{i=1}^{r} G_A K_i$ has a thick part $\longleftrightarrow$ there are no disjoint topologically syndetic sets.

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Theorem (Kechris, Pestov, and Todorčević)

G is extremely amenable iff finitely generated substructures of $\mathcal{A}$ form a rigid Ramsey class.

RAMSEY CLASSES

- finite linear orders (Ramsey);
- finite linearly ordered graphs (Nešetřil and Rödl);
- finite linearly ordered metric spaces (Nešetřil);
- finite Boolean algebras (Graham and Rothschild).

Examples

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EXTREMELY AMENABLE GROUPS

1. $\mathrm{Aut}(\mathbb{Q}, <)$(Pestov);
2. $\mathrm{Aut}(\mathcal{R}, <)$ = group of automorphisms of the random ordered graph (KPT);
3. $\mathrm{Iso}(\mathbb{U}, \mathrm{d})$ = group of isometries of the Urysohn space (Pestov);
4. $U(l_2)$ = group of unitaries of the separable Hilbert space (Gromov + Milman);
5. $\mathrm{LIso}(\mathbb{G})$ = group of linear isometries of the Gurarij space (B + López-Abad + Mbombo).

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Theorem (Melleray-Tsankov)

For M approximately ultrahomogeneous, $\mathrm{Iso}(M)$ *is extremely amenable* $\longleftrightarrow$ *finitely-generated substructures satisfy the approximate Ramsey property (ARP).*

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$\mathrm{Iso}(U, d)$ is e.a. $\longleftrightarrow$ finite metric spaces satisfy ARP.

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$\mathrm{Iso}(U, d)$ is e.a. $\longleftrightarrow$ finite metric spaces satisfy ARP.

PREFIRST EXAMPLE (Gromov + Milman)
$U(l_2)$ is e.a. $\longleftrightarrow$ finite dimensional inner spaces satisfy ARP.

FIRST COMBINATORIAL PROOF (B + LA + M)
$\mathrm{Iso}_l(\mathbb{G})$ is e.a. $\longleftrightarrow$ finite dimensional Banach spaces satisfy ARP.

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KUBIŚ-SOLECKI; HENSON

Simple proof - metric Fraïssé theory.

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Katětov construction

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$\mathrm{Iso}_l(\mathbb{G})$ *is extremely amenable.*

About the proof

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GENERAL CASE
Discretize and use dual Ramsey theorem of Graham and Rothschild.

Further structures

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p – extreme point in p
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Theorem (B+Kwiatkowska)

$Homeo(L, <)$ is e.a. $\longleftrightarrow$ generalization of Gowers' Hindman's theorem.

Big problem

P – pseudoarc
$p \in P$

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$$\text{Is Homeo}(P, p) \text{ e.a.?}$$

The end

THANK YOU!