Partition properties for simply definable colourings

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Introduction

We are interested in questions asking whether certain complicated mathematical objects can be defined by simple formulas.

In this talk, we focus on the definability of colourings of pairs of ordinals without large homogeneous sets.

If X is a set, then we let $[X]^2$ denote the set consisting of all two-element subsets of X.

Given a function c with domain $[X]^2$, we say that a subset H of X is c-*homogeneous* if $c \restriction [H]^2$ is constant.

Classical results of Erdős and Tarski show that an uncountable cardinal κ is weakly compact if and only if for every colouring $c : [\kappa]^2 \longrightarrow 2$, there is a c-homogenoues subset of κ of cardinality κ.

Colourings witnessing failures of weak compactness are usually constructed using objects like Aronszajn trees or injections into the power sets of smaller cardinals.

For example, if we want to show that ω_1 is not weakly compact, then we fix an injection $\iota : \omega_1 \longrightarrow \mathbb{R}$ and let $c : [\omega_1]^2 \longrightarrow 2$ denote the unique function with

$$c(\alpha, \beta) = 1 \iff \iota(\alpha) < \iota(\beta)$$

for all $\alpha < \beta < \omega_1$.

The work presented in this paper is motivated by the question whether colourings witnessing a failure of weak compactness can be simply definable, i.e. whether they can be defined by formulas of low quantifier complexity that use parameters of low hereditary cardinality.

Definition

Given $n < \omega$ and sets $z_0, \ldots, z_{n-1}$, a class X is $\Sigma_n(z_0, \ldots, z_{n-1})$-*definable* if there is a Σ_n-formula $\varphi(v_0, \ldots, v_n)$ with

$$X = \{x \mid \varphi(x, z_0, \ldots, z_{n-1})\}.$$

Definition

Given a set z, an infinite cardinal κ *has the* $\Sigma_n(z)$*-partition property* if, for every $\Sigma_n(\kappa, z)$-definable function $c : [\kappa]^2 \longrightarrow 2$, there is a c-homogeneous set of cardinality κ.

Definition

An infinite regular cardinal κ is $\boldsymbol{\Sigma}_n$-weakly compact if κ has the $\Sigma_n(z)$-partition property for every $z \in \mathrm{H}(\kappa)$.

Definable colourings of pairs of countable ordinals

The following observations show that the statement that ω_1 is $\mathbf{\Sigma}_1$-weakly compact is independent of the axioms of $\mathbf{ZFC}$.

Proposition

If μ is an infinite regular cardinal, $\kappa > \mu$ is a weakly compact cardinal and G is $\mathrm{Col}(\mu, {<}\kappa)$-generic over V, then κ is $\mathbf{\Sigma}_n$-weakly compact for all $n < \omega$ in $\mathrm{V}[G]$.

Proof.

Let $z \in \mathrm{H}(\kappa)^{\mathrm{V}[G]}$ and let $c : [\kappa]^2 \longrightarrow 2$ be a $\Sigma_n(\kappa, z)$-definable colouring in $\mathrm{V}[G]$. Then there is an inaccessible cardinal $\mu < \nu < \kappa$ and H $\mathrm{Col}(\mu, {<}\nu)$-generic over V such that $z \in \mathrm{V}[H]$ and $\mathrm{V}[G]$ is a $\mathrm{Col}(\mu, {<}\kappa)$-generic extension of $\mathrm{V}[H]$. In this situation, the homogeneity of $\mathrm{Col}(\mu, {<}\kappa)$ in $\mathrm{V}[H]$ implies that $c \in \mathrm{V}[H]$. Since κ is weakly compact in $\mathrm{V}[H]$, there is a c-homogeneous set of size κ in $\mathrm{V}[H]$. □

In contrast, a variation of the above argument showing that ω_1 is not weakly compact provides us with examples of non-$\mathbf{\Sigma}_1$-weakly compact cardinals.

Proposition

Let κ be an uncountable regular cardinal. If there is a cardinal $\nu < \kappa$ and an injection $\iota : \kappa \longrightarrow \mathcal{P}(\nu)$ that is $\mathbf{\Sigma}_n(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)$, then κ is not $\mathbf{\Sigma}_n$-weakly compact.

Corollary

If κ is a $\mathbf{\Sigma}_1$-weakly compact, then κ is inaccessible in L.

Corollary (Friedman–Holy, L.)

If ν is an uncountable cardinal with $\nu = \nu^{<\nu}$ and $2^\nu = \nu^+$, then there is a $<\nu$-closed, ν^+-c.c. partial order forcing that ν^+ is not $\mathbf{\Sigma}_1$-weakly compact.

In contrast to the above consistency results, it turns out that many canonical extensions of $\mathbf{ZFC}$ imply that ω_1 has strong partition properties for definable colourings.

Theorem

Assume that one of the following assumptions holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- *There is a measurable cardinal and a precipitous ideal on ω_1.*
- $\mathbf{BMM}$ *holds and the nonstationary ideal on ω_1 is precipitous.*
- *Woodin's Axiom $(*)$ holds.*

Then ω_1 is $\mathbf{\Sigma}_1$-weakly compact.

The proof of the above result is based on the following notion.

Definition

Given $0 < n < \omega$, an uncountable regular cardinal κ has the $\boldsymbol{\Sigma}_n$-*club property* if every subset A of κ with the property that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)$ either contains a club subset of κ or is disjoint from a club in κ.

An easy argument shows that the above property implies a strengthening of $\boldsymbol{\Sigma}_n$-weak compactness.

Proposition

Let κ be an uncountable regular cardinal and let $c : [\kappa]^2 \longrightarrow 2$ be a colouring with the property that for every $\alpha < \kappa$, there is an $i_\alpha < 2$ and a club subset C_α of κ such $c(\alpha, \beta) = i_\alpha$ holds for all $\beta \in C_\alpha$.

Then there is a stationary subset of κ that is c-homogeneous.

Proof.

Define $C = \triangle_{\alpha<\kappa} C_\alpha$. Then C is a club in κ and there is a stationary subset S of C and $i < 2$ such that $i_\alpha = i$ for all $\alpha \in S$.

If $\alpha, \beta \in S$ with $\alpha < \beta$, then $\beta \in C_\alpha$ and $c(\alpha, \beta) = i_\alpha = i$. □

Lemma

If κ has the $\boldsymbol{\Sigma}_n$-club property and $c : [\kappa]^2 \longrightarrow 2$ is $\Sigma_n(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)$, then there is stationary subset of κ that is c-homogeneous.

In particular, every cardinal with the $\boldsymbol{\Sigma}_n$-club property is $\boldsymbol{\Sigma}_n$-weakly compact.

Proof.

Let $c : [\kappa]^2 \longrightarrow 2$ be $\Sigma_n(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)$. Given $\alpha < \kappa$, define

$$A_\alpha \;=\; \{\beta \in (\alpha, \kappa) \mid c(\alpha, \beta) = 0\}.$$

Then the set $\{A_\alpha\}$ is $\Sigma_n(\kappa, \alpha, z)$-definable for all $\alpha < \kappa$. In this situation, the $\boldsymbol{\Sigma}_n$-club property implies that for every $\alpha < \kappa$, there is an $i_\alpha < \kappa$ and a club subset C_α of κ such $c(\alpha, \beta) = i_\alpha$ holds for all $\beta \in C_\alpha$. By the above proposition, there is a stationary subset of κ that is c-homogeneous. $\square$

The above theorem about the $\boldsymbol{\Sigma}_1$-weak compactness of ω_1 is now a direct consequence of the following result.

Theorem

Assume that one of the following assumptions holds:

- *There is a measurable cardinal above a Woodin cardinal.*
- *There is a measurable cardinal and a precipitous ideal on ω_1.*
- $\mathbf{BMM}$ *holds and the nonstationary ideal on ω_1 is precipitous.*
- *Woodin's Axiom $(*)$ holds.*

Then ω_1 has the $\boldsymbol{\Sigma}_1$-club property.

In the following, we present the proof of a weakening of the first implication that uses results of Woodin on the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$.

Proposition

Assume that there are infinitely many Woodin cardinals with a measurable cardinal above them all. Then ω_1 has the $\mathbf{\Sigma}_1$-club property.

Proof.

Fix a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and $z \in \mathrm{H}(\omega_1)$. Assume, towards a contradiction, that there is a unique subset A of ω_1 with $\varphi(A, \omega_1, z)$ and this subset A is a bistationary subset of ω_1.

Let G be $\mathbb{P}_{max}$-generic over $\mathrm{L}(\mathbb{R})$. By the Π_2-maximality of the $\mathbb{P}_{max}$-extension of $\mathrm{L}(\mathbb{R})$, there is $B \in \mathcal{P}(\omega_1)^{\mathrm{L}(\mathbb{R})[G]}$ such that B is bistationary subset of ω_1 in $\mathrm{L}(\mathbb{R})[G]$ and B is the unique subset of ω_1 with $\varphi(B, \omega_1, z)$ in $\mathrm{L}(\mathbb{R})[G]$. Since the partial order $\mathbb{P}_{max}$ is weakly homogeneous in $\mathrm{L}(\mathbb{R})$, we have $B \in \mathrm{L}(\mathbb{R})$.

Our assumptions imply that AD holds in $\mathrm{L}(\mathbb{R})$ and therefore the clubfilter on ω_1 is an ultrafilter in $\mathrm{L}(\mathbb{R})$. But this contradicts the bistationarity of B in $\mathrm{L}(\mathbb{R})[G]$. $\square$

The following result allows us to derive the above implication from a weaker assumption.

Lemma (L.–Schindler–Schlicht)

Assume that either $M_1^{\#}(A)$ exists for every $A \subseteq \omega_1$ or there is a measurable cardinal and a precipitous ideal on ω_1. Then the following statements hold for every Σ_1-formula $\varphi(v_0, v_1, v_2)$ and all $z \in \mathrm{H}(\omega_1)$:

- *If there is a stationary subset A of ω_1 such that $\varphi(A, \omega_1, z)$ holds, then there is an element B of the club filter on ω_1 such that $\varphi(B, \omega_1, z)$ holds.*
- *If there is a costationary subset A of ω_1 such that $\varphi(A, \omega_1, z)$ holds, then there is an element B of the non-stationary ideal on ω_1 such that $\varphi(B, \omega_1, z)$ holds.*

The proof of this result uses iterated generic ultrapowers and Woodin's countable stationary tower forcing.

Definable colourings of $[\omega_2]^2$

In contrast to the above results, neither large cardinal axioms nor forcing axioms imply that ω_2 is $\mathbf{\Sigma}_1$-weakly compact.

Theorem

If $\mathbf{BPFA}$ holds, then ω_2 is not $\mathbf{\Sigma}_1$-weakly compact.

This statement is a direct consequence of the following results.

Theorem (Caicedo–Veličković)

If $\mathbf{BPFA}$ holds, then there is a well-ordering $\lhd$ of the reals of length ω_2 with the property that the set of all initial segments of $\lhd$ is $\Sigma_1(z)$-definable for some $z \subseteq \omega_1$.

This result shows that $\mathbf{BPFA}$ implies the existence of an injection $\iota : \omega_2 \longrightarrow \mathbb{R}$ that is $\Sigma_1(\omega_2, z)$-definable for some $z \subseteq \omega_1$ and such an injection directly yields a failure of $\mathbf{\Sigma}_1$-weak compactness.

With the help of results of Caicedo–Veličković and Larson, it is possible to show that all large cardinal axioms are also compatible with simpler failures of the $\mathbf{\Sigma}_1$-weak compactness of ω_2.

Theorem

Assume that there is a measurable cardinal above a supercompact limit of supercompact cardinals. Then there is a semi-proper partial order $\mathbb{P}$ with the property that the following statements hold in every $\mathbb{P}$-generic extension of the ground model:

- $\mathbf{MM}^{+\omega}$ *holds.*
- *There is a $\Sigma_1(\omega_2)$-definable colouring $c : [\omega_2]^2 \longrightarrow 2$ with the property that every c-homogeneous subset of ω_2 is countable.*
- *There is a $\Sigma_1(\omega_2)$-definable well-ordering of the reals.*

The following theorem is the first ingredient used in the proof of the above result.

Theorem (Caicedo–Veličković)

Assume that $\mathbf{BPFA}$ *holds. If* M *is an inner model of* $\mathbf{ZFC} + \mathbf{BPFA}$ *with* $\omega_2 = \omega_2^M$*, then* $\mathcal{P}(\omega_1) \subseteq M$*.*

This theorem allows us to derive the following statement.

Lemma

Assume that $\mathbf{BPFA}$ *holds and there is a measurable cardinal. Then the set* $\{\mathrm{H}(\omega_2)\}$ *is* $\Sigma_1(\omega_2)$*-definable.*

Proof.

Let $\mathcal{A}$ denote the collection of all sets A with the property that there is a transitive model M of $\mathbf{ZFC}^-$ and $\delta \in M$ such that δ is a measurable cardinal in M, $\mathrm{H}(\delta)^M$ is a model of $\mathbf{ZFC} + \mathbf{BPFA}$, $\omega_2 = \omega_2^M$ and $A = \mathrm{H}(\omega_2)^M$. Then $\mathcal{A}$ is $\Sigma_1(\omega_2)$-definable and $\mathrm{H}(\omega_2) \in \mathcal{A}$.

In the other direction, let M and δ witness that A is an element of $\mathcal{A}$. Pick a normal ultrafilter U on δ in M and let

$$\langle \langle M_\alpha \mid \alpha \in \mathrm{On} \rangle, \ \langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta \in \mathrm{On} \rangle \rangle$$

denote the system of ultrapowers and elementary embeddings induced by $\langle M, \in, U \rangle$. Define

$$N \ = \ \bigcup \{ j_{0,\alpha}(\mathrm{H}(\delta)^M) \mid \alpha \in \mathrm{On} \}.$$

Then N is an inner model of $\mathbf{ZFC} + \mathbf{BPFA}$ with $\omega_2^N = \omega_2^M = \omega_2$. Hence we have $\mathcal{P}(\omega_1) \subseteq N$ and therefore $A = \mathrm{H}(\omega_2)^M = \mathrm{H}(\omega_2)^N = \mathrm{H}(\omega_2)$. $\square$

The second ingredient for the above result is the following consistency result of Larson.

Theorem (Larson)

Assume that there exists a supercompact limit of supercompact cardinals. Then there is a semi-proper partial order $\mathbb{P}$ with the property that the following statements hold in every $\mathbb{P}$-generic extension $\mathrm{V}[G]$ of the ground model V:

- $\mathbf{MM}^{+\omega}$ *holds.*
- *There is a well-ordering of the reals that is definable in $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula without parameters.*

Proof of the Theorem.

Assume that there is a measurable cardinal above a supercompact limit of supercompact cardinals. Force with the partial order given by Larson's theorem. Then the following statements hold in the corresponding generic extension:

- There is a measurable cardinal.
- $\mathbf{MM}^{+\omega}$ holds.
- There is a well-ordering of the reals that is definable in $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula without parameters.

Then there is an injection $\iota : \omega_2 \longrightarrow \mathbb{R}$ that is definable in $\langle \mathrm{H}(\omega_2), \in \rangle$ by a formula without parameters. By the above lemma, this injection is $\Sigma_1(\omega_2)$-definable. This injection can now be used to construct a $\Sigma_1(\omega_2)$-definable colouring $c : [\omega_2] \longrightarrow 2$ with the property that every c-homogeneous subset of ω_2 is countable. $\square$

The above construction shows that large cardinal axioms do not imply partition relations for $\Sigma_1(\omega_2)$-definable colourings of $[\omega_2]^2$.

It leaves open the question whether a strengthening of $\mathrm{MM}^{+\omega}$ could imply such statements.

Question

Does MM^{++} imply that for every $\Sigma_1(\omega_2)$-definable colouring $c : [\omega_2]^2 \longrightarrow 2$, there is a c-homogeneous subset of ω_2 of size ω_2?

Inaccessible cardinals

Next, we consider inaccessible $\mathbf{\Sigma}_1$-weakly compact cardinals.

The following observations shows that the first inaccessible $\mathbf{\Sigma}_1$-weakly compact cardinal is much smaller than the first weakly compact cardinal.

Note that the first $\mathbf{\Sigma}_2$-weakly compact cardinal can be the first weakly compact cardinal.

Lemma

Let κ be a weakly compact cardinal. Then every $\mathbf{\Pi}^1_1$-statement that holds in $\mathrm{H}(\kappa)$ reflects to an inaccessible $\mathbf{\Sigma}_1$-weakly compact cardinal less than κ.

Proof.

Fix a Π^1_1-formula $\Psi(v)$ and $A \subseteq \kappa$ with $\mathrm{H}(\kappa) \models \Psi(A)$. Pick an elementary submodel M of $\mathrm{H}(\kappa^+)$ of cardinality κ with $\mathrm{H}(\kappa) \cup \{A\} \subseteq M$ and ${}^{<\kappa}M \subseteq M$. By the *Hauser characterization* of weak compactness, there is a transitive set $N \in \mathrm{H}(\kappa^+)$ and an elementary embedding $j : M \longrightarrow N$ with critical point κ and $M \in N$. Then κ is inaccessible in N, $A = j(A) \cap \kappa$, $\mathrm{H}(\kappa) \in N$ and Π^1_1-downwards absoluteness implies that $\mathrm{H}(\kappa) \models \Psi(A)$ holds in N.

The above construction ensures that κ is weakly compact in M and all Σ_1-formulas with parameters in M are absolute between M and N. In particular, every colouring $c : [\kappa]^2 \longrightarrow 2$ that is definable in N by a Σ_1-formula with parameters in $\mathrm{H}(\kappa) \cup \{\kappa\}$ is definable in M by the same formula. Hence there is a c-homogeneous set of cardinality κ in $M \subseteq N$ for every such colouring.

This shows that κ is $\boldsymbol{\Sigma}_1$-weakly compact in N. With the help of a universal Σ_1-formula this yields the statement of the theorem. □

Lemma

If κ is a regular limit of measurable cardinals, then κ has the $\mathbf{\Sigma}_1$-club property.

Proof.

Fix a Σ_1-formula $\varphi(v_0, v_1, v_2)$ and $z \in \mathrm{H}(\kappa)$ with the property that there is a unique subset A of κ such that $\varphi(A, \kappa, z)$ holds. Pick a measurable cardinal $\delta < \kappa$ with $z \in \mathrm{H}(\delta)$ and a normal ultrafilter U on δ. Let

$$\langle \langle M_\alpha \mid \alpha \in \mathrm{On} \rangle, \langle j_{\alpha,\beta} : M_\alpha \longrightarrow M_\beta \mid \alpha \leq \beta \in \mathrm{On} \rangle \rangle$$

denote the system of ultrapowers and elementary embeddings induced by $\langle \mathrm{V}, \in, U \rangle$. Then $j_{0,\alpha}(\kappa) = \kappa$ and $j_{0,\alpha}(z) = z$ holds for all $\alpha < \kappa$. By Σ_1-upwards absoluteness, this implies that $\varphi(j_{0,\alpha}(A), \kappa, z)$ holds for all $\alpha < \kappa$. But this shows that $A = j_{0,\alpha}(A)$ holds for all $\alpha < \kappa$.

This allows us to conclude that the club $\{j_{0,\alpha}(\delta) \mid \alpha < \kappa\}$ is either contained in A or disjoint from A. □

The following large cardinal property provides us with more interesting examples of of inaccessible $\mathbf{\Sigma}_1$-weakly compact cardinals that are not weakly compact.

Definition (Sharpe & Welch)

Let κ be an uncountable cardinal.

- A *weak κ-model* is a transitive model M of ZFC^- of size κ with $\kappa \in M$.
- The cardinal κ is *ω_1-iterable* if for every subset A of κ there is a weak κ-model M and a weakly amenable M-ultrafilter U on κ such that $A \in M$ and $\langle M, \in, U \rangle$ is ω_1-iterable.

Lemma (L.–Schindler–Schlicht)

If κ is a regular cardinal that is either ω_1-iterable or a stationary limit of ω_1-iterable cardinals, then κ has the $\mathbf{\Sigma}_1$-club property.

If $\mathrm{V} = \mathrm{L}$ and κ is an uncountable regular cardinal, then there is a bistationary subset A of κ such that the set $\{A\}$ is $\Sigma_1(\kappa)$-definable.

Such subsets can be constructed from the canonical $\Diamond_\kappa$-sequence in L, using the facts that this sequence is definable over $\langle \mathrm{L}_\kappa, \in \rangle$ by a formula without parameters and the set $\{\mathrm{L}_\kappa\}$ is $\Sigma_1(\kappa)$-definable.

This shows that all large cardinal properties that are compatible with the assumption $\mathrm{V} = \mathrm{L}$ do not imply the $\mathbf{\Sigma}_1$-club property.

Successors of singular cardinals

The following result shows that partition relations for definable colourings can consistently hold at successors of singular cardinals of countable cofinality.

Theorem (Cummings–Friedman–Magidor–Rinot–Sinapova)

Assume that ν is a singular limit of supercompact cardinals with $\mathrm{cof}(\nu) = \omega$ and $\kappa > \nu$ is supercompact. Then there is a partial order $\mathbb{P}$ with the property that the following statements hold in every $\mathbb{P}$-generic extension $\mathrm{V}[G]$ of the ground model V:

- *The models V and $\mathrm{V}[G]$ have the same bounded subsets of ν.*
- *Every infinite cardinal μ with $\mu \leq \nu$ or $\mu \geq \kappa$ is preserved in $\mathrm{V}[G]$.*
- $\kappa = (\nu^+)^{\mathrm{V}[G]}$.
- *If $x \in \mathcal{P}(\nu)^{\mathrm{V}[G]}$, then κ is supercompact in $\mathrm{HOD}_x^{\mathrm{V}[G]}$.*

In particular, the cardinal ν^+ is $\mathbf{\Sigma}_n$-weakly compact for all $n < \omega$ in the above forcing extension.

In contrast, a small modification of a result of Shelah, stating that $\mathrm{L}(\mathcal{P}(\nu))$ is a model of $\mathbf{ZFC}$ for every singular strong limit cardinal of uncountable cofinality, yields the following result.

Theorem

If ν is a singular strong limit cardinal of uncountable cofinality, then ν^+ is not $\mathbf{\Sigma}_2$-weakly compact.

Proof of the Theorem.

Set $\mu = \mathrm{cof}(\nu)$ and fix an injection $\iota : \mathrm{H}(\nu) \longrightarrow \nu$.

Given $f, g \in {}^{\mu}\nu$, we write $f \lhd g$ if $f(\alpha) < g(\alpha)$ for eventually all $\alpha < \nu$. Since μ is uncountable, the ordering $\lhd$ is well-founded. Given $\alpha \in \mathrm{On}$, we let R_α denote the set of all elements of ${}^{\mu}\nu$ of $\lhd$-rank α. Then the proof of Shelah's result shows that it is possible to use the Erdős–Rado theorem to conclude that $|R_\alpha| < 2^\mu$ for all $\alpha \in \mathrm{On}$.

Given $\alpha < \nu^+$, set $A_\alpha = \bigcup\{\mathrm{ran}(f) \mid f \in R_\alpha\} \subseteq \nu$ and let $\pi_\alpha : A_\alpha \longrightarrow \lambda_\alpha$ denote the corresponding transitive collapse. Then the above computations show that $\lambda_\alpha < \nu$ and $\pi_\alpha \circ f \in \mathrm{H}(\nu)$ for all $\alpha < \nu^+$ and $f \in R_\alpha$. For each $\alpha < \nu^+$, pick $f_\alpha \in R_\alpha$ with $\iota(\pi_\alpha \circ f_\alpha)$ minimal.

Then the resulting injection

$$i : \nu^+ \longrightarrow {}^{\mu}\nu;\ \alpha \longmapsto f_\alpha$$

is definable by a Σ_2-formula with parameters in $\mathrm{H}(\nu^+)$ and this injection directly implies a failure of the $\mathbf{\Sigma}_2$-weak compactness of ν^+. □

The above result leaves open the following question.

Question

Is it consistent that the successor of a singular strong limit cardinal of uncountable cofinality is Σ_1-weakly compact?

Σ_1-weakly compact cardinals in inner models

We conclude this talk by discussing large cardinal properties of $\mathbf{\Sigma_1}$-weakly compact cardinals in canonical inner models.

This result make use of the following lemma that transfers classical characterization of weak compactness to the definable context.

Lemma

The following statements are equivalent for every uncountable regular cardinal κ:

- *κ has the $\Sigma_n(z)$-partition property.*
- *If $\iota : \kappa \longrightarrow {}^{<\kappa}2$ is a $\Sigma_n(\kappa, z)$-definable injection with the property that $\mathrm{ran}(\iota)$ is a subtree of ${}^{<\kappa}2$ of height κ, then there is a cofinal branch through this subtree.*
- *If $A \subseteq \kappa$ has the property that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable, then there is a weak κ-model M, a transitive set N and an elementary embedding $j : M \longrightarrow N$ with critical point κ such that $A \in M$ and κ is inaccessible in M.*

These characterizations allow us to derive the following result.

Theorem

If κ is $\mathbf{\Sigma}_n$-weakly compact and A is a subset of κ with the property that the set $\{A\}$ is $\Sigma_n(\kappa, z)$-definable for some $z \in \mathrm{H}(\kappa)$, then κ is a Mahlo cardinal in $\mathrm{L}[A]$.

Proof.

First, κ is inaccessible in $\mathrm{L}[A]$, because otherwise there is a $\Sigma_n(\kappa, \nu, z)$-definable injection $\iota : \kappa \longrightarrow \mathcal{P}(\nu)$ for some $\nu < \kappa$.

Now, assume that κ is not Mahlo in $\mathrm{L}[A]$. Then a theorem of Todorčević implies that there is a special κ-Aronszajn trees in $\mathrm{L}[A]$. Let ι_* denote the $<_{\mathrm{L}[A]}$-least function from κ to ${}^{<\kappa}2$ in $\mathrm{L}[A]$ with the property that $\mathrm{ran}(\iota_*)$ is a special κ-Aronszajn subtree of ${}^{<\kappa}2$.

Then the set $\{\iota_*\}$ is $\Sigma_1(\kappa, z)$-definable and $\mathrm{ran}(\iota_*)$ is a subtree of ${}^{<\kappa}2$ without a cofinal branch. In this situation, the above lemma implies that κ is not $\mathbf{\Sigma}_n$-weakly compact. $\square$

Remember that, given a cardinal κ and an ordinal α, we say that κ is an α-*Mahlo* if κ is a Mahlo cardinal and for every $\bar{\alpha} < \alpha$, the set $\{\nu < \kappa \mid \nu \textit{ is an } \bar{\alpha}\textit{-Mahlo cardinal}\}$ is stationary in κ.

Theorem

If $\mathrm{V} = \mathrm{L}$ holds, then every Σ_1-weakly compact cardinal is κ-Mahlo.

The proof of this result relies on the following lemma.

Lemma

Assume that $\mathrm{V} = \mathrm{L}$. Let κ be an inaccessible cardinal and let $\langle S_\alpha \mid \alpha < \lambda \rangle$ be a sequence of stationary subsets of κ with $\lambda < \kappa$ such that the following statements hold:

- *The set $\{\langle \alpha, \gamma \rangle \mid \alpha < \lambda, \ \gamma \in S_\alpha\}$ is $\Delta_1(\kappa, z)$-definable.*
- *The set*

  $$\{\nu \in \mathrm{Lim} \cap \kappa \mid \mathrm{cof}(\nu) = \nu, \ S_\alpha \cap \nu \text{ is stationary in } \nu \text{ for all } \alpha < \lambda\}$$

  is not stationary in κ.

Then κ does not have the $\Sigma_1(z)$-partition property.

Sketch of the proof.

Let C denote the $<_{\mathrm{L}}$-least club in κ with the property that for every regular $\nu \in C$, there is an $\alpha < \lambda$ with the property that the set $S_\alpha \cap \mathrm{Lim}(C) \cap \nu$ is not stationary in ν.

Let $\vec{C} = \langle C_\gamma \mid \gamma < \kappa \rangle$ be the unique C-sequence of length κ with the property that for every $\gamma \in \mathrm{Lim} \cap \kappa$, the club C_γ is $<_{\mathrm{L}}$-minimal with the following properties:

- *If γ is singular, then $\mathrm{otp}(C_\gamma) = \mathrm{cof}(\gamma) < \min(C_\gamma)$.*
- *If $\gamma = \mu^+$ for a cardinal μ, then $C_\gamma = (\mu, \gamma)$.*
- *If γ is an inaccessible cardinal, then there is $\alpha(\gamma) < \lambda$ with*

$$C_\gamma \cap S_{\alpha(\gamma)} \cap \mathrm{Lim}(C) \ = \ \emptyset.$$

Then the set $\{\vec{C}\}$ is $\Sigma_1(\kappa, z)$-definable.

Sketch of the proof (cont.)

Using techniques developed by Todorčević, we can construct a slim tree $\mathbb{T} = \mathbb{T}(\rho_0^{\vec{C}})$ of height κ with the following properties:

- $\mathbb{T}$ is a $\Delta_1(\kappa, z)$-definable subset of $\mathrm{H}(\kappa)$.
- $\mathbb{T}$ has a cofinal branch if and only if there is a $\xi < \kappa$ and a club D in κ such that for every $\xi < \gamma \in \mathrm{Lim}(D)$, there is a $\gamma \leq \delta(\gamma) < \kappa$ with

$$D \cap \gamma \ = \ C_{\delta(\gamma)} \cap [\xi, \gamma).$$

Assume that D witnesses that $\mathbb{T}$ has a cofinal branch. Then there is a club $D_* \subseteq \mathrm{Lim}(D)$ consisting of strong limit cardinals such that $\delta(\gamma)$ is inaccessible for every $\gamma \in C$. But this yields an $\alpha < \lambda$ with $\alpha = \alpha(\delta(\gamma))$ for stationary-many $\gamma \in D_*$ and hence $D \cap S_\alpha \cap \mathrm{Lim}(C) = \emptyset$, a contradiction.

This shows that $\mathbb{T}$ is a κ-Aronszajn tree and we can use this tree to construct a counterexample to the $\Sigma_1(z)$-partition property. $\square$

The above theorem then follows from this Lemma by applying it to the sets

$$S_\alpha = \{\nu < \kappa \mid \nu \textit{ is } \alpha\textit{-Mahlo}\}$$

for all $\alpha < \kappa$.

This result leaves open the following question.

Question

If $\mathrm{V} = \mathrm{L}$, is every $\mathbf{\Sigma}_1$-weakly compact cardinal κ^+-Mahlo?

Thank you for listening!