Descriptive Set Theory: an elementary introduction

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1 Introduction

These notes are meant as an introduction to descriptive set theory. The goal is to provide a propaedeutic tour through the basics of the subject that does not assume any background in Set Theory (well, almost no background). The material is based on some notes taken by Bjork at a course given by Foreman at UC Irvine, and heavily rewritten by Foreman with editing help from the students in a graduate course at the Hebrew University of Jerusalem.

Originally intended as an appendix for the paper *The conjugacy problem in ergodic theory*, by Foreman, Rudolph and Weiss, the notes contain adequate background for the descriptive set theoretic portion of that paper and covers some additional topics, such as *norms*, useful for other applications in analysis.

Most of the material is taken from the book *Classical Descriptive Set Theory*, by A. Kechris [1], as well as the lecture notes from D. Marker ([2]) that are on the web. The notes here are much less complete than either of those sources.

1.1 Set Theoretic Background and notation

In this section we briefly describe the Set Theoretic background we will use. It can be safely skipped and referred back to when necessary.

We will avoid the controversy about whether 0 is a natural number by using the notation ω for the set $\{0, 1, 2, 3, \ldots\}$. People uncomfortable with this notation can systematically substitute $\mathbb{N}$, with the convention that $\mathbb{N}$ contains 0.

We use the von Neumann definition of natural numbers and ordinals. In particular we identify 0 with the empty set. Recursively, we identify the number n with the set $\{0, 1, 2, \ldots n-1\}$. For example, the number 4 is the set $\{0, 1, 2, 3\}$. We note that the number n has exactly n elements.

Well-orderings in general, and ordinals in particular play an important role in Descriptive Set Theory. Recall that a well-ordering is a linear ordering $(I, <_I)$ with the property that if $A \subseteq I$ is a non-empty set then A contains an $<_I$ minimal element. Any standard set theory book (such as Levy [3]) contains the pertinent information.

We summarize the relevant facts:

- Each ordinal α is the set of smaller ordinals. The first few ordinals are

$$0, 1, 2, 3, \ldots$$

  The first infinite ordinal is ω.

- If α is an ordinal, the least ordinal greater than α is $\alpha \cup \{\alpha\}$ which is denoted $\alpha + 1$. Ordinals of the form $\alpha + 1$ are *successor* ordinals, the other ordinals are *limit* ordinals.

- α is an ordinal iff

  1. $\in$ linearly orders α (i.e. $(\alpha, \in)$ is a linear ordering) and
  2. If $x \in \alpha$ then $\in$ linearly orders x.

- We will write OR for the class of ordinals.

By far the most important property of ordinals is that they give canonical examples of well-orderings. This is summarized in the following proposition:

Proposition 1 *Suppose that $(I, <_I)$ is a well-ordering. Then there is a unique ordinal α and a unique bijection $f : I \to \alpha$ such that*

$$i <_I j \text{ iff } f(i) \in f(j).$$

In other words there is a unique ordinal α such that $(\alpha, \in)$ is isomorphic with $(I, <_I)$ and the isomorphism is unique.

In the same constellation of results we have the following definition and proposition.

Definition 2 *Let X be a set and $R \subseteq X \times X$ be a relation. Then R is* well-founded *iff every non-empty subset $A \subseteq X$ has an R-minimal element; i.e. there is an $a \in A$ such that for all $b \in A$, $(b, a) \notin R$.*

It is worth noting that the following proposition does not use the *Axiom of Choice*.

Proposition 3 *Let $R \subseteq X \times X$ be a relation. Then R is well-founded iff there is a function $f : X \to OR$ such that for all $a, b \in X$ if $(a, b) \in R$ then $f(a) \in f(b)$.*

We will mostly be interested in countable ordinals. The set of countable ordinals is itself an ordinal. This ordinal is the least uncountable ordinal and will be denoted ω_1.

Other set theoretic notation we will use includes:

1. We will write A^B for the collection of functions from B to A.
2. *Warning:* Many (most?) texts write ${}^B A$ for the functions from B to A.
3. Examples of this include 2^ω, the collection of functions from ω to $\{0, 1\}$ and ω^ω, the collection of functions from ω to ω.
4. We will consider *finite sequences* of elements of B to be functions $s : n \to B$ where $n \in \omega$.
5. The *collection* of finite sequences of elements of B will be denoted $B^{<\omega}$. The collection of sequences of elements of B of length n is denoted B^n.

1.2 Polish Topologies

Definition 4 *A topological space (X, τ) is called Polish if τ is separable and if there is a complete metric d on X which generates the topology τ.*

This definition is intended to be sufficiently concrete to be able to prove non-trivial theorems, while being abstract enough to encompass, not only the obvious examples such as $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{T}$ (etc.) but also more exotic spaces such as spaces of compact sets.

We explicitly *allow* our spaces to have isolated points. For example ω with the discrete topology is a Polish space.

A Polish space without isolated points will be called *perfect*. While the empty set technically fulfills this criterion, we will ignore that problem in practice.

We note that being Polish is a property of the topology, not of the metric. So the open unit interval $(0,1)$ is a Polish space (as it is homeomorphic to $\mathbb{R}$) even though the usual metric on $(0,1)$ is not complete. Indeed Polish spaces can admit many complete separable metrics. The next proposition shows that a Polish space always admits a complete metric bounded by 1.

Proposition 5 *Suppose (X,τ) is a Polish space. Let d be any complete metric on X generating τ. Let*

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)}.$$

Then d' is also a complete metric on X which generates τ.

The metric d' is sometimes called the "nearsighted" metric associated with τ.

We will use the standard fact that open sets in a separable metric space are countable unions of closed sets (i.e. $\mathcal{F}_\sigma$) and, by taking complements, closed sets are countable intersections of open sets (i.e. $\mathcal{G}_\delta$).

1.3 Product topologies

Let (X_i,τ_i) be a topological space for each $i \in I$. The product topology on $X = \Pi_i X_i$ has basic open sets those $O = \Pi_i O_i$, where $O_i = X_i$ for all but finitely many indices i, and otherwise O_i is open in X_i. The following is an easy exercise.

Theorem 6 *Let X_i be a Polish space for all $i \in \omega$. Then $X = \Pi_{i\in\omega} X_i$ is Polish.*

We can explicitly describe the complete separable metric on $\prod X_i$. Let d_i be a complete metric generating the Polish topology of X_i and bounded by 1. Define the following metric on X:

$$d(x,y) = \sum_{i\in\omega} \frac{d_i(x_i,y_i)}{2^{i+1}}$$

Then d is a complete separable metric generating the product topology on X.

Notice that the product topology is the "topology of pointwise convergence": given sequences $\vec{x}^n = \langle x_i^n : i \in \omega\rangle$, then $\vec{x}^n$ converges to $\vec{y}$ iff for each i, x_i^n converges in X_i to y_i.

If all of the X_i are the same space X then we can identify $\prod_{i\in I} X_i$ with X^I. It is particularly common to identify ω^ω with $\prod_{i\in\omega}\omega$ and 2^ω with $\prod_{i\in\omega} 2$.

1.4 The Cantor Space and the Baire Space

There are two spaces that (along with the discrete space ω) will play a particularly important role for us. This is because of their *universality properties* and because they are very easy to use for *coding* and *diagonal arguments*.

The first such space is the Baire Space. We note that ω^ω can be canonically identified with $\Pi_{n\in\omega}\omega$. If we equip ω with the discrete topology, then the Baire Space is ω^ω with the product topology. It is denoted by $\mathcal{N}$ in some texts.

The Baire Space is a very familiar space in another guise: "continued fraction" expansions of irrationals in $(0,1)$ give elements of ω^ω and the map sending an irrational to its continued fraction expansion is a homeomorphism between $(0,1)\setminus\mathbb{Q}$ (with the induced topology from $(0,1)$) and ω^ω.

Example 7 *Consider the following metric on ω^ω:*

$$d(x,y)=\begin{cases}\frac{1}{n+1} & \textit{where } n \textit{ is the first coordinate such that } x_n\neq y_n\\ 0 & \textit{if no such } n \textit{ exists}\end{cases}$$

Then this metric is complete and generates the product topology on ω^ω.

This metric will be convenient when we want to discuss the "radius" of sets in the Baire Space.

We will see later (Theorem 26) that there is always a continuous surjection from the Baire Space onto any Polish Space.

The other special space we will consider is the Cantor Space. We will view the Cantor Space as the set 2^ω consisting of infinite sequences of 0's and 1's. Viewing 2^ω as the product space $\prod_{i\in\omega}2$ we can equip 2 with the discrete topology and 2^ω with the product topology.

Clearly $2^\omega\subseteq\omega^\omega$ and the Cantor Space is identified with a closed compact subset of the Baire Space.

The Cantor Space is homeomorphic with the usual binary Cantor Set in $(0,1)$. It is universal in the sense that any (non-empty) Polish space with a perfect subset contains a closed subspace homeomorphic to The Cantor Space. (See Theorem 29.)

Remark 8 *The spaces ω^ω and 2^ω are quite flexible. For example it is easy to verify the following:*

1. *Let $b:\omega\to\omega\times\omega$ be a bijection. Then the map $\varphi_b:\omega^{\omega\times\omega}\to\omega^\omega$ defined by $\varphi_b(f)(n)=f(b(n))$ is a homeomorphism that restricts to a homeomorphism of $2^{\omega\times\omega}$ with 2^ω.*

2. *The map* $\langle f_n : n \in \omega\rangle \mapsto g$ *defined by* $g(n,m) = f_n(m)$ *is a homeomorphism between* $(\omega^\omega)^\omega =_{def} \prod_{n\in\omega} \omega^\omega$ *and* $\omega^{\omega\times\omega}$ *that restricts to a homeomorphism of* $(2^\omega)^\omega$ *and* 2^ω.

3. *Define a map* $\varphi : \omega^\omega \times \omega^\omega \to \omega^\omega$ *by*

$$\varphi(f,g)(k) = \begin{cases} f(\frac{k}{2}) & \text{if } k \text{ is even} \\ g(\frac{n-1}{2}) & \text{if } k \text{ is odd.} \end{cases}$$

   Then φ *is a homeomorphism that restricts to a homeomorphism between* $2^\omega \times 2^\omega$ *and* 2^ω.

4. *The map* $(k, f) \mapsto g$ *where*

$$g(n) = \begin{cases} k & \text{if } n = 0 \\ f(n-1) & \text{otherwise} \end{cases}$$

   is a homeomorphism between $\omega \times \omega^\omega$ *and* ω^ω.

The point of this remark is that the product topology is given by "finite information" on a discrete set.

1.5 Finite Sequences and canonical bases

One of the reasons that the Baire Space and the Cantor Space are particularly useful is that their topologies can be understood explicitly as combinatorial objects.

Recall that for any set A, we write $A^{<\omega}$ for the set of finite sequences of elements of A viewed as functions $s : \{0, 1, 2, \ldots, k-1\} \to A$ for some k. If we give A the discrete topology then we can identify a base for the product topology on A^ω as follows.

If $s \in A^{<\omega}$ has length k, then we write $s = (s_0, s_1, \ldots, s_{k-1})$ and $lh(s) = k$. If $a \in A$, we will write $s^\frown a$ for the sequence $\langle s_0, \ldots s_{k-1}, a\rangle$. If $f \in A^\omega$ we will write $f|k$ for the sequence $\langle f(0), f(1), \ldots f(k-1)\rangle$. For $s \in A^{<\omega}$ of length k, let $[s] = \{f \in A^\omega \mid f|k = s\}$.

Proposition 9 *The sets of the form* $[s]$ *form a clopen basis for the product topology on* A^ω.

In particular A^ω is zero-dimensional (totally disconnected).

For these notes we will be considering the case where A is countable or finite; however it is one of the deep insights of Descriptive Set Theory, in the context of

Determinacy, that for uncountable A closed sets in A^ω code complicated subsets of the Baire Space.

If A is countable or finite, then $A^{<\omega}$ is countable. Hence the basis we have given is countable.

We will frequently use a combinatorial characterization of convergence in A^ω. From the description of the basis, we see immediately that:

Proposition 10 *Let A be a set with the discrete topology. Then $\langle x_n : n \in \omega\rangle$ converges to y in A^ω iff for each $k \in \omega$ and all large enough $n, x_n \restriction k = y \restriction k$.*

1.6 Infinite trees

We will analyze closed subsets of A^ω in terms of *branches through trees*.

Definition 11 *Let A be a set. We say that $T \subseteq A^{<\omega}$ is a* tree *if and only if for every $s \in T$, if s has length k and if $j \leq k$, then $(s_0, \ldots, s_{j-1}) \in T$. A node $s \in T$ is* terminal *iff it has no proper extension in T. A tree T is* pruned *iff it has no terminal nodes.*

Definition 12 *Let $T \subseteq A^{<\omega}$ be a tree. A function $f : \omega \to A$ is an* infinite path *through T iff for all $k \in \omega$, $(f(0), f(1), \ldots, f(k-1)) \in T$. We write $[T] = \{f \mid f$ is an infinite path through $T\}$.*

The next proposition characterizes closed sets as the collections of paths through pruned trees.

Proposition 13 *Let $C \subseteq A^\omega$. Then the following are equivalent:*

1. *C is closed.*
2. *There is a tree T such that $C = [T]$.*
3. *There is a pruned tree T such that $C = [T]$.*

We note that the pruned tree corresponding to C is canonical.

If $T \subseteq A^{<\omega}$ is a tree and $s \in T$, then s is a *splitting node* of T iff there are $a \neq b$ in A such that both $s^\frown a, s^\frown b \in T$.

Proposition 14 *Let C be a closed set and T a pruned tree with $C = [T]$. Then C is perfect iff for all $t \in T$ there is an $s \in T$ extending t that is a splitting node.*

1.7 "Lightface" theory

There is a recursive enumeration $\langle s_n : n \in \omega \rangle$ of finite sequences of natural numbers. In view of Proposition 13, we can identify closed subsets of the Baire Space with sets of finite sequences of natural numbers, and thus with sets of natural numbers.

By this device we can make sense of such ideas as "recursive" closed (or open) sets in the Baire Space. By using a recursive pairing function we can make sense of the notions of "recursive unions" and other natural operations. Moreover, as we will see, we can identify continuous functions with their neighborhood diagrams and hence we get a well defined notion of a "recursive" function from ω^ω to ω^ω.

These ideas turn out to be very useful and important in many contexts, but will not play an significant role in these notes.

1.8 Universality properties of Polish Spaces

We begin with a discussion that, in some sense, characterizes Polish spaces.

Definition 15 *The infinite dimensional Hilbert cube is $\mathbb{H} = [0,1]^\omega$ in the product topology, where $[0,1]$ is equipped with the usual topology coming from the real numbers.*

Theorem 16 *Every Polish space is homeomorphic to a subspace of the Hilbert cube.*

$\vdash$ Let X be Polish, and $D = \{x_i \mid i \in \omega\}$ be any dense subset. Let d_X be a complete metric on X compatible with the Polish topology of X such that $|d_X| \leq 1$. Define the following function $f : X \to \mathbb{H}$ coordinatewise by:

$$f(x)(i) = d_X(x, x_i)$$

Claim 17 *f is uniformly continuous.*

$\vdash$ [Proof of Claim 17] Let $\varepsilon > 0$ and $\delta < \frac{\varepsilon}{2}$. Suppose $d_X(x,y) < \delta$. For each i, $d_X(x,x_i) \leq d_X(x,y) + d_X(y,x_i)$ and $d_X(y,x_i) \leq d_X(y,x) + d_X(x,x_i)$. Hence

$$|d_X(x,x_i) - d_X(y,x_i)| \leq d_X(x,y) < \frac{\varepsilon}{2}$$

It follows that

$$\begin{aligned} d_{\mathbb{H}}(f(x), f(y)) &= \sum_{i \in \omega} \frac{|d_X(x,x_i) - d_X(y,x_i)|}{2^{i+1}} \\ &\leq \sum_{i \in \omega} \frac{\varepsilon}{2^{i+2}} \\ &< \varepsilon \end{aligned}$$

⊣

Claim 18 *f is injective.*

⊢ [Proof of Claim 18] Suppose $x \neq y$. By the density of D, choose x_i such that $d_X(x, x_i) < \frac{d_X(x,y)}{2}$. Then $d_X(x, x_i) \neq d_X(y, x_i)$ so that $f(x)(i) \neq f(y)(i)$. ⊣

Claim 19 *f^{-1} is continuous on the range of f.*

⊢ [Proof of Claim 19] Let $\varepsilon > 0$, $x \in X$. Choose $n \in \omega$ such that $d_X(x, x_n) < \frac{\varepsilon}{3}$. Let $\delta = \frac{\varepsilon}{3 \cdot 2^{n+1}}$. Assume, for a proof by contraposition, that $d_X(x, y) \geq \varepsilon$ for some $y \in X$. Then the triangle inequality implies that $d_X(y, x_n) \geq \frac{2\varepsilon}{3}$ so that

$$\begin{aligned} d_{\mathbb{H}}(f(y), f(x)) &\geq \frac{|d_X(x, x_n) - d_X(y, x_n)|}{2^{n+1}} \\ &\geq \frac{\frac{\varepsilon}{3}}{2^{n+1}} \\ &= \delta \end{aligned}$$

⊣

Thus X is homeomorphic to $f(X) \subseteq \mathbb{H}$. ⊣

We will see shortly (Theorem 35) that a subset of a Polish space is Polish with the induced topology iff it is a $\mathcal{G}_\delta$ set. Once we have that result we can draw the following corollary:

Corollary 20 *Let (X, τ) be a topological space. Then X is Polish iff it is homeomorphic to a $\mathcal{G}_\delta$ subset of the Hilbert cube.*

We now discuss the universality properties of ω^ω. We will use these to reduce questions about arbitrary Polish Spaces to questions about ω^ω.

Lemma 21 *Let X be a Polish space, d a complete metric, $D \subseteq X$ be an $\mathcal{F}_\sigma$ set and $\varepsilon > 0$. Then there is a pairwise disjoint sequence of $\mathcal{F}_\sigma$ sets $\langle D_i \mid i \in \omega \rangle$ such that*

$$D = \bigcup_{i \in \omega} D_i = \bigcup_{i \in \omega} \overline{D_i}$$

and $diam(D_i) < \epsilon$ for each $i \in \omega$.

⊢ Since each closed set can be written as a countable union of closed sets of radius less than ϵ we can assume that $D = \bigcup_{j\in\omega} D'_j$ where each D'_j has diameter less than ϵ. Let $D_i = D'_i \setminus \bigcup_{j<i} D_j = D'_i \setminus \bigcup_{j<i} D'_j$. Then each D_i is the intersection of an open set with a closed set and hence is an $\mathcal{F}_\sigma$ set. ⊣

We will frequently be building "Schemes" in our constructions. These are combinatorial objects usually consisting of trees labelled with sets that have certain properties. Here is our first example.

Proposition 22 *Let X be a Polish space with complete separable metric d. There is a sequence of $\mathcal{F}_\sigma$ sets $\langle D_s \mid s \in \omega^{<\omega}\rangle$ with the following properties:*

1. $D_\emptyset = X$
2. $diam(D_s) < \frac{1}{lh(s)}$
3. $\overline{D_\tau} \subseteq D_s$ *whenever* τ *extends* s
4. $D_s = \bigcup_{i\in\omega} D_{s^\frown i}$
5. *If* $i \neq j$, *then* $D_{s^\frown i} \cap D_{s^\frown j} = \emptyset$

⊢ Construct D_s by induction on the length of s. At successor stages, use Lemma 21. ⊣

Theorem 23 *Suppose that X is a Polish Space. Then there is a closed set $C \subseteq \omega^\omega$ and a continuous bijection $f : C \to X$.*

⊢ Fix a complete separable metric d on X. Let $\langle D_i : i \in \omega\rangle$ be a scheme as in Lemma 22 . Let

$$C = \{a \in \omega^\omega : \bigcap_{n\in\omega} D_{a\restriction n} \neq \emptyset\}.$$

We claim that C is closed. Let $\langle a_n\rangle \to b$ with $a_n \in C$. We need to see that $b \in C$, i.e. that $\bigcap_{n\in\omega} D_{b\restriction n} \neq \emptyset$. By passing to a subsequence we can assume that for all n and all $m \geq n, a_m\restriction n = b\restriction n$.

Choose an $x_n \in D_{a_n\restriction n}$. Since the $D_{a_n\restriction n}$ are nested and have diameters going to zero, the x_n form a Cauchy sequence. Let $y = \lim_n x_n$. Then for all n:

$$y \in \overline{D_{a_n\restriction n}} = \overline{D_{b\restriction n}}.$$

Again by the nesting properties of the scheme, we see that

$$\bigcap_n D_{b\restriction n} = \bigcap_n \overline{D_{b\restriction n}}.$$

Hence $y \in \bigcap_n D_{b\restriction n}$ and so $b \in C$.

We now note that since the diameter of $D_{a\restriction n} < \frac{1}{n}$, if $a \in C$ there is a unique element x_a of $\bigcap_n D_{a\restriction n}$. Let $f : C \to X$ be defined by $a \mapsto x_a$.

We claim that f is a continuous bijection. Let $a \neq a'$ be elements of C. Choose an n such that $a\restriction n \neq a'\restriction n$. Then $D_{a\restriction n} \cap D_{a'\restriction n} = \emptyset$. Hence $f(a) \neq f(a')$. To see f is onto we do some Boolean algebra. We will frequently use this type of calculation in the sequel so we do it explicitly once here.

Claim 24

$$\bigcap_n \bigcup_{s\in\omega^n} D_s = \bigcup_{a\in\omega^\omega} \bigcap_n D_{a\restriction n}.$$

$\vdash$ (Claim 24) Let $x \in \bigcap_n \bigcup_{s\in\omega^n} D_s$. Using the fact that the D's are nested we can inductively construct a sequence of numbers $a_0, a_1, \ldots a_i \ldots$ such that for all n, $x \in D_{\langle a_0,\ldots a_{n-1}\rangle}$. Define $a \in \omega^\omega$ by setting $a(i) = a_i$. Then

$$x \in \bigcap_{n\in\omega} D_{a\restriction n}.$$

On the other hand if $x \in \bigcup_{a\in\omega^\omega} \bigcap_n D_{a\restriction n}$, we can let $s = a\restriction n$ to see that for all $n, x \in \bigcup_{s\in\omega^n} D_s$. $\dashv$

We now see that f is onto: since $D_s = \bigcup_{i\in\omega} D_{s^\frown i}$, we can inductively show that for all $n, \bigcup_{s\in\omega^n} D_s = X$.

To see the continuity of f, let $a \in \omega^\omega$ and $\epsilon > 0$. Choose n so large that $1/n < \epsilon$. Since the diameter of $D_{a\restriction n} < 1/n < \epsilon$, we see that for all $b \in [a\restriction n]$ we have $d(a, b) < \epsilon$. $\dashv$

We remark that we cannot prove, in general, that f is a homeomorphism, since closed subspaces of ω^ω are completely disconnected.

Proposition 25 *Let $C \subseteq \omega^\omega$ be closed. Then there is a continuous map $g : \omega^\omega \to C$ such that $g\restriction C$ is the identity (i.e. g is a* retract *of ω^ω to C).*

$\vdash$ By Proposition 13 we can find a pruned tree T such that $C = [T]$. Since T is pruned, we can recursively define a map $G : \omega^{<\omega} \to T$ with the following properties:

1. $lh(s) = lh(G(s))$,

2. if $s \in T$, then $G(s) = s$,

3. for all i, there is a j such that $G(s^\frown i) = G(s)^\frown j$.

By the last clause if $a \in \omega^\omega$ then $\langle G(a \restriction n) : n \in \omega\rangle$ is a coherent collection of elements of T whose union is an infinite branch through T. Hence if we define $g(a) = \bigcup_n G(a \restriction n)$, g is a function from ω^ω to $[T] = C$ which is the identity on C.

We must check that g is continuous. Let $c \in C$. Then a basic open neighborhood of c in C is of the form $[c \restriction n] \cap C$ where $[c \restriction n]$ is the usual neighborhood in ω^ω. If $g(a) = c$ and $b \in [a \restriction n]$, we know that $g(b) \in [c \restriction n] \cap C$. $\dashv$

Combining Theorem 23 and Proposition 25, we easily see:

Theorem 26 *If X is a Polish space, then there is a continuous surjection $\varphi : \omega^\omega \to X$.*

We now turn to the Cantor Space. Using very similar ideas to Theorem 26 we prove:

Theorem 27 *Let X be a compact Polish space. Then there is a continuous surjection*

$$\varphi : 2^\omega \to X.$$

$\vdash$ Inductively build a scheme of closed sets $\langle C_s : s \in \omega^{<\omega}\rangle$ such that

1. for all $s \in \omega^{<\omega}, C_s = \bigcup_{n\in\omega} C_{s^\frown n}$,

2. for all $s \in \omega^{<\omega}, n \in \omega$, the diameter of $C_{s^\frown n}$ is less than $1/lh(s)$,

3. for all $s \in \omega^{<\omega}$, the set of n such that $C_{s^\frown n} \neq \emptyset$ is finite.

To pass from C_s to $C_{s^\frown n}$, consider a finite cover $\langle B_i : i < n = n(s)\rangle$ of C_s by open sets of diameter less than $1/(lh(s) + 1)$. Then $C_{s^\frown i} = C_s \cap \overline{B_i}$ for $i < n(s)$ and $C_{s^\frown i} = \emptyset$ otherwise.

Let $T = \{s \in \omega^{<\omega} : C_s \neq \emptyset\}$. Then T is a finitely branching tree. Define a surjection $p : 2^{<\omega} \to T$ by induction on $lh(s)$ so that for all $s \in 2^{<\omega}$

1. $s \subseteq t$ implies $p(s) \subseteq p(t)$,

   and there are distinct $t_0, \ldots t_{k-1}$ strictly extending s such that

2. $[s] = \bigcup_{i<k}[t_i] \subseteq 2^\omega$ and

3. $\{p(t_i) : i < k\} = \{p(s)^\frown n : p(s)^\frown n \in T\}$.

Then for all $a \in 2^\omega, \bigcup_{k\in\omega} p(a \restriction k) \in [T]$. Moreover if $a \in 2^\omega$ the sequence $\langle C_{p(a|k)} : k \in \omega\rangle$ is a decreasing sequence of non-empty compact sets. As a consequence $\bigcap_{k\in\omega} C_{p(a|k)} \neq \emptyset$. Now define $\varphi : 2^\omega \to X$ by

$$\varphi(a) = \bigcap_{k\in\omega} C_{p(a|k)}.$$

$\dashv$

An interested reader will expand the proof of Theorem 29 to show the following result:

Proposition 28 *Suppose that X is a compact, zero-dimensional, non-empty, perfect Polish space. Then X is homeomorphic to the Cantor Space.*

We next inject 2^ω into each perfect Polish space.

Theorem 29 *If X is a non-empty perfect Polish space, then there is a homeomorphism from 2^ω to a subspace of X.*

$\vdash$ We build a scheme of sets $\langle U_s \mid s \in 2^{<\omega}\rangle$ with the following properties:

1. $U_\emptyset = X$
2. U_s is open in X and non-empty.
3. $diam(U_s) < \frac{1}{lh(s)}$
4. $\overline{U}_{s\hat{}i} \subseteq U_s$
5. $U_{s\hat{}0} \bigcap U_{s\hat{}1} = \emptyset$

To build this scheme recursively, notice that if U_s is defined, then it is a non-empty open set containing no isolated points. So there are points $x \neq y$ in U_s. Then we can choose open sets $U_{s\hat{}0}$ and $U_{s\hat{}1}$ that do not meet, separate x from y and have very small diameter. Since X is complete, for any $a \in 2^\omega$, we have that

$$\bigcap_{i\in\omega} \overline{U_{a|i}} = \{x_a\}$$

for some $x_a \in X$. So let $\varphi : 2^\omega \to X$ be given by $\varphi(a) = x_a$.

To see that φ is continuous, let $\varepsilon > 0$. Let $n \in \omega$ be so large that $\frac{1}{n} < \varepsilon$. Fix $a \in 2^\omega$ and let $s = a{\restriction}n$. Then for all $b \in [s]$, we have

$$d(\varphi(a), \varphi(b)) < \frac{1}{lh(s)} < \varepsilon$$

To see that φ is injective, suppose that $a \neq b$. Then there is some $n \in \omega$ such that $a_n \neq b_n$. By property (5) above, we have that $U_{a{\restriction}n+1} \bigcap U_{b{\restriction}n+1} = \emptyset$. Hence $\varphi(a) \neq \varphi(b)$.

Since the Cantor Space is compact, a 1-1 continuous map from 2^ω into X is a homeomorphism onto its range. $\dashv$

Showing that a subset of a topological space contains a perfect subset says more than that it is uncountable or even that it has power at least $2^{\aleph_0}$, it is a refined version of the Continuum Hypothesis.

Corollary 30 *Any infinite perfect Polish space has cardinality* $2^{\aleph_0}$.

Remark 31 *We note that the range of the function φ constructed in Theorem 29 is compact, since φ is a homeomorphism. As a consequence the φ image of a Borel set is Borel.*

1.9 Subspaces of Polish spaces

In this section we will find necessary and sufficient conditions for a subspace of a Polish space to be Polish. We will also give a tool for refining Polish topologies while still remaining Polish.

Recall the following definition:

Definition 32 *Let (X, d) be a metric space and $A \subseteq X$. Then for any $x \in X$, $d(x, A) =_{def} \inf_{a \in A}\{d(x, a)\}$.*

Definition 33 *Let (X, τ) be a topological space and $A \subseteq X$. We write $\tau \restriction A$ for the topology whose open sets are:*

$$U \in \tau \restriction A \iff U = A \cap O \text{ for some } O \in \tau.$$

If (X, τ) is a Polish space with compatible complete separable metric d, and $C \subseteq X$ is closed, then $d \restriction (C \times C)$ is a complete separable metric on C. In particular the induced topology on a closed subset of a Polish space is Polish. For open sets one must work a bit harder:

Lemma 34 *Let (X, τ) be a Polish space, and $O \subseteq X$ be open. Then $(O, \tau \restriction O)$ is Polish.*

$\vdash$ Let d be any complete nearsighted metric on X generating τ. Define the following metric on O:

$$\hat{d}(x,y) = d(x,y) + \left| \frac{1}{d(x, X \setminus O)} - \frac{1}{d(y, X \setminus O)} \right|$$

It is left as an exercise to verify that $\hat{d}$ is a metric.

We need to show that $\hat{d}$ is a complete metric on O, and that $\hat{d}$ generates $\tau \restriction O$. For the latter, notice that for any x, y we have that $\hat{d}(x,y) \geq d(x,y)$. It follows that for any $x \in O$ and any $\varepsilon > 0$, $B_{\hat{d}}(x, \varepsilon) \subseteq B_d(x, \varepsilon)$. So assume that $x \in O$ and let $\varepsilon > 0$. We need to find some $\delta > 0$ such that $B_d(x, \delta) \subseteq B_{\hat{d}}(x, \varepsilon)$. Since O is open, there is an $r > 0$ such that $B_d(x, r) \subseteq O$. Choose $0 < \delta < r$ such that

$$\delta + \frac{\delta}{r(r-\delta)} < \varepsilon$$

If $y \in O$ is such that $d(x,y) < \delta$, then $d(y, X \setminus O) > r - \delta$. We then have:

$$\begin{aligned}
\hat{d}(x,y) &= d(x,y) + \left| \frac{1}{d(x, X \setminus O)} - \frac{1}{d(y, X \setminus O)} \right| \\
&= d(x,y) + \left| \frac{d(y, X \setminus O) - d(x, X \setminus O)}{d(x, X \setminus O) d(y, X \setminus O)} \right| \\
&\leq \delta + \left| \frac{d(y, X \setminus O) - d(x, X \setminus O)}{r(r-\delta)} \right| \\
&\leq \delta + \frac{\delta}{r(r-\delta)} \\
&< \varepsilon
\end{aligned}$$

To see that $\hat{d}$ is complete, let $\langle x_n \mid n \in \omega \rangle$ be a Cauchy sequence with respect to $\hat{d}$. Since $\hat{d} \geq d$, this sequence is Cauchy with respect to d. Since d is complete in X, there is an $x \in X$ that this sequence converges to x in the d metric. The metrics d and $\hat{d}$ have the same convergent sequences so it suffices to show that $x \in O$. Since the sequence is Cauchy with respect to $\hat{d}$, we have:

$$\lim_{n,m \to \infty} \left| \frac{1}{d(x_n, X \setminus O)} - \frac{1}{d(x_m, X \setminus O)} \right| = 0$$

Thus $\lim_{n \to \infty} \frac{1}{d(x_n, X \setminus O)}$ exists and is finite. Call this limit $r \geq 1$. We now have that

$$0 < \frac{1}{r} = \lim_{n \to \infty} d(x_n, X \setminus O) = d(x, X \setminus O)$$

so that $x \in O$.

$\dashv$

Theorem 35 *Let (X, τ) be a Polish space, and $Y \subseteq X$. Then $(Y, \tau \restriction Y)$ is Polish if and only if Y is a $\mathcal{G}_\delta$ set in X.*

$\vdash$ Suppose first that Y is a $\mathcal{G}_\delta$ set in X, say $Y = \bigcap_{n\in\omega} U_n$ for some U_n open in X. Without loss of generality we can assume that for all $n, U_{n+1} \subseteq U_n$. By the previous Lemma, there is a complete metric d_n on U_n which generates the subspace topology and is bounded by 1. Define the metric $d : Y \times Y \to [0, +\infty)$ by $d(x, y) = \sum_{n\in\omega} \frac{d_n(x,y)}{2^n}$. Since $\frac{d_n}{2^n} \leq d$, the topology generated by d is finer than the subspace topology.

To see that they are the same topology, let $\varepsilon > 0$, $x \in Y$. We need to find an open set O in the subspace topology that is included in $B_d(x, \epsilon)$ and contains x. Choose N so large that

$$\sum_{n\geq N} \frac{1}{2^n} < \frac{\varepsilon}{6}$$

Let $O = \bigcap_{n=1}^{N-1} B_{d_n}(x, \frac{1}{2^n}\frac{\varepsilon}{6})$. Then $O \cap Y$ is an open set in the subspace topology on Y. Fix $y \in Y \cap O$. Then

$$\begin{aligned} d(x, y) &= \sum_{n\in\omega} \frac{d_n(x, y)}{2^n} \\ &\leq \sum_{n=1}^{N-1} \frac{d_n(x, y)}{2^n} + \frac{\varepsilon}{6} \\ &\leq \frac{1}{6} \sum_{n=1}^{N-1} \frac{\varepsilon}{2^n} + \frac{\varepsilon}{6} \\ &< \frac{\varepsilon}{3} \end{aligned}$$

Hence $O \subseteq B_d(x, \varepsilon)$.

To see that d is complete, let $\langle x_k \mid k \in \omega \rangle$ be a Cauchy sequence with respect to d. Then for each $n \in \omega$, $\langle x_k \mid k \in \omega \rangle$ is a Cauchy sequence with respect to d_n. Since all of these metrics are compatible with τ, these Cauchy sequences must converge to the same point, say x. Let $\varepsilon > 0$ and $N \in \omega$ be such that $\sum_{n\geq N} \frac{1}{2^n} < \frac{\varepsilon}{3}$. Let k be so large that if $1 \leq n < N$ and $k' \geq k$ then $d_n(x_{k'}, x) < \frac{1}{2^n}\frac{\varepsilon}{3}$. Then

$$\begin{aligned} d(x_k, x) &= \sum_{n=1}^{N-1} \frac{d_n(x_k, x)}{2^n} + \sum_{n \geq N} \frac{d_n(x_k, x)}{2^n} \\ &< \frac{1}{3}\sum_{n=1}^{N-1} \frac{\varepsilon}{2^n} + \sum_{n \geq N} \frac{1}{2^n} \\ &< \varepsilon \end{aligned}$$

Suppose now that Y is a Polish subspace of X. Since closed subsets of a Polish space are Polish, we can assume that X is the closure of Y. Let $\langle U_i \mid i \in \omega \rangle$ enumerate a basis for the topology of X. Let d be a complete bounded metric on Y generating the subspace topology and d_X be a metric on X generating τ. Fix $x \in Y$ and $\varepsilon > 0$. Then there is an $i \in \omega$ such that $x \in U_i$, U_i has d_X-diameter less than ϵ and $U_i \cap Y$ has d-diameter less than ϵ. Thus

$$Y \subseteq \{x \in X \mid \forall \varepsilon > 0\ \exists n \in \omega\ x \in U_n \wedge diam_d(Y \cap U_n) < \varepsilon \wedge diam_{d_X}(U_n) < \epsilon\}$$

Call this set A. Since

$$A = \bigcap_{m \in \omega} \bigcup \{U_n : diam_d(Y \cap U_n) < \frac{1}{m} \wedge diam_{d_X}(U_n) < \frac{1}{m}\},$$

A is a $\mathcal{G}_\delta$ set. It suffices to show that $A \subseteq Y$.

Let $x \in A$. For each m choose $n(m) \in \omega$ such that $x \in U_{n(m)}$, $diam_d(Y \cap U_{n(m)}) < \frac{1}{m}$ and $diam_{d_X}(U_{n(m)}) < \frac{1}{m}$. Let $x_m \in U_{n(1)} \cap \cdots \cap U_{n(m)} \cap Y$. Such an x_m exists as X is the closure of Y and the intersection of the $U_{n(i)}$'s is non-empty. Since the diameter of the $U_{n(m)}$'s approach 0 as m approaches infinity, the sequence $\langle x_m \mid m \in \omega \rangle$ is Cauchy with respect to d. Hence there is a $z \in Y$ to which this sequence converges. In X, this sequence converges to x, and so $x = z$, and hence $x \in Y$. $\dashv$

2 Borel Sets and the Borel Hierarchy

2.1 σ-Algebras

We start with a long collection of definitions that can be skipped and referred back to.

Definition 36 *Let X be a set. A collection $\mathfrak{A}$ of subsets of X is called a $\sigma-$algebra iff*

1. *$X \in \mathfrak{A}$*
2. *If $A \in \mathfrak{A}$, then $X \setminus A \in \mathfrak{A}$.*
3. *If $\langle A_i \mid i \in \omega \rangle$ is a sequence of sets in $\mathfrak{A}$, then both*

$$\bigcup_{i\in\omega} A_i \in \mathfrak{A} \text{ and } \bigcap_{i\in\omega} A_i \in \mathfrak{A}$$

We write $(X, \mathfrak{A})$ when talking about the structure of the $\sigma-$algebra $\mathfrak{A}$.

We note that to verify that a collection of sets is a σ-algebra it suffices to show that it is closed under complements and *either* countable unions or countable intersections.

Definition 37 *If $\mathcal{A}$ is any collection of subsets of X, we write $\sigma(\mathcal{A})$ for the smallest $\sigma-$algebra of sets containing $\mathcal{A}$.*

If $\mathcal{A}$ is a collection of subsets of X, then the collection of *all* subsets of X, called the *Power Set* of X or $P(X)$, is a σ-algebra including $\mathcal{A}$. We can characterize $\sigma(A)$ "from the outside" by noting that it is the intersection of all σ-algebras that include $\mathcal{A}$. We will get more information by "building" the σ-algebra by closing under the operations of countable union and complement.

Definition 38 *If (X, τ) is a topological space, the* Borel $\sigma-$algebra *of X is defined to be $\sigma(\tau)$, the smallest $\sigma-$algebra containing all open sets of X. We will sometimes denote this as $\mathcal{B}(\tau)$. The members of $\mathcal{B}(\tau)$ are called Borel sets.*

Definition 39 *Let $\mathfrak{A} \subseteq P(X)$ and $\mathfrak{B} \subseteq P(Y)$ be σ-algebras, and $f : X \to Y$. We say that f is $\mathfrak{A}$-measurable iff for every $B \in \mathfrak{B}$, $f^{-1}(B) \in \mathfrak{A}$. A pair $(Y, \mathfrak{A})$ is a* standard Borel space *iff there is Polish space (X, τ) and a bimeasurable bijection $f : (X, \mathcal{B}(\tau)) \to (Y, \mathfrak{A})$.*

If f is a function with domain a topological space, then f is Borel measurable if it is measurable with respect to Borel $\sigma-$algebras.

2.2 Some easy facts about Borel sets and functions

If f is a function then f^{-1} preserves both finitary and infinitary Boolean operations. As a consequence:

Remark 40 *Suppose (X, τ) and (Y, σ) are topological spaces. Then*

1. *$f : X \to Y$ is Borel measurable iff for every $A \in \sigma$, $f^{-1}(A) \in \mathcal{B}(\tau)$.*
2. *If Y has a countable base $\langle B_i : i \in \omega \rangle$ then $f : X \to Y$ is Borel measurable iff $f^{-1}(B_i) \in \mathcal{B}(\tau)$ for every $i \in \omega$.*

We use this to show:

Proposition 41 *Suppose Y is a second countable Hausdorff topological space and X is a Hausdorff topological space. If $f : X \to Y$ is Borel measurable, then $graph(f) = \{(x, y) \in X \times Y \mid y = f(x)\}$ is a Borel set in $X \times Y$.*

$\vdash$ Let $\langle A_i \mid i \in \omega \rangle$ be an enumeration of a basis for the topology of Y. If $A \subseteq Y$ is open and $B \subseteq X$ is Borel, then $B \times A$ is Borel in $X \times Y$. Thus, for each $i \in \omega$, the set $f^{-1}(A_i) \times A_i$ is Borel in $X \times Y$. Since $X \times (Y \setminus A_i)$ is closed, we have that

$$\bigcap_{i \in \omega} \big((X \times (Y \setminus A_i)) \cup \big(f^{-1}(A_i) \times A_i \big) \big) \in \mathcal{B}(X \times Y)$$

we claim that this is exactly $graph(f)$.

Suppose $f(x) = y$ and $i \in \omega$. If $y \notin A_i$, then $(x, y) \in X \times (Y \setminus A_i)$. Otherwise, $x \in f^{-1}(A_i)$.

Conversely, suppose that for all $i \in \omega$, $(x, y) \in (X \times (Y \setminus A_i)) \cup (f^{-1}(A_i) \times A_i)$. If $y' = f(x)$ for some $y' \neq y$, then there is an $i \in \omega$ such that $y \in A_i$ and $y' \notin A_i$. But then $(x, y) \notin X \times (Y \setminus A_i)$, so that $(x, y) \in f^{-1}(A_i) \times A_i$. Then $y' = f(x) \in A_i$, a contradiction. $\dashv$

2.3 The Borel Hierarchy

We now give an inductive construction of the Borel sets that begins with the open sets. To keep track of the level of complexity of the set, we introduce the following "logical" notation:

Definition 42 *Let (X, τ) be a Polish topological space. The levels of the Borel hierarchy are as follows:*

- $\mathbf{\Sigma}^0_1$ *sets are the open subsets of* X.
- $\mathbf{\Pi}^0_1$ *sets are the closed sets.*
- *Suppose the hierarchy has been defined up to (but not including) the ordinal level* $\alpha < \omega_1$. *Then* $A \in \mathbf{\Sigma}^0_\alpha$ *if and only if there is a sequence* $\langle B_i \mid i \in \omega \rangle$ *of subsets of* X *with* $B_i \in \mathbf{\Pi}^0_{\beta_i}$ *such that for each* $i \in \omega$, $\beta_i < \alpha$ *and*

$$A = \bigcup_{i \in \omega} B_i$$

- $B \in \mathbf{\Pi}^0_\alpha$ *if and only if there is a subset of* X, $A \in \mathbf{\Sigma}^0_\alpha$ *such that* $B = X \setminus A$.
- *We write* $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$.

We note that this is the "boldface" theory, indicated by the use of boldface capital Greek letters. In many texts this is written with a "tilde" (as in $\underset{\sim}{\Sigma}$) to emphasize the difference between the boldface and the lightface theory.

Remark 43 *In more classical language,* $\mathbf{\Pi}^0_2$ *sets are often called* $\mathcal{G}_\delta$ *subsets of a space* X. *The* $\mathbf{\Sigma}^0_3$ *sets are called the* $\mathcal{G}_{\delta\sigma}$ *sets. Similarly,* $\mathbf{\Sigma}^0_2$ *sets are called* $\mathcal{F}_\sigma$ *sets, etc.*

We will also abuse the language in a standard way by using $\mathbf{\Sigma}^0_\alpha$ (and $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$) both as an adjective and to stand for the collection of sets that are $\mathbf{\Sigma}^0_\alpha$ (and $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Delta}^0_\alpha$). We will also frequently use these notations to mean the classes of all subsets of all Polish spaces that have a given complexity. If we want to point to the collection of $\mathbf{\Sigma}^0_\alpha$ subsets of a particular Polish space X we will sometimes write $\mathbf{\Sigma}^0_\alpha(X)$.

The following result justifies Definition 42 :

Lemma 44 *For all* $0 < \alpha < \omega_1$ *we have that* $\mathbf{\Sigma}^0_\alpha \cup \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Delta}^0_{\alpha+1}$. *The Borel sets of a Polish space* X *are precisely* $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)$.

$\vdash$ We first note that $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Sigma}^0_2$. This is the fact that every open set is a countable union of closed sets, i.e. an $\mathcal{F}_\sigma$ set. It now follows immediately that $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Sigma}^0_{\alpha'}$ for all $\alpha < \alpha' < \omega_1$.

By passing to complements we see that $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Pi}^0_{\alpha+1}$. Since it is immediate from the definition that $\mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_{\alpha+1}$ for all α, we see that for all $\alpha, \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Delta}^0_{\alpha+1}$. Since $\mathbf{\Delta}^0_{\alpha+1}$ is closed under complements, we see that $\mathbf{\Sigma}^0_\alpha \subseteq \mathbf{\Delta}^0_{\alpha+1}$.

From these observations it follows that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ is closed under complements. To see that it is closed under countable unions, suppose that $A_i \in \mathbf{\Sigma}^0_{\alpha_i}$ for each $i \in \omega$, where each α_i is a countable ordinal. Then, since a countable union of countable sets is countable (or weaker: ω_1 is regular), there is an ordinal α^* such that for all $i, \alpha_i + 1 < \alpha^*$. By the first assertion, $A_i \in \mathbf{\Pi}^0_{\alpha_i+1}$, and hence $\bigcup_i A_i \in \mathbf{\Sigma}^0_{\alpha^*}$.

We have shown that $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ is a σ-algebra containing the open sets and hence is a σ-algebra containing all of the Borel sets.

On the other hand every member of $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha$ is built from open sets by taking countable unions and complements, and hence is Borel. $\dashv$

A more refined version of Lemma 44 is given by:

Lemma 45 *Let $0 < \alpha < \omega_1$.*

1. *$\mathbf{\Sigma}^0_\alpha$ is closed under countable unions and finite intersections.*
2. *$\mathbf{\Pi}^0_\alpha$ is closed under countable intersections and finite unions.*
3. *$\mathbf{\Delta}^0_\alpha$ is closed under finite intersections and unions, as well as taking complements.*
4. *Let $f : X \to Y$ be a continuous function between Polish spaces. If $A \in \mathbf{\Sigma}^0_\alpha(Y)$ (respectively $\mathbf{\Pi}$ or $\mathbf{\Delta}$), then $f^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(X)$ (respectively $\mathbf{\Pi}$ or $\mathbf{\Delta}$).*

$\vdash$ Inspection of the proof of Lemma 44 yield the first items. To see the last, we note again that the inverses of functions preserve Boolean operations. If f is continuous, then the inverse image of an open set is open. The rest of the claim follows by induction. $\dashv$

Corollary 46 *Let X and Y be Polish spaces. If $A \in \mathbf{\Sigma}^0_\alpha(X \times Y)$ (respectively $\mathbf{\Pi}$ or $\mathbf{\Delta}$), then for all $x \in X$ the set $A_x =_{def} \{y \in Y \mid (x, y) \in A\} \in \mathbf{\Sigma}^0_\alpha(Y)$ (respectively $\mathbf{\Pi}$ or $\mathbf{\Delta}$).*

$\vdash$ Let $f : Y \to X \times Y$ be given by $f(y) = (x, y)$. Then $A_x = f^{-1}(A)$. $\dashv$

We will often compute the complexity of a Borel set by writing its definition as a logical formula not involving the implication symbol. In such a formula, $\exists n \in \omega$ corresponds to a countable union and $\forall n \in \omega$ corresponds to a countable intersection.

A pitfall of this method is that the implication symbol in

$$(\exists n \in \omega)(\varphi) \implies (\exists n \in \omega)(\psi)$$

must be rewritten as

$$(\neg(\exists n \in \omega)(\varphi) \vee (\exists n \in \omega)(\psi))$$

before its complexity can be recognized. The complexity is that of the formula

$$(\forall n \in \omega)(\neg\varphi) \vee (\exists n \in \omega)(\psi).$$

A clear understanding of this is not essential at this point; we will illustrate the relevance of the logical notation mostly with examples.

Example 47 *1. Let X be a Polish space and $F \subseteq X$ be countable, then $F \in \mathbf{\Sigma}^0_2(X)$.*

2. Let $A \subseteq \omega^\omega$ be the set of functions that are eventually injective. Then $f \in A$ if and only if

$$\exists n \forall m_1, m_2 > n \left(f(m_1) \neq f(m_2) \vee m_1 = m_2 \right).$$

This can be written as:

$$\bigcup_{n \in \omega} \bigcap_{\substack{m_1, m_2 > n \\ m_1 \neq m_2}} \{f : f(m_1) \neq f(m_2)\}.$$

Thus A is a $\mathbf{\Sigma}^0_2$ set.

An important example involves the group of permutations of the natural numbers denoted as S_∞:

Example 48 *We can view S_∞ as a subset of the Baire Space, since $f \in S_\infty$ if and only if f is a bijection from ω to ω. Now the set of injective functions in ω^ω is closed: f is injective if and only if $\forall m_1, m_2 \left(f(m_1) \neq f(m_2) \vee m_1 = m_2 \right)$. Thus this is a countable intersection of closed sets of the form*

$$\{f : f(m_1) \neq f(m_2)\}$$

Furthermore, f is onto if and only if $\forall n \exists m (f(m) = n)$, hence a countable intersection of open sets of the form $\{f : \exists m (f(m) = n)\}$. This implies that the set of onto functions is a $\mathbf{\Pi}^0_2$ set. S_∞ is therefore the intersection of a closed set with a $\mathbf{\Pi}^0_2$ set, which is $\mathbf{\Pi}^0_2$, by Lemmas 44 and 45.

Exercise 49 *1. Show that S_∞ is a Polish group.*

2. Show that S_∞ is not a $\mathbf{\Sigma}^0_2$ subset of ω^ω.

(Hint: Use the Baire Category Theorem.)

2.4 Examples of coding

Here are some important examples of coding:

Example 50 *Let $R \subseteq \omega \times \omega$. Then we can associate an element of $2^{\omega\times\omega}$ to R by considering its characteristic function $\chi_R : \omega \times \omega \to 2$ defined by*

$$\chi_R(m,n) = \begin{cases} 1 & \textit{if } (m,n) \in R \\ 0 & \textit{otherwise.} \end{cases}$$

If $f \in 2^{\omega\times\omega}$, we will say that f codes *R if $R = \{(m,n) : f(m,n) = 1\}$. Then*

$$\mathcal{LO} =_{def} \{f \in 2^{\omega\times\omega} : f \textit{ codes a linear ordering of a subset of } \omega\}$$

is a closed set.

Example 51 *Let $\langle \sigma_n : n \in \omega\rangle$ be a 1-1 enumeration of $\omega^{<\omega}$ such that $\sigma_m \subsetneq \sigma_n$ implies $m \leq n$. (Shorter sequences come first.) For $T \subseteq \omega^{<\omega}$, let $f_T \in 2^\omega$ be the characteristic function of $\{n : \sigma_n \in T\}$. Then:*

$$\{f_T : T \textit{ is an infinite tree}\}$$

is a $\mathbf{\Pi}^0_2$ subset of 2^ω

Exercise 52 *What is the complexity of*

$$\{f_T : T \textit{ is a finitely branching tree}\}?$$

$$\{f_T : T \textit{ is a tree}\}?$$

We will refer to the space of trees with the topology induced from 2^ω as $\mathcal{T}rees$.

Using the fact that $2^\omega \cong \prod_{n\in\omega}(2^{\omega^n})^\omega$, we can similarly code any structure in a countable first order language. For an EC_Δ class of structures, the set of codes is a Borel set. To give a simple example:

Example 53 *Let $G = \langle g_n : n \in \omega\rangle$ be a countable group with $g_0 = e_G$. Associate to G an element $\chi_G \in 2^{\omega\times\omega\times\omega}$ by setting $\chi_G(m,n,p) = 1$ iff $g_m \cdot_G g_n = g_p$. Then the collection of χ_G for G a countable group is a $\mathbf{\Pi}^0_2$ set.*

Note that we can let S_∞ act on $2^{\omega\times\omega\times\omega}$ by setting $(g\cdot x)(k,l,m) = x(g^{-1}k, g^{-1}l, g^{-1}m)$ for $g \in S_\infty$ and $x \in 2^{\omega\times\omega\times\omega}$. Then for countable groups G, H we have $G \cong H$ iff there is a $g \in S_\infty$,

$$g\chi_G = \chi_H.$$

More generally this S_∞ action on $2^\omega \cong \prod_{n\in\omega}(2^{\omega^n})^\omega$ codes isomorphism for any class of countable structures, a fact that is relevant to model theory.

Example 54 *Let* $\varphi : \omega^\omega \to 2^{\omega\times\omega}$ *be defined by* $f \mapsto \chi_f$*, where* $\chi_f(m,n) = 1$ *iff* $f(m) = n$*. Then* φ *is continuous and 1-1. Moreover* φ^{-1} *is continuous on the range of* φ*. In fact the range is* $\mathcal{G}_\delta$*. This can be seen using Theorem 35, or by the following computation:*

If $a \in 2^{\omega\times\omega}$ *then* a *is in the range of* φ *iff*

1. $(\forall m)(\exists n)a(m,n) = 1$

2. $(\forall m)(\forall n)(\forall p)((a(m,n) = 1 \wedge a(m,p) = 1) \to n = p)$

We note that this shows that if X *is perfect Polish then there is a* $\mathcal{G}_\delta$ *subset of* X *that is homeomorphic to* ω^ω*.*

The next exercise will be useful when we consider analytic and co-analytics sets:

Exercise 55 *Let* X *be a perfect Polish space. Then there is a bijection* $f : X \to \omega^\omega$ *such that* f *and* f^{-1} *are Borel measurable. (Hint: Build Borel injections in each direction and then use the Cantor-Schroeder-Bernstein theorem.)*

2.5 Universal Sets

Definition 56 *Let* X*,* Y *be any spaces and* Γ *be a point class of* X*, that is,* Γ *is a collection of subsets of* X*. Let* $A \subseteq Y \times X$*. Then* A *is universal for* Γ *if and only if* $\Gamma = \{A_y \mid y \in Y\}$*.*

Theorem 57 *Let* X *be a second countable metric space. For* $1 \leq \alpha < \omega_1$ *there is a* $U_\alpha \in \mathbf{\Sigma}^0_\alpha(2^\omega \times X)$ *that is universal for* $\mathbf{\Sigma}^0_\alpha(X)$*.*

$\vdash$ We proceed by induction on α. Let $\langle B_i \mid i \in \omega\rangle$ be a basis for the topology of X. Let

$$U_1 = \bigcup_{n\in\omega}\{(f,x) \mid f(n) = 1 \wedge x \in B_n\}$$

It is clear from the definition that U_1 is open. To see that it is universal, let $O \subseteq X$ be open. Define the following function $f : \omega \to 2$ by $f(n) = 1$ if and only if $B_n \subseteq O$. We claim that $O = (U_1)_f$.

If $x \in (U_1)_f$, then there is an $n \in \omega$ such that $f(n) = 1$ and $x \in B_n$. Hence $B_n \subseteq O$, and thus $x \in O$. Conversely, since O is open, if $x \in O$ there is an $i \in \omega$ such that $x \in B_i \subseteq O$. But then $f(i) = 1$ and $x \in B_i$, so that $x \in (U_1)_f$.

Now suppose that for each $\beta < \alpha$ we have a $U_\beta \in \mathbf{\Sigma}^0_\beta(2^\omega \times X)$ which is universal for $\mathbf{\Sigma}^0_\beta(X)$. Let $V_\beta = (2^\omega \times X) \setminus U_\beta$. It is easy to see that $V_\beta \in \mathbf{\Pi}^0_\beta(2^\omega \times X)$ and is universal for $\mathbf{\Pi}^0_\beta(X)$. If α is a limit ordinal, let $\langle \beta_i \mid i \in \omega \rangle$ be an increasing sequence of ordinals approaching α. If α is a successor ordinal, say $\alpha = \beta + 1$, set $\beta_i = \beta$ for all $i \in \omega$.

Fix any homeomorphism $F : 2^\omega \to (2^\omega)^\omega$. For $f \in 2^\omega$, denote $F(f) = \langle (f)_i \mid i \in \omega \rangle$. Define U_α to be:

$$U_\alpha = \{(f, x) \mid x \in \bigcup_{i \in \omega} (V_{\beta_i})_{(f)_i}\}.$$

We first claim that U_α is $\mathbf{\Sigma}^0_\alpha$. Since

$$\{(f, x) \mid x \in \bigcup_{i \in \omega} (V_{\beta_i})_{(f)_i}\} = \bigcup_{i \in \omega} \left\{(f, x) \mid x \in (V_{\beta_i})_{(f)_i}\right\}$$

it suffices to show that for a fixed i, $B_i =_{def} \{(f, x) : x \in (V_{\beta_i})_{(f)_i}\}$ is $\mathbf{\Pi}^0_{\beta_i}$.

Let $Y = (2^\omega)^\omega \times X$ and $G : 2^\omega \times X \to Y$ be given by $(f, x) \mapsto (\langle (f)_i \rangle, x)$. Then for each i, $\{(\vec{f}, x) \in Y : x \in (V_{\beta_i})_{(f)_i}\} \in \mathbf{\Pi}^0_{\beta_i}$. Hence $G^{-1}(\{(\vec{f}, x) \in Y : x \in (V_{\beta_i})_{(f)_i}\}) = B_i \in \mathbf{\Pi}^0_{\beta_i}$.

It remains to show that U_α is universal. Let $A \in \mathbf{\Sigma}^0_\alpha(X)$. Then there is a sequence of sets $\langle B_i : i \in \omega \rangle$ such that for some $\delta_i < \alpha, B_i \in \mathbf{\Pi}^0_{\delta_i}$ and $A = \bigcup_i B_i$. Since the sequence $\langle \beta_i \rangle$ is not bounded in α and the $\mathbf{\Pi}^0_\xi$'s increase with ξ, we can assume that $\delta_i = \beta_i$. Choose $f_i \in 2^\omega$ such that $B_i = (V_{\beta_i})_{f_i}$. Let $f \in 2^\omega$ be such that $(f)_i = f_i$. We see that

$$\begin{aligned} x \in (U_\alpha)_f &\Leftrightarrow \exists i \; x \in (V_{\beta_i})_{f_i} \\ &\Leftrightarrow \exists i \; x \in B_i \\ &\Leftrightarrow x \in A \end{aligned}$$

Thus $A = (U_\alpha)_f$. ⊣

Corollary 58 *Let X be a perfect Polish space. Then for all $0 < \alpha < \omega_1$ there is a set $U_\alpha \in \mathbf{\Sigma}^0_\alpha(X \times X)$ that is universal for $\mathbf{\Sigma}^0_\alpha(X)$.*

⊢ Let $\varphi : 2^\omega \to X$ be a homeomorphism of 2^ω into X. Then the range of φ is compact, and hence closed. If we let $F : 2^\omega \times X \to X \times X$ be defined by setting $f(a, x) = (\varphi(a), x)$, then it is easy to check that F takes closed sets to closed sets. Let V_1' be a $\mathbf{\Pi}^0_1(2^\omega \times X)$ set that is universal for $\mathbf{\Pi}^0_1$ subsets of X. Then $V_1 = F[V_1']$ is $\mathbf{\Pi}^0_1$ and is universal for $\mathbf{\Pi}^0_1$ subsets of X. If we let $U_1 = (X \times X) \setminus V_1$, then U_1 is $\mathbf{\Sigma}^0_1(X \times X)$ and universal for $\mathbf{\Sigma}^0_1$ subsets of X.

For $\alpha > 1$, life is easier: Let $U'_\alpha \in \mathbf{\Sigma}^0_\alpha(2^\omega \times X)$ be universal for $\mathbf{\Sigma}^0_\alpha(X)$. Then $U_\alpha =_{def} F[U'_\alpha]$ is the intersection of a $\mathbf{\Sigma}^0_\alpha(X \times X)$ set with the range of F. Since the range of F is closed and $\alpha \geq 2$, this is $\mathbf{\Sigma}^0_\alpha$ and is universal for $\mathbf{\Sigma}^0_\alpha$ subsets of X. $\dashv$

The reader ambitious enough to try to directly construct universal subsets of $X \times X$ for an arbitrary X will begin to appreciate the utility of working directly with 2^ω.

The next argument is Cantor's classical diagonal argument cast in this setting.

Corollary 59 *Let X be a (non-empty) perfect Polish space. Then*

$$\left\{ \mathbf{\Sigma}^0_\alpha(X) \mid 1 \leq \alpha < \omega_1 \right\}$$

is a strictly increasing collection of subsets of $P(X)$.

$\vdash$ Assume towards a contradiction that $\mathbf{\Sigma}^0_{\alpha+1}(X) \subseteq \mathbf{\Sigma}^0_\alpha(X)$. Then $\mathbf{\Pi}^0_{\alpha+1}(X) \subseteq \mathbf{\Pi}^0_\alpha(X)$. Consider a set $U_{\alpha+1} \in \mathbf{\Sigma}^0_{\alpha+1}(X \times X)$ universal for $\mathbf{\Sigma}^0_{\alpha+1}(X)$ and let

$$D = \{x \mid x \notin (U_{\alpha+1})_x\}.$$

Then $D \in \mathbf{\Pi}^0_{\alpha+1}$. By our assumption, we have:

$$\mathbf{\Pi}^0_{\alpha+1} \subseteq \mathbf{\Pi}^0_\alpha \subseteq \mathbf{\Sigma}^0_{\alpha+1}$$

so that $D \in \mathbf{\Sigma}^0_{\alpha+1}$. By the universality of $U_{\alpha+1}$ there is a y such that $D = (U_{\alpha+1})_y$.

Following Cantor:

$$y \in D \iff (y, y) \in U_{\alpha+1} \iff y \notin D.$$

$\dashv$

Exercise 60 *Let X be perfect Polish.*

1. *Deduce that for all $0 < \alpha < \omega_1$, $\mathbf{\Sigma}^0_{\alpha+1}(X) \nsubseteq \mathbf{\Sigma}^0_\alpha(X)$ directly from the statement that $\mathbf{\Sigma}^0_{\alpha+1}(2^\omega) \nsubseteq \mathbf{\Sigma}^0_\alpha(2^\omega)$.*

2. *Show that for all $0 < \alpha < \omega_1$ there is no $\mathbf{\Delta}^0_\alpha(X \times X)$ that is universal for $\mathbf{\Delta}^0_\alpha(X)$.*

2.6 Changing Topologies

For the proof of the next theorem, we need the following Lemma:

Lemma 61 *Let (X, τ) be a Polish space, and suppose that $\langle \tau_n \mid n \in \omega \rangle$ is a sequence of Polish topologies on X with $\tau \subseteq \tau_n$ for each n. Then the topology τ_∞ which is generated by $\cup \tau_n$ is Polish. Moreover, if $\mathcal{B}(\tau_n) = \mathcal{B}(\tau)$ for each n, then $\mathcal{B}(\tau_\infty) = \mathcal{B}(\tau)$.*

$\vdash$ Write X_n for the space (X, τ_n) and X_∞ for the product space $\prod_{n \in \omega} X_n$. By our earlier results, X_∞ is a Polish space. Let $\varphi : X \to X_\infty$ be defined by $\varphi(z) = (z, z, z, \ldots)$.

We claim that $\varphi[X]$ is closed in X_∞. To see this, let $y \in X_\infty$ with $y \notin \varphi[X]$. There are then $n, m \in \omega$ with $y(n) \neq y(m)$. Choose open sets U_n, U_m such that $y(n) \in U_n$, $y(m) \in U_m$, and $U_n \cap U_m = \emptyset$. Let $V \subseteq X_\infty$ be the open set $V = X \times \cdots \times X \times U_n \times X \times \cdots \times X \times U_m \times X \times \ldots$, where U_i occurs in the ith place. Then V is an open neighborhood of y that is disjoint from $\varphi[X]$.

Since $\varphi[X]$ is closed, $\varphi[X]$ is a Polish space in the induced topology ν from the product space. The lemma follows if we can show that ν coincides with the topology given by $\cup_n \tau_n$. Let O be a basic open set for ν. Then there are $O_1 \in \tau_1$, $O_2 \in \tau_2$, $\ldots$, $O_n \in \tau_n$ for some $n \in \omega$ such that $O = \{x \in X \mid \varphi(x) \in O_1 \times O_2 \times \cdots \times O_n \times X \times \ldots\}$. Thus $O = O_1 \cap O_2 \cap \cdots \cap O_n$ and hence O is in the topology generated by the τ_n's. $\dashv$

Exercise 62 *Show that in the previous Lemma, if $\mathcal{B}(\tau_n) = \mathcal{B}(\tau)$ for each n, then $\mathcal{B}(\tau_\infty) = \mathcal{B}(\tau)$.*

Theorem 63 *Suppose that (X, τ) is Polish and $A \subseteq X$ is Borel. Then there is a Polish topology $\tau_A \supseteq \tau \cup \{A\}$ that has the same Borel sets as τ.*

$\vdash$ We first consider the case where A is a closed set F. Let $S_1 = (F, \tau \restriction F)$ and $S_2 = (X \setminus F, \tau \restriction X \setminus F)$, which are both Polish spaces. Let d_1 be a complete compatible metric on S_1 and d_2 a complete compatible metric on S_2, both bounded by 1. Define

$$d_3(x, y) = \begin{cases} d_1(x, y) & \text{if both } x, y \in F \\ d_2(x, y) & \text{if both } x, y \in X \setminus F \\ 2 & \text{otherwise.} \end{cases}$$

This metric is Polish on $S_1 \oplus S_2$ and generates a topology on X that contains both τ and F. Thus for any Polish space (X, τ) and any closed set F we can extend the topology τ to contain F, be Polish and have the same Borel sets as τ.

Let $G = \{A \subseteq X \mid$ there is a Polish topology $\tau_A \supseteq \tau$ with the same Borel sets as τ with $A \in \tau_A\}$. We want to show that every Borel set belongs to G. Since G contains all the open and closed sets, we need only show that G is closed under countable unions and complementation.

Suppose $A \in G$. Then (X, τ_A) is Polish with $A \in \tau_A$. So $X \setminus A$ is τ_A-closed, and by the first paragraph of the proof we can extend τ_A to $\tau_{A, X\setminus A}$. G is therefore closed under taking complements. Suppose $\langle A_n \mid n \in \omega \rangle$ is a sequence in G. Then τ_{A_n} satisfy the requirements of Lemma 61, hence $\cup_{n\in\omega} A_n \in G$ with Polish topology τ_∞ on X. $\dashv$

Corollary 64 *If (X, τ) is Polish and $Y \subseteq X$ is Borel, then $(Y, \{B \cap Y \mid B \in \mathcal{B}(\tau)\})$ is a standard Borel space.*

The next example is cautionary:

Example 65 *Let $X = [0, 1]$ and τ be the usual topology. Then $A =_{def} \mathbb{Q} \cap X$ is an $\mathcal{F}_\sigma$-set. Let σ be the topology generated by A and τ. We claim that σ is* not *Polish.*

For if it were, the Baire Category theorem would apply to this topology. If we write $A = \bigcup_{q\in A}\{q\}$, then, as each $\{q\}$ is a closed set, one of the $\{q\}$ must have non-empty interior. The only way this could happen is if $\{q\}$ is an open set. But an easy argument shows that there are no singleton open sets in σ.

3 Analytic Sets

Definition 66 *Let X be a Polish space. A set $A \subseteq X$ is analytic if and only if it is the continuous image of a Polish space Y.*

If $Z = X \times Y$ then $\Pi_X : Z \to X$ is the map $(x, y) \mapsto x$ and Π_Y is $(x, y) \mapsto y$. Note that if $A \subseteq Z, \Pi_X[A] = \{x : (\exists y)\ (x, y) \in A\}$

We will denote the collection of analytic subsets of X by $\mathbf{\Sigma}^1_1(X)$, with the usual abuses of language. There are many equivalent ways of thinking of analytic sets. The following Proposition gives several of them:

Proposition 67 *The following are equivalent:*

1. *A is analytic.*
2. *A is the continuous image of ω^ω.*

3. *There is a Polish space Y and a closed set $F \subseteq X \times Y$ such that*

$$A = \Pi_X(F)$$

4. *There is a closed set $F \subseteq X \times \omega^\omega$ such that*

$$A = \Pi_X(F)$$

5. *There is a Borel set $B \subseteq X \times \omega^\omega$ such that*

$$A = \Pi_X(B)$$

$\vdash$ The equivalence of the first two conditions is immediate because ω^ω is Polish and every Polish space Y is the continuous image of ω^ω.

For the proof of (1) iff (3), assume A is analytic and $f : Y \to X$ is continuous with Y Polish and $A = rng(f)$. Then the inverse graph of $f \subseteq X \times Y$ is closed. Moreover,

$$\begin{aligned} x \in A &\Leftrightarrow (\exists y) f(y) = x \\ &\Leftrightarrow x \in \Pi_X(graph^{-1}(f)) \end{aligned}$$

Conversely, suppose $F \subseteq X \times Y$ is closed. Then F is a Polish space with the induced topology. The function $p : F \to X$ defined by $p((x, y)) = x$ is continuous with $A = rng(p)$ as desired.

The proof that (2) implies (4) is exactly parallel to the proof that (1) implies (3). Moreover it is obvious that (4) implies (5). To see that (5) implies (1) we fix a Borel set $B \subseteq X \times \omega^\omega$ with $A = \Pi_X(B)$. By Theorem 63 we can extend the topology on $X \times \omega^\omega$ to a larger topology making a space Y with underlying set $X \times \omega^\omega$ in which B is closed. Since the topology on Y is finer than the original topology on $X \times \omega^\omega$ the map $\Pi_X : Y \to X$ is still continuous. Since B is closed, it is Polish with the induced topology. Hence $\Pi_X : B \to X$ is a continuous map from a Polish space to X. Finally we note that A is the range of $\Pi_X \restriction B$. $\dashv$

From Theorem 63, it follows immediately that:

Corollary 68 *If (X, τ) is Polish and $B \subseteq X$ is Borel then B is analytic. Moreover continuous images of Borel sets are analytic.*

$\vdash$ The second statement is stronger. Let $f : Y \to X$ and $A = f(B)$. By Theorem 63, we can retopologize X with a σ so that B is open and $\sigma \restriction B$ is Polish. Since we have added open sets $f \restriction B$ is still continuous. Hence A is the image of a Polish space under a continuous function. $\dashv$

Example 69 *In the notation of Example 51, show that*

$$\{f_T : T \subseteq \omega^{<\omega} \textit{is a tree with an infinite branch}\}$$

is a $\mathbf{\Sigma}^1_1$ *subset of* 2^ω. *Why does this differ from Example 51?*

Example 70 *In the notation of Example 50, show that*

$$\{f \in \mathcal{LO} : f \textit{ is not a well-ordering}\}$$

is a $\mathbf{\Sigma}^1_1$*-set.*

3.1 Universal analytic sets and a non-Borel analytic set

Theorem 71 *For any Polish space* X *there is an analytic set* $U \subseteq \omega^\omega \times X$ *such that, if* $A \subseteq X$ *is analytic, then there is an* $f \in \omega^\omega$ *with* $A = U_f = \{x \mid (f, x) \in U\}$.

In other words, there is an analytic set that is universal for the class of analytic sets of a Polish space.

$\vdash$ Let $V \subseteq \omega^\omega \times (\omega^\omega \times X)$ be closed and universal for closed sets in $(\omega^\omega \times X)$. Let $U = \{(f, x) \mid \exists g \in \omega^\omega\ (f, g, x) \in V\}$, which is analytic. To show that U is universal, let $A \subseteq X$ be analytic. Then there is a closed $F \subseteq \omega^\omega \times X$ such that $A = \{x \in X \mid \exists g \in \omega^\omega\ (g, x) \in F\}$. Since F is closed, there is an $f \in \omega^\omega$ such that $F = V_f$. But then

$$\begin{aligned} U_f &= \{x \mid \exists g\ (f, g, x) \in V\} \\ &= \{x \mid \exists g\ (g, x) \in F\} \\ &= A \end{aligned}$$

$\dashv$

Exercise 72 *Use Example 54 to show that if* X *is a perfect Polish space, then there is a set* $A \subseteq X \times X$ *that is universal for analytic subsets of* X.

Definition 73 *Let* $\langle X_n : n \in \omega \rangle$ *be a sequence of Polish spaces and assume that* $\langle d_n \rangle$ *is a sequence of complete metrics witnessing that the* X_n*'s are Polish that are bounded by* 1. *Let* $Y = \bigcup_{n \in \omega} X_n$ *and define a metric on* Y *by setting:*

$$d_Y(x, y) = \begin{cases} d_n(x, y) & \textit{if both } x, y \in X_n \textit{ for some } n \\ 2 & \textit{otherwise.} \end{cases}$$

Then the separated sum *of the* X_n*'s is the space* (Y, τ) *where* τ *is the topology of* d_Y.

Theorem 74 *Let X be a Polish space, and let $\mathbf{\Sigma}^1_1(X) = \{A \subseteq X \mid A \text{ is analytic}\}$. Then $\mathbf{\Sigma}^1_1(X)$ is closed under countable unions and intersections.*

$\vdash$ Let $\langle A_n \mid n \in \omega\rangle$ be a sequence of analytic sets.

We first show that $\bigcup_n A_n$ is analytic. For each n there is a Polish space X_n and a closed set $F_n \subseteq X \times X_n$ such that $A_n = \Pi_X(F_n)$. Let Y be the separated sum of the X_n's. Let $F_\infty = \cup_{n\in\omega} F_n$. We claim that F_∞ is closed in $X \times Y$.

To see this, let $\langle (x_n, y_n) \mid n \in \omega\rangle$ be a sequence in $X \times Y$ converging to some (x, y) such that for every $n \in \omega$, $(x_n, y_n) \in F_\infty$. Since the y_n converge there is an N such that $y_n \in X_N$ for all large n. Hence for all large $n, (x_n, y_n) \in F_N$. Since F_N is closed we must have $(x, y) \in F_N \subseteq F_\infty$.

Now $x \in \Pi_X(F_\infty)$ if and only if for some n, $x \in \Pi_X(F_n)$, that is to say if and only if $x \in \cup_{n\in\omega} A_n$.

For countable intersections, let $V \subseteq \omega^\omega \times (\omega^\omega \times X)$ be a closed universal set for $\mathbf{\Pi}^0_1(\omega^\omega \times X)$. Then for every $n \in \omega$ there is an $f_n \in \omega^\omega$ such that

$$A_n = \{x \in X \mid \exists g \in \omega^\omega \ (f_n, g, x) \in V\}.$$

Fix a homeomorphism $g \mapsto \langle (g)_n : n \in \omega\rangle$ between ω^ω and $(\omega^\omega)^\omega$. Let V' be such that

$$(g, x) \in V' \Leftrightarrow \ \forall n \ (f_n, (g)_n, x) \in V$$

Then V' is closed as $\{(g, x) \mid (f_n, (g)_n, x) \in V\}$ is closed.

Now $x \in \cap_{n\in\omega} A_n$ if and only if for every n there is an h_n such that $(f_n, h_n, x) \in V$. Suppose that there is such a sequence $\langle h_n : n \in \omega\rangle$. Let h be such that $(h)_n = h_n$ for every n. Then $(h, x) \in V'$. On the other hand if $(h, x) \in V'$ then for all $n, (f_n, (h)_n, x) \in V$. Thus $x \in \cap_{n\in\omega} A_n$ if and only if there is a g such that $(g, x) \in V'$, whereby $\cap_{n\in\omega} A_n$ is analytic. $\dashv$

Lemma 75 *Let X, Y be Polish spaces. If $A \subseteq X$ is an analytic subset of X and $f : Y \to X$ is continuous, then $f^{-1}(A)$ is analytic in Y.*

$\vdash$ Let $C \subseteq \omega^\omega \times X$ be closed such that $A = \mathbf{\Pi}_X(C)$. Define $D \subseteq \omega^\omega \times Y$ to be the set $D = \{(w, y) \mid (w, f(y)) \in C\}$. Notice that D is a closed set and that

$$\begin{aligned} y \in f^{-1}(A) &\Leftrightarrow f(y) \in A \\ &\Leftrightarrow \exists w \ (w, f(y)) \in C \\ &\Leftrightarrow \exists w \ (w, y) \in D \end{aligned}$$

so that $f^{-1}(A)$ is the projection of a closed subset of a Polish space.

$\dashv$

It is easy to see that the continuous image of an analytic set is analytic; in particular, the projection of an analytic set is analytic.

The following theorem shows that in a perfect Polish space there are strictly more analytic sets than Borel sets.

Theorem 76 *If X is a perfect Polish space, then there is an analytic set $A \subseteq X$ such that $X \setminus A$ is not analytic.*

$\vdash$ It is enough to show the result for $X = 2^\omega$. Let $U \subseteq 2^\omega \times 2^\omega$ be universal for $\mathbf{\Sigma}^1_1(2^\omega)$. Let $f : 2^\omega \to 2^\omega \times 2^\omega$ be the function $f(x) = (x, x)$. Since U is analytic, $f^{-1}(U)$ is analytic. Let $A = f^{-1}(U)$. Suppose for a contradiction that $D = X \setminus A = \{x \mid (x, x) \notin U\}$ is analytic. Then there is a $y \in 2^\omega$ such that $D = U_y$. Then again:

$$y \in D \iff (y, y) \in U \iff y \notin D.$$

$\dashv$

3.2 Understanding analytic sets in terms of trees

In Proposition 13, we understood closed subsets of A^ω as the collection of branches through a pruned tree. We now continue this analysis for analytic sets.

Given sets A, A' we can identify sequences in $(A \times A')^{<\omega}$ with pairs of sequences (σ, τ) with $\sigma \in A^{<\omega}$ and $\tau \in (A')^{<\omega}$. Moreover for an arbitrary tree $T \subseteq (A \times A')^{<\omega}$, we can identify elements of $[T]$ with pairs of elements in $A^\omega \times (A')^\omega$. For such a tree, let

$$p[T] = \{x \in A^\omega | (\exists y \in (A')^\omega)(x, y) \in [T].$$

By Proposition 67, if $S \subseteq \omega^\omega$ is analytic there is a closed set $C \subseteq \omega^\omega \times \omega^\omega$ such that $S = \{f : (\exists g)(f, g) \in C\}$. Since $\omega^\omega \times \omega^\omega$ is canonically homeomorphic to $(\omega \times \omega)^\omega$ we can apply Proposition 13 to see that there is a pruned tree $T \subseteq (\omega \times \omega)^{<\omega}$ (with $A = \omega \times \omega$) such that $C = [T]$. Then

$$\begin{aligned} f \in S \quad & \text{iff} \quad (\exists g)(f, g) \in C \\ & \text{iff} \quad (\exists g)(f, g) \in [T] \\ & \text{iff} \quad f \in p[T]. \end{aligned}$$

3.3 The CH for analytic sets

It is a somewhat strange fact that all concrete (e.g. definable) classes of sets satisfy a strong version of CH, at least assuming the appropriate axioms. We present here two early examples of this type of theorem. We first present the theorem for closed sets to illustrate the method and then for analytic sets. The attentive reader will note that Theorem 77 implies the same result for Borel sets using Theorem 63.

If (X, τ) is Polish and $A \subseteq X$ then we say that A *contains a perfect set* if there is a closed (with respect to τ) non-empty perfect subset $P \subseteq A$. This is equivalent to the existence of a continuous, 1-1 map $\varphi : 2^\omega \to X$ with the range φ included in A.

Theorem 77 *Let (X, τ) be a Polish space and suppose that $C \subseteq X$ is closed and uncountable. Then C contains a perfect set.*

$\vdash$ Let d be a complete compatible metric. Let $\langle U_i : i \in \omega\rangle$ be a base for τ. We build a scheme of open sets $\langle O_s : s \in 2^{<\omega}\rangle$ such that:

1. $O_s \supseteq \overline{O}_{s^\frown i}$,
2. $O_{s^\frown 0} \cap O_{s^\frown 1} = \emptyset$,
3. O_s has diameter less than or equal to $\frac{1}{lh(s)}$,
4. $O_s \cap C$ is uncountable.

We do this inductively, setting $O_\emptyset = X$. To complete the construction we need the following:

Claim: Given O_s there are $O_{s^\frown 0}$ and $O_{s^\frown 1}$ that satisfy 2-4.
$\vdash$ (Claim) If not we can choose a decreasing sequence of open sets $\langle V_n : n \in \omega\rangle$ such that $V_0 = O_s$, $diam(V_n) < \frac{1}{n}, |V_n \cap C| > \omega$ and $(O_s \setminus V_n) \cap C$ is countable. But then

$$O_s \cap C \subseteq \bigcup_{n\in\omega}((O_s \setminus V_n) \cap C) \cup \bigcap_{n\in\omega} V_n.$$

Since $diam(V_n) \to 0$, the intersection $\bigcap_n V_n$ has at most one element. But this implies that $O_s \cap C$ is countable, a contradiction. $\dashv$

Continuing the proof of Theorem 77, we first remark that if $a \in 2^\omega$, then $\bigcap_n O_{a\restriction n}$ is a singleton and a member of C. To see this, for each n choose $x_n \in O_{a\restriction n} \cap C$. Then $\langle x_n\rangle$ is a Cauchy sequence of elements converging to some $y \in C$. For each n, $\langle x_m : m > n\rangle \subseteq O_{a\restriction n}$, hence $y \in \overline{O}_{a\restriction n}$. Since the O's are nested, $y \in \bigcap_n O_{a\restriction n}$.

Define $\varphi(a) = a_x$ where $x \in \bigcap_n O_{a\restriction n}$. Then φ is clearly a 1-1 continuous map. $\dashv$

The following is also true, but requires a finer analysis.

Exercise 78 *Let X be a Polish space. Show that if $C \subseteq X$ is closed, then there is a countable set S and a (possibly empty) closed perfect set P disjoint from S such that $C = S \cup P$, and that this decomposition is unique.*

(Hint: remove the isolated points of C. Then remove the isolated points of the result. Keep going transfinitely.)

We now try to understand the analysis of Theorem 77 for analytic sets. Suppose that $A \subseteq \omega^\omega$ is analytic and uncountable. Then there is a closed set $C \subseteq \omega^\omega \times \omega^\omega$ such that $A = \Pi_X(C)$, where Π_X is the projection to the first coordinate. Since A is uncountable, C must be uncountable and hence C contains a perfect set. Unfortunately this does not imply that A contains a perfect set. (For example $C = \{a\} \times \omega^\omega$ contains a perfect set, but its projection is $\{a\}$.) Instead we need to do the previous argument more carefully.

If $T \subseteq (S)^{<\omega}$ and $\sigma \in (S)^{<\omega}$ we let

$$T_\sigma = \{\sigma' \in T : \sigma \subseteq \sigma' \text{ or } \sigma' \subseteq \sigma\}.$$

Theorem 79 *Suppose that $A \subseteq \omega^\omega$ is an uncountable analytic set. Then A contains a perfect subset.*

$\vdash$ Let $C \subseteq \omega^\omega \times \omega^\omega$ be a closed set such that $A = \Pi_X(C)$. Let $T \subseteq (\omega \times \omega)^{<\omega}$ be a tree such that $A = p[T]$.

Claim Suppose that $p[T_\sigma]$ is uncountable. Then there are $\sigma_0, \sigma_1 \in T_\sigma$ such that

1. $p[T_{\sigma_i}]$ is uncountable for each i and
2. $p[T_{\sigma_0}] \cap p[T_{\sigma_1}] = \emptyset$.

Suppose that $\sigma = (\rho, \tau)$. A simple counting argument shows that there must be $\eta \supseteq \rho$ and $m \neq n$ $|p[T_\sigma] \cap [\eta^\frown m]| > \omega$ and $|p[T_\sigma] \cap [\eta^\frown n]| > \omega$. Another counting argument shows that there must be $\tau_0 \supseteq \tau$ and $\tau_1 \supseteq \tau$ such that both $p[T_{\eta^\frown m, \tau_0)}]$ and $p[T_{(\eta^\frown n, \tau_1)}]$ are uncountable. Letting $\sigma_0 = (\eta^\frown m, \tau_0)$ and $\sigma_1 = (\eta^\frown n, \tau_1)$ we see the claim.

We can now use the claim to produce a collection $\langle \sigma_s : s \in 2^{<\omega}\rangle$ such that:

1. $\sigma_{s^\frown i} \supseteq \sigma_s$,
2. $p[T_{\sigma_s}]$ is uncountable for all s and

3. $p[T_{\sigma_s \frown 0}] \cap p[T_{\sigma_s \frown 1}] = \emptyset$.

Define $\varphi : 2^\omega \to [T]$ by $\varphi(a) \restriction n = \sigma_{a \restriction n}$. Then φ is a continuous map and for $a \neq b \in 2^\omega$, we see that $\Pi_X(\varphi(a)) \neq \Pi_X(\varphi(b))$ as needed. $\dashv$

Exercise 80 *Show that if X is a Polish space and $A \subseteq X$ is an uncountable analytic set then A contains a perfect subset.*

The next exercise extends 55:

Exercise 81 *Suppose that X is an uncountable Polish space. Then there is a bijection $f : X \to \omega^\omega$ such that f and f^{-1} are Borel measurable.*

3.4 Operation $\mathcal{A}$

This section is about Suslin's operation $\mathcal{A}$.

Definition 82 *Let X be perfect Polish and $\mathcal{S} = \langle C_s : s \in \omega^{<\omega} \rangle$ be a scheme. Define $\mathcal{A}(\mathcal{S}) = \bigcup_{f \in \omega^\omega} \bigcap_{n \in \omega} C_{f \restriction n}$. If $\mathcal{R} \subseteq P(X)$ is any collection of sets we let $\mathcal{A}(\mathcal{R})$ be the result of applying operation $\mathcal{A}$ to schemes of sets in $\mathcal{R}$.*

Note the form of the operation:

$$x \in \mathcal{A}(\mathcal{S}) \text{ iff } (\exists f \in \omega^\omega)(\forall n \in \omega) x \in C_{f \restriction n}.$$

Theorem 83 *Let X be Polish and $\mathcal{R} \subseteq P(X)$. Then $\mathcal{A}(\mathcal{A}(\mathcal{R})) = \mathcal{A}(\mathcal{R})$.*

$\vdash$ It suffices to show that $\mathcal{A}(\mathcal{A}(\mathcal{R})) \subseteq \mathcal{A}(\mathcal{R})$.

Let $\langle B_\sigma : \sigma \in \omega^{<\omega} \rangle$ be a sequence of $\mathcal{A}(\mathcal{R})$ sets. We need to see that $A = \bigcup_{f \in \omega^\omega} \bigcap_{n \in \omega} B_{f \restriction n} \in \mathcal{A}(\mathcal{R})$. Write each $B_\sigma = \bigcup_{g \in \omega^\omega} \bigcap_{m \in \omega} C_{\sigma, g \restriction m}$.

Note that

$$\begin{aligned} A &= \{x : (\exists f \in \omega^\omega)(\forall n)(\exists g \in \omega^\omega)(\forall m)(x \in C_{f \restriction n, g \restriction m})\} && (3.1) \\ &= \{x : (\exists f \in \omega^\omega)(\exists \langle g_n : n \in \omega \rangle)(\forall n)(\forall m)(x \in C_{f \restriction n, g_n \restriction m})\} && (3.2) \\ &= \{x : (\exists h)(\forall n)(\forall m)(x \in C_{h^* \restriction n, (h)_n \restriction m})\} && (3.3) \end{aligned}$$

where $h \mapsto (h^*, \langle (h)_n : n \in \omega \rangle)$ is a homeomorphism of ω^ω with $\omega^\omega \times (\omega^\omega)^\omega$. Unfortunately this is not *exactly* the form we need.

Let $\langle \cdot, \cdot \rangle : \omega \times \omega \to \omega$ be a bijection such that for all m, n, we have $n \leq \langle n, m \rangle, m \leq \langle n, m \rangle$ and $\langle m, \cdot \rangle : \omega \to \omega$ is order preserving. Let $n \mapsto ((n)_0, (n)_1)$ be the inverse map.

We define a special homeomorphism from $\varphi : \omega^\omega \times (\omega^\omega)^\omega \to \omega^\omega$ by sending $\varphi(f, \langle g_n : n \in \omega \rangle) = h$ where

$$h(k) = \langle f(k), g_{(k)_0}((k)_1) \rangle.$$

We note that if we know $h \restriction \langle n, m \rangle$ then we can recover $f \restriction n$ and $g_n \restriction m$. Given $\tau \in \omega^{<\omega}$ of length $k = \langle n, m \rangle$ we treat it as $h \restriction \langle n, m \rangle$ and define

$$D_\tau = C_{f \restriction n, g_n \restriction m}.$$

We now check that $\mathcal{A}(\langle D_\tau : \tau \in \omega^{<\omega} \rangle) = A$. Suppose that $x \in A$. Let f and $\langle g_n : n \in \omega \rangle$ be as in Equation 3.2. Let $h = \varphi(f, \langle g_n : n \in \omega \rangle)$. Then

$$(\forall n)(\forall m)(x \in C_{f \restriction n, g_n \restriction m})$$

so

$$(\forall k)(x \in D_{h \restriction k}).$$

On the other hand, suppose that $x \in \mathcal{A}(\langle D_\tau \rangle)$. Then there is an h such that for all $k, x \in D_{h \restriction k}$. Let $(f, \langle g_n : n \in \omega \rangle) = \varphi^{-1}(h)$. Then:

$$(\forall n)(\forall m)(x \in C_{f \restriction n, g_n \restriction m}).$$

$\dashv$

Theorem 84 *Let X be perfect Polish and $A \subseteq X$. Then the following are equivalent:*

1. *A is analytic*
2. *A is the result of applying operation $\mathcal{A}$ to a scheme of Borel sets*
3. *A is the result of applying operation $\mathcal{A}$ to a scheme of closed sets.*

In particular, the analytic sets are closed under operation $\mathcal{A}$.

$\vdash$ In view of Theorem 83, it suffices to show that 1 is equivalent to 3. Suppose A is analytic. Then there is a closed set $C \subseteq X \times \omega^\omega$ such that $A = \Pi_X(C)$. Let C_s be the closure of $\{x : (\exists y)((x,y) \in C \text{ and } y \in [s])\}$. We claim that $A = \mathcal{A}(\langle C_s : s \in \omega^{<\omega}\rangle)$. Clearly $A \subseteq \mathcal{A}(\langle C_s : s \in \omega^{<\omega}\rangle)$. So suppose that $x \in \mathcal{A}(\langle C_s : s \in \omega^{<\omega}\rangle)$. Then we can find a $g \in \omega^\omega$ such that for all n, there is a $y_n, (x, y_n) \in \overline{\{(x,y) : (x,y) \in C \text{ and } y \upharpoonright n = g \upharpoonright n\}}$. Hence we can find x_n, y_n such that $x_n \to x$ and $(x_n, y_n) \in C$ and $y_n \upharpoonright n = g \upharpoonright n$. Then $y_n \to g$, so $\langle x_n, y_n\rangle \to (x, g)$. Since C is closed, we have $(x, g) \in C$ and $x \in A$.

Now suppose that $A = \mathcal{A}(\langle D_s : s \in \omega^{<\omega}\rangle)$ where each D_s is closed. Define $C = \{(x, f) : (\forall n)(x \in D_{f \upharpoonright n})\}$. Then C is closed and $A = \Pi_X(C)$. $\dashv$

Inspecting the proof of Theorem 84, we see that if we let $B_s = \{x : (\exists y)((x,y) \in C \text{ and } y \in [s])\}$ then $A = B_\emptyset = \mathcal{A}(\langle B_s : s \in \omega^{<\omega}\rangle) = \mathcal{A}(\langle C_s : s \in \omega^{<\omega}\rangle)$. This allows us to give an easy proof of:

Theorem 85 *Let μ be a regular, non-atomic probability measure on X and A be an analytic set. Then A is μ-measurable.*

$\vdash$ We use the notation of the proof of Theorem 84. As μ is regular we can find Borel sets $D_s \supseteq B_s$ such that the outer measure of B_s is $\mu(D_s)$. One way of saying this is that if $E \subseteq D_s \setminus B_s$ is measurable, then $\mu(E) = 0$. Without loss of generality we can assume that $B_s \subseteq D_s \subseteq C_s$ and thus $A = B_\emptyset = \mathcal{A}(\langle D_s : s \in \omega^{<\omega}\rangle)$.

We claim that $\mu(D_\emptyset \setminus B_\emptyset)) = 0$. This clearly suffices. To see that this is true, note that

$$D_\emptyset \setminus B_\emptyset \subseteq \bigcup_{s \in \omega^{<\omega}} (D_s \setminus \bigcup_n D_{s^\frown n}).$$

This follows since if $x \in D_\emptyset$ but $x \notin \bigcup_{s\in\omega^{<\omega}}(D_s \setminus \bigcup_n D_{s^\frown n})$ we can inductively build $f \in \omega^\omega$ such that $x \in \bigcap_n D_{f \upharpoonright n}$. However, we know that $\mathcal{A}(\langle D_s : s \in \omega^{<\omega}\rangle) = \mathcal{A}(\langle B_s : s \in \omega^{<\omega}\rangle)$. Hence $x \in B_\emptyset$.

Thus it suffices to show that for each fixed $s \in \omega^{<\omega}, \mu(D_s \setminus \bigcup_n D_{s^\frown n}) = 0$. Let $E \subseteq D_s \setminus \bigcup_n D_{s^\frown n}$ be Borel. Then

$$E \subseteq D_s \setminus \bigcup_n D_{s^\frown n} \subseteq D_s \setminus \bigcup_n B_{s^\frown n} \subseteq D_s \setminus B_s.$$

Hence $\mu(E) = 0$. $\dashv$

Exercise 86 *Show that if $A \subseteq X$ is analytic and X is Polish, then A has the Property of Baire.*

3.5 Lusin's separation theorem and $\mathbf{\Delta}^1_1$ sets

Theorem 87 (Lusin's Separation Theorem) *Let X be a Polish space, A and B be disjoint analytic subsets of X. Then there is a Borel set C with $A \subseteq C$ and $C \cap B = \emptyset$.*

$\vdash$ Let f, g be continuous functions from ω^ω to X such that $A = range(f)$ and $B = range(g)$. For $s \in \omega^{<\omega}$, let $A_s = f([s])$ and $B_s = g([s])$. Then $A_s = \bigcup_{i \in \omega} A_{s^\frown i}$ and $B_s = \bigcup_{j \in \omega} B_{s^\frown j}$.

Observation: Suppose that for all $i, j \in \omega$ there is a Borel set C_{ij} such that $A_{\sigma^\frown i} \subseteq C_{ij}$ and $B_{\tau^\frown j} \cap C_{ij} = \emptyset$. Then $A_\sigma \subseteq \bigcup_i \bigcap_j C_{ij}$ and $\bigcup_i \bigcap_j C_{ij} \cap B_\tau = \emptyset$.

Let us assume, for a contradiction, that A and B cannot be separated by any Borel set C. Using the claim, we can inductively build sequences $\langle \sigma_k \mid k \in \omega \rangle$ and $\langle \tau_k \mid k \in \omega \rangle$ with the following properties:

1. $\sigma_k, \tau_k \in \omega^k$
2. $\sigma_k \subseteq \sigma_{k+1}$ $\tau_k \subseteq \tau_{k+1}$
3. A_{σ_k} can't be separated from B_{τ_k} by any Borel set

Let $x = \bigcup_{k \in \omega} \sigma_k$ and $y = \bigcup_{k \in \omega} \tau_k$. Consider $f(x) \in A$ and $g(y) \in B$. We can find open sets U, V in X with $f(x) \in U$, $g(y) \in V$ and $U \cap V = \emptyset$.

By the continuity of f and g, there is a $k \in \omega$ with

$$f([\sigma_k]) \subseteq U \text{ and } g([\tau_k]) \subseteq V$$

Then A_{σ_k} and B_{τ_k} can be separated by these U, V, both of which are Borel, a contradiction. $\dashv$

Corollary 88 (Suslin's Theorem) *Let $A \subseteq X$. If both A and $X \setminus A$ are analytic, then A is Borel.*

$\vdash$ Use the Lusin separation theorem on A and $X \setminus A$. $\dashv$

Corollary 89 *Suppose that X, Y are Polish spaces, and that $f : X \to Y$. Then the following are equivalent:*

1. *f is Borel measurable.*

2. *graph(f) is a Borel subset of $X \times Y$.*

3. *graph(f) is an Analytic subset of $X \times Y$.*

$\vdash$ (1) $\Rightarrow$ (2) is from Proposition (41). (2) $\Rightarrow$ (3) is trivial. It remains to show that (3) $\Rightarrow$ (1). Let $O \subseteq Y$ be open. We will show that $f^{-1}(O)$ is Borel in X. In particular, we show that it and its complement are both analytic. Now $f^{-1}(O) = \{x \in X \mid \exists y\,(y = f(x) \wedge y \in O)\}$ and $X \setminus f^{-1}(O) = \{x \in X \mid \exists y\,(y = f(x) \wedge y \notin O)\}$, both of which are analytic sets. $\dashv$

Theorem 90 *Let X, Y be Polish spaces and $B \subseteq X$ be a Borel set. If $f : X \to Y$ is continuous and $f \restriction B$ is injective, then $f(B) \subseteq Y$ is Borel.*

$\vdash$ By Theorem 63 , we can retopologize X to make B a closed set in a Polish topology finer than the original topology on X. By Theorem 23, there is a closed set $C \subseteq \omega^\omega$ and a function $g : \omega^\omega \to B$ that is continuous with respect to the finer topology such that $f \restriction C$ is a one to one map onto B. Since g is continuous with respect to the finer topology, g is continuous with respect to the original topology on X. Composing f with g we get a continuous map of ω^ω to Y and a closed set C such that $f \circ g$ is a one one map from C to $f(B)$. Thus it is enough to consider the case where $X = \omega^\omega$ and B closed. Let $T \subseteq \omega^{<\omega}$ be a pruned tree such that $B = [T]$.

Let $A = f(B)$, and for every $s \in T$, let $A_s = f(B \cap [s])$. Fix a complete compatible metric on Y. Then

1. $A_{s^\frown i} \cap A_{s^\frown j} = \emptyset$ for $i \neq j$ since f is 1-1,

2. $A_{s^\frown i} \subseteq A_s$,

3. each A_s is analytic,

4. for every $x \in B$, $\operatorname{diam}(A_{x \restriction n}) \to 0$.

Using the Lusin separation theorem to separate $A_{s^\frown i}$ from $\bigcup_{j \neq i} A_{s^\frown j}$ we can recursively construct a scheme of Borel sets $\langle A'_s : s \in \omega^{<\omega} \rangle$ such that

a. $A_s \subseteq A'_s$.

b. $A'_{s^\frown i} \cap A'_{s^\frown j} = \emptyset$

c. $A'_{s^\frown i} \subseteq A'_s$

If we set $A^*_s = A'_s \cap \overline{A_s}$ then we get another scheme of Borel sets that satisfies (a)-(c) and for all $x \in B$, $\operatorname{diam}(A^*_{x\restriction n}) \to 0$. The following claim suffices to show the theorem.

Claim 91

$$f(B) = \bigcap_{k\in\omega} \bigcup_{s\in(T\cap\omega^k)} A^*_s$$

$\vdash$ [Claim] Let $x \in B$. Then $f(x) \in \bigcap_{k\in\omega} A_{x\restriction k} \subseteq \bigcap_{k\in\omega} \bigcup_{s\in(T\cap\omega^k)} A^*_s$. Hence $f(B) \subseteq \bigcap_{k\in\omega} \bigcup_{s\in(T\cap\omega^k)} A^*_s$.

Now assume that $y \in \bigcap_{k\in\omega} \bigcup_{s\in(T\cap\omega^k)} A^*_s$. Then for each k there is a unique $s_k \in \omega^k$ with $y \in A^*_{s_k}$. By our tree construction, $s_k \subseteq s_{k+1}$, so we can let $x = \bigcup_{k\in\omega} s_k$.

Since x is a branch through T, $x \in B$. We claim that $f(x) = y$. If not, let $z = f(x)$. Then $z \in \bigcap_{k\in\omega} A^*_{x\restriction k}$. Since $\operatorname{diam}(A^*_{x\restriction k}) \to 0$, we have $d(y,z) = 0$. $\dashv$

We introduce the following definition here, but will only use it later.

Definition 92 *Let X be a Polish space, and $A \subseteq X$. A is called co-analytic if $X \setminus A$ is analytic. The class of co-analytic sets of X is denoted by $\mathbf{\Pi}^1_1(X)$. We denote $\mathbf{\Pi}^1_1 \cap \mathbf{\Sigma}^1_1$ by $\mathbf{\Delta}^1_1$.*

In this notation:

$$\mathbf{\Sigma}^1_1(X) \cap \mathbf{\Pi}^1_1(X) = \mathbf{\Delta}^1_1(X) = \text{Borel}(X).$$

Remark 93 *In light of the results in this and earlier sections the properties of analytic and co-analytic depend largely on the Borel structure rather than the particular topology on a space X. Thus we can rephrase many of these results in terms of* standard Borel spaces.

4 Reductions

Definition 94 *Let X, Y be Polish spaces with $A \subseteq X$ and $B \subseteq Y$. A function $f : X \to Y$ is called a reduction if and only if*

$$\forall x \in X \ (x \in A \Leftrightarrow f(x) \in B)$$

We write $A \leq B$.

A reduction f is called a Borel reduction provided f is Borel measurable and write $A \leq_{\mathbb{B}} B$. If f is continuous, f is called a continuous reduction and we write $\leq_c$.

We note that "reductions" reduce the question of membership in A to membership in B and that the relations $\leq_{\mathbb{B}}$ and $\leq_c$ are both transitive.

Definition 95 *Let X, Y be Polish spaces, Γ be a collection of subsets of X, and B be a subset of Y. Then B is* complete *for Γ via Borel (or continuous) reduction if and only if for all $A \in \Gamma$ there is a Borel (or continuous) reduction of A to B.*

If Γ is a collection of subsets of Polish spaces, we will say that B is a complete Γ set *iff B is a Γ set and is complete for the Γ subsets of each Polish space.*

Example 96

1. *Let $U \in \mathbf{\Sigma}^0_\alpha(\omega^\omega \times X)$ be universal for $\mathbf{\Sigma}^0_\alpha(X)$. Then U is complete for $\mathbf{\Sigma}^0_\alpha$ subsets of X via continuous reduction.*
2. *Let $X = \omega^\omega$. Then there is a $\mathbf{\Sigma}^0_\alpha$ set $U' \subseteq \omega^\omega$ that is complete via continuous reductions for the $\mathbf{\Sigma}^0_\alpha$ subsets of ω^ω.*
3. *The same results are true for $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$.*

$\vdash$ To see the first claim in the example, we first take a $\mathbf{\Sigma}^0_\alpha$ set $A \subseteq X$. Then there is an $x_0 \in \omega^\omega$ such that $A = U_{x_0}$. Define a function $R : X \to \omega^\omega \times X$ by $y \mapsto (x_0, y)$. It is easy to check that R is a reduction and continuous.

For the second claim, let $F : \omega^\omega \times \omega^\omega \to \omega^\omega$ be a homeomorphism. Let $U' = F[U]$. Given $A \subseteq \omega^\omega$ be $\mathbf{\Sigma}^0_\alpha$ and suppose that $A = U_{x_0}$. Define $R : \omega^\omega \to \omega^\omega$ by $R(y) = F(x_0, y)$. As before R is a continuous reduction.

The third claim is proved along the same lines by taking U to be universal for different pointclasses. $\dashv$

4.1 Reducing analytic sets to ill-founded trees

Definition 97 *Let A be a set and $T \subseteq A^{<\omega}$ be a tree. We call T* well-founded *if and only if T has no infinite branches. If T does have an infinite branch, then we call T* ill-founded.

If $T \subseteq A^{<\omega}$ is a tree, we define $<_T$ on T by setting $\sigma <_T \tau$ iff $\sigma \supset \tau$. With this partial ordering T is well-founded iff the ordering $<_T$ is well-founded in the sense of Definition 2. As a consequence T is well-founded iff there is a function:

$$\rho : T \to OR$$

such that $\sigma \subseteq \tau$ and $\sigma \neq \tau$ implies $\rho(\sigma) > \rho(\tau)$. We call the least ordinal α such that there is a $<_T$-preserving function $\rho : T \to \alpha$ the *height* of T. We note that if $T \subseteq A^{<\omega}$ then the height of T has cardinality at most $|A| + \aleph_0$.

Exercise 98 *Let α be a countable ordinal. Then there is a well-founded tree $T \subseteq \omega^{<\omega}$ such that T has height α.*

Now, if $A \subseteq \omega^\omega$ is analytic, then there is a closed set $C \subseteq \omega^\omega \times \omega^\omega$ such that $A = \{x \mid (\exists f \in \omega^\omega)\ (f, x) \in C\}$. Let $T \subseteq \omega^{<\omega} \times \omega^{<\omega}$ be defined by:

$$T = \{(s, t) \mid lh(s) = lh(t) \ \wedge \ ([s] \times [t]) \cap C \neq \emptyset\}$$

Then $C = \{(f, x) \mid \forall n (f \restriction n, x \restriction n) \in T\}$

Given $x \in \omega^\omega$, let $T^x = \{s \mid (s, x \restriction lh(s)) \in T\}$. Then T^x is a tree and

$$\begin{aligned} x \in A \ &\Leftrightarrow \ \exists f \forall n \ (f \restriction n, x \restriction n) \in T \\ &\Leftrightarrow \ \exists f \forall n \ f \restriction n \in T^x \end{aligned}$$

Thus $x \in A$ if and only if T^x is ill-founded.

As we saw in Example 51 , the collection of trees can be represented as a closed subset of 2^ω, via an enumeration $\langle \sigma_n : n \in \omega \rangle$ of $\omega^{<\omega}$. Hence we can view the collection of trees (or the collection of infinite trees) as a Polish space, which we named $\mathcal{T}rees$. For convenience we will view 2^ω as $2^{\omega^{<\omega}}$

Lemma 99 *The map $R : \omega^\omega \to \mathcal{T}rees$ given by $R(x) = T^x$ is continuous.*

$\vdash$ Basic open sets for $\mathcal{T}rees$ have the form $B(\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l) = \{T \mid \forall i < k, j < l \ \sigma_i \in T \ \wedge \ \tau_j \notin T\}$.

Fix $\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l$. Let $n = \max_{i<k, j<l}\{lh(\sigma_i), lh(\tau_j)\}$. Suppose that $x, y \in 2^\omega$ are such that $x \restriction n = y \restriction n$. Then $R(x) \in B(\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l)$ iff $R(y) \in B(\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l)$. Thus $R^{-1}(B(\sigma_1, \ldots, \sigma_k, \tau_1, \ldots, \tau_l)$ is a clopen set. $\dashv$

Hence we have shown that if A is analytic, then there is a continuous function $R : \omega^\omega \to \mathcal{T}rees$ such that $x \in A$ if and only if T^x is ill-founded.

Theorem 100 *The set $\{$ill-founded trees$\} \subseteq \mathcal{T}rees$ is an analytic set and every analytic subset of ω^ω can be continuously reduced to it. Moreover if $A \subseteq X$ is an analytic subset of a Polish space X, then there is a Borel reduction of A to $\mathcal{T}rees$.*

Comment: Kechris has shown that every set that is complete via Borel reductions is also complete via continuous reductions, but we don't prove or use that fact in these notes.

$\vdash$ We have shown that every analytic subset of ω^ω can be continuously reduced to $\mathcal{T}rees$ so what remains is to reduce arbitrary analytic sets to $\mathcal{T}rees$. We can assume that X is uncountable, since every subset of a countable Polish space is an $\mathcal{F}_\sigma$-set.

Claim: Let $A \subseteq X$ be an analytic subset of an uncountable Polish space. Let $\langle U_n \mid n \in \omega \rangle$ be an enumeration of a basis for the topology of X. Define a map $c : X \to 2^\omega$ by $c(x) = \chi_{\{n \mid x \in U_n\}}$. Then c is a Borel measurable injection.

To see the claim, we note that If $[s]$ is a basic open set in 2^ω, then

$$c^{-1}([s]) = \bigcap_{s(n)=1} U_n \cap \bigcap_{s(m)=0} U_m^c$$

which is Borel.

Let A, X and c be as in the above proposition. Let $A' = c(A)$, which is an analytic subset of 2^ω. Let $R : A' \to \mathcal{T}rees$ be a continuous reduction of A' to ill-founded trees. Then

$$\begin{aligned} x \in A \quad &\Leftrightarrow \quad c(x) \in A' \\ &\Leftrightarrow \quad R(c(x)) \text{ is ill-founded} \end{aligned}$$

so that $R \circ c : X \to \mathcal{T}rees$ is a Borel measurable reduction. $\dashv$

Corollary 101 *The set $\{$well-founded trees$\} \subseteq \mathcal{T}rees$ is a complete co-analytic set.*

4.2 Linear Orderings and Well Orderings

Definition 102 *Consider relations $R \subseteq \omega \times \omega$ that are linear orderings. We call $\mathcal{LO} = \{\chi_R \in 2^{\omega\times\omega} \mid R \text{ is a linear ordering}\}$.*

Similarly, we write $\mathcal{WO}$ for the set of characteristic functions of well orderings on ω.

If T is a tree we want to be able to systematically linearize $<_T$ in such a way that T is well-founded iff its linearization is a well-ordering. The *Kleene-Brower* ordering does this with a local definition. (In other contexts this is sometimes called the *Lusin-Sierpinski* ordering.)

Definition 103 *The KB relation on $\omega^{<\omega}$ is defined as follows:*

$s \prec_{KB} t$ if and only if either $t \subseteq s$ or if there is an $n < lh(s)$ such that for every $i < n$ we have that $s(i) = t(i)$ and $s(n) < t(n)$.

We can extend the KB relation to a relation on $(\omega\times\omega)^{<\omega}$ (or $(\omega^n)^{<\omega}$) by setting

$$(\sigma,\tau) \prec_{KB} (\sigma',\tau')$$

$$\text{iff}$$

$$(\sigma(0),\tau(0),\sigma(1),\tau(1),\ldots\sigma(n-1),\tau(n-1)) \prec_{KB} (\sigma'(0),\tau'(0),\ldots\sigma'(m-1),\tau'(m-1))$$

where $n = lh(\sigma)$ and $m = lh(\sigma')$.

Proposition 104 *Let α be any ordinal. Then:*

- *$\prec_{KB}$ is a linear ordering of $\alpha^{<\omega}$ with greatest element $\emptyset$,*
- *If $T \subseteq \alpha^{<\omega}$ is a tree, then T is well-founded if and only if $(T,\prec_{KB})$ is a well ordering.*

$\vdash$ To check that $\prec_{KB}$ is a linear ordering is a routine exercise left for the reader.

For the second clause, suppose that T is ill-founded. Let $f:\omega\to\omega$ be such that for every n, $f{\restriction}n\in T$. Then $\{f{\restriction}n \mid n\in\omega\}$ is $\prec_{KB}$-decreasing, so that $\prec_{KB}$ cannot be a well-ordering.

Now suppose that $(T,\prec_{KB})$ is not a well ordering. Let $\langle\sigma_k \mid k\in\omega\rangle$ be a $\prec_{KB}$-decreasing sequence. Define the function $f:\omega\to\alpha$ by induction:

$$\begin{aligned} f(0) &= \text{the least } \beta \text{ such that } \exists k\ \sigma_k(0)=\beta \\ f(n) &= \begin{cases} \text{the least } \beta & \text{such that } \exists k\ \sigma_k{\restriction}n = f{\restriction}n\ \wedge \sigma_k(n)=\beta \\ \text{undefined} & \text{otherwise} \end{cases} \end{aligned}$$

We show by induction on n that $f{\restriction}n$ is the $\prec_{KB}$-least element of

$$\{\tau \mid \text{for some } k,\ \tau\subseteq\sigma_k \text{ and } lh(\tau)=n\}$$

This will imply that $f(n)$ is always defined.

Suppose we have shown the statement to be true for n. Suppose that $f\restriction n = \sigma_k\restriction n$. Since $\sigma_{k+1}\prec_{KB}\sigma_k$ and $\sigma_{k+1}\restriction n \not\prec_{KB} \sigma_k\restriction n$, we must have σ_{k+1} strictly extends σ_k and hence we know that $f(n)$ exists. Suppose for a contradiction that $\sigma_l{\restriction}n{+}1 \prec_{KB} f{\restriction}n{+}1$. Then either $\sigma_l{\restriction}n \prec_{KB} f{\restriction}n$ (which is impossible by our induction hypothesis), or $\sigma_l(n) < f(n)$, which cannot be by the definition of f.

Now from the definition of f, it is now clear that $f{\restriction}n\in T$ for each n. Hence f is an infinite branch through T. $\dashv$

For the following result, let $\langle\sigma_n \mid n\in\omega\rangle$ be an enumeration of $\omega^{<\omega}$ such that $\sigma_n\not\subseteq\sigma_m$ whenever $m<n$. Define $R:\mathcal{T}rees\to\mathcal{LO}$ to be:

$$R(T)(m,n) = \begin{cases} 1 & \text{if } \sigma_n,\sigma_m\in T \text{ and } \sigma_n\prec_{KB}\sigma_m \\ 0 & \text{otherwise} \end{cases}$$

Theorem 105 *R is a continuous reduction from $\{$well-founded trees$\} \subseteq \mathcal{T}rees$ to $\mathcal{WO} \subseteq \mathcal{LO}$.*

$\vdash$ That R is a reduction is clear from the previous proposition. We need to see that R is continuous.

Let $s : r \times r \to 2$ determine a basic open interval in $\mathcal{LO}$. Then $R^{-1}(\langle s \rangle)$ is:

$$\{T : \text{for all } m, n < r((\sigma_m, \sigma_n \in T \text{ and } \sigma_m \prec_{KB} \sigma_n) \text{ iff } s(m,n) = 1)\}$$

Since $\prec_{KB}$ is independent of T, if s coincides with $\prec_{KB}$ then $R^{-1}[s] = \{T : n = 0 \text{ or } (\text{for all } n < r)\sigma_n \in T \text{ iff } s(n,0) = 1\}$. $\dashv$

Corollary 106 *The set $\mathcal{WO}$ is a co-analytic set that is complete via continuous reduction for co-analytic subsets of ω^ω.*

4.3 Rank on the family of trees

Definition 107 *Let $T \subseteq \omega^{<\omega}$ be a tree.*

1. *Define the following sequence of trees $\langle T_\alpha \mid \alpha \in \omega_1 \rangle$:*

$$\begin{aligned} T_0 &= T \\ T_{\alpha+1} &= \{\sigma \mid \sigma \textit{ is not a terminal node of } T_\alpha\} \\ T_\lambda &= \bigcap_{\beta<\lambda} T_\beta \quad \textit{if } \lambda \textit{ is a limit ordinal} \end{aligned}$$

2. *Define the following function $\rho_T : \omega^{<\omega} \to \omega_1 \cup \{\infty\} \cup \{-1\}$ by:*

$$\rho_T(\sigma) = \begin{cases} -1 & \textit{if } \sigma \notin T \\ \textit{the least } \alpha & \textit{such that } \sigma \notin T_{\alpha+1} \\ \infty & \textit{if no such } \alpha \textit{ exists} \end{cases}$$

3. *if T is well-founded and non-empty, we let $\rho(T) = \rho_T(\emptyset)$ and -1 otherwise.*

4. *If T is ill-founded we let $\rho(T) = \infty$.*

5. *For any given $\sigma \in \omega^{<\omega}$, let $T(\sigma) = \{\tau \mid \sigma^\frown\tau \in T\}$.*

Notice that since $|T| = \aleph_0$, there is an $\alpha < \omega_1$ such that for all $\beta > \alpha$, $T_\beta = T_\alpha$. We sometimes denote this T_α by T_∞.

Exercise 108

1. $$\rho_T(\sigma) = \begin{cases} \sup_{i\in\omega}\{\rho_T(\sigma^\frown i)+1 \mid \sigma^\frown i \in T\} & \textit{if } \sigma \in T \\ -1 & \textit{otherwise} \end{cases}$$

2. *If $\sigma \subseteq \tau$, then $\rho_T(\sigma) \geq \rho_T(\tau)$.*

3. *If $\sigma \subseteq \tau$, $\sigma \neq \tau$ and $0 \leq \rho_T(\tau) < \infty$, then $\rho_T(\sigma) > \rho_T(\tau)$.*

4. *Show that for every $\sigma \in \omega^{<\omega}$, the function $\varphi_\sigma : \mathcal{T}rees \to \mathcal{T}rees$ given by*

$$\varphi_\sigma(T) = T(\sigma)$$

is continuous and $\rho(T(\sigma)) = \rho_T(\sigma)$.

Definition 109 1. *We denote by $\mathcal{WF}_\alpha$ the set of trees T with $\rho(T) < \alpha$. We say these are well founded trees of rank α.*

2. *Let $S, T \subseteq \omega^{<\omega}$ be trees. A function $f : S \to T$ is said to be order preserving if and only if for all $\sigma, \tau \in S$, if $\sigma \subseteq \tau$, then $f(\sigma) \subseteq f(\tau)$.*

Exercise 110 1. *Show that $T \in \mathcal{WF}_\alpha$ iff there is an $<_T$-order preserving function $f : T \to \alpha$.*

2. *From this deduce that for every $\alpha < \omega_1$, $\mathcal{WF}_{\alpha+1} \setminus \mathcal{WF}_\alpha \neq \emptyset$.*

Lemma 111 *Let $S, T \subseteq \omega^{<\omega}$ be trees.*

1. *$\rho(S) \leq \rho(T)$ if and only if there exists an order preserving $f : S \to T$. Moreover, if f exists and $S \neq \emptyset$ we can assume that $f(\emptyset) = \emptyset$.*

2. *If T is well founded, then*

$$\rho(S) < \rho(T)$$

iff

$$[(S = \emptyset \wedge T \neq \emptyset)$$

or

there is an $n \in \omega$, $\langle n\rangle \in T$ and an order preserving $f : S \to T$ with $f(\emptyset) = \langle n\rangle$].

$\vdash$ To see (1), suppose first that $f : S \to T$ is order preserving. Without loss of generality we can assume that T is well-founded. We show by induction on α that if $\rho_S(\sigma) = \alpha$, then $\rho_T(f(\sigma)) \geq \alpha$. This will suffice as $\rho(S) = \rho_S(\emptyset) \leq \rho_T(f(\emptyset)) \leq \rho_T(\emptyset) = \rho(T)$.

The case $\alpha = 0$ is trivial. Suppose the induction hypothesis holds for all $\beta < \alpha$. Let σ be such that $\rho_S(\sigma) = \alpha$. Then by hypothesis

$$\begin{aligned}\rho_S(\sigma) &= \sup\{\rho_S(\sigma^\frown i) + 1 \mid \sigma^\frown i \in S \wedge i \in \omega\} \\ &\leq \sup\{\rho_T(f(\sigma^\frown i)) + 1 \mid \sigma^\frown i \in S \wedge i \in \omega\} \\ &\leq \rho_T(f(\sigma))\end{aligned}$$

The last inequality follows from the fact that $\rho_T(f(\sigma^\frown i)) \leq \rho_T(f(\sigma)^\frown i)$ for every i as $f(\sigma) \subseteq f(\sigma^\frown i)$.

Conversely, suppose that $\rho(S) \leq \rho(T)$.

Case 1: T is well-founded. Define $f : S \to T$ by induction on $lh(\sigma)$ for $\sigma \in S$ with the property that for every σ, $\rho_S(\sigma) \leq \rho_T(f(\sigma))$. If $lh(\sigma) = 0$, then $\sigma = \emptyset$, so let $f(\emptyset) = \emptyset$. Suppose we have defined f on all sequences of length n in S, and fix $\sigma \in S$ with $lh(\sigma) = n$. Then $\rho_S(\sigma) = \sup\{\rho_S(\sigma^\frown i) + 1 \mid i \in \omega\}$ and $\rho_T(f(\sigma)) = \sup\{\rho_T(f(\sigma)^\frown j) + 1 \mid j \in \omega\}$. By the induction hypothesis, for every i there is a j_i such that $\rho_S(\sigma^\frown i) \leq \rho_T(f(\sigma)^\frown j_i)$. So let $f(\sigma^\frown i) = f(\sigma)^\frown j_i$. This function is as desired.

Case 2: T is ill-founded. Let $b : \omega \to \omega$ be a branch through T. Define $f : S \to T$ by $f(\sigma) = \langle b(0), b(1), \ldots b(lh(\sigma) - 1)\rangle$.

To prove (2), let T be well founded. Suppose first that $\rho(S) < \rho(T)$ and S is not empty. Then

$$\begin{aligned}\rho(T) &= \rho_T(\emptyset) \\ &= \sup_{n\in\omega}\{\rho_T(\langle n\rangle) + 1\} \\ &= \sup_{n\in\omega}\{\rho(T(\langle n\rangle) + 1\}.\end{aligned}$$

So there is an $n \in \omega$ such that $\rho(S) \leq \rho(T(\langle n\rangle))$. By (1), construct an order preserving $f_0 : S \to T(\langle n\rangle)$ with $f_0(\emptyset) = \emptyset$. The f_0 induces an order preserving $f : S \to T$ with $f(\emptyset) = \langle n\rangle$.

Conversely, suppose $f : S \to T(\langle n\rangle)$ is order preserving. Then $\rho(S) \leq \rho(T(\langle n\rangle)) < \rho(T)$ because T is well founded. $\dashv$

Lemma 112 *$\mathcal{WF}_\alpha$ is Borel.*

$\vdash$ We proceed by induction on α. Since $\emptyset$ is Borel, the case $\alpha = 1$ is clear. For any ordinal $\alpha < \omega_1$, we have

$$\mathcal{WF}_{\alpha+1} = \bigcap_{n\in\omega} \{T \mid \langle n \rangle \notin T \vee T(\langle n \rangle) \in \mathcal{WF}_\alpha\}$$

which is Borel.

Suppose α is a limit ordinal. Then $\mathcal{WF}_\alpha = \bigcup_{\beta<\alpha} \mathcal{WF}_\beta$ which is also Borel. $\dashv$

Theorem 113 *Every $\mathbf{\Pi}^1_1$ set is a union of ω_1 many Borel sets.*

$\vdash$ Let $A \subseteq X$ be a $\mathbf{\Pi}^1_1$ set. Let $f : X \to \mathcal{T}rees$ be a Borel reduction of A to the well founded trees. Then $A = \bigcup_{\alpha\in\omega_1} f^{-1}(\mathcal{WF}_\alpha)$. $\dashv$

Exercise 114 *Show that if $A \subseteq \omega^\omega$ is a $\mathbf{\Pi}^1_1$ set and $|A| > \omega_1$ then $|A| = 2^{\aleph_0}$.*

Exercise 115 *Show that every analytic subset of ω^ω is a union of ω_1 Borel sets. (**Hint:** Suppose that $A = p[T]$. Let $B_\alpha = \{x : (\rho(T^x) \not< \alpha) \wedge (\forall s \in \omega^{<\omega})(\rho(T^x(s)) \neq \alpha)\}$. Show that B_α is Borel and $A = \bigcup B_\alpha$.)*

Lemma 116 *1. $\{(S, T) \subseteq \omega^{<\omega} \times \omega^{<\omega} \mid \rho(S) \leq \rho(T)\}$ is a $\mathbf{\Sigma}^1_1$ subset of the Cartesian product of the space of trees with itself.*

2. There is a $\mathbf{\Sigma}^1_1$ set $R \subseteq \mathcal{T}rees \times \mathcal{T}rees$ such that if T is well founded, then

$$\{S \mid (S, T) \in R\} = \{S \mid \rho(S) < \rho(T)\}$$

$\vdash$ For (1), we have that $\rho(S) \leq \rho(T)$ if and only if there is a $f : \omega^{<\omega} \to \omega^{<\omega}$ such that f is order preserving from S to T. In other words,

$$\rho(S) \leq \rho(T)$$

iff

$(\exists F \in 2^{\omega^{<\omega}\times\omega^{<\omega}})((F$ is a function with domain $S)$ and $(F$ is order preserving) and

$$\forall\sigma(\sigma \in S \Rightarrow F(\sigma) \in T))$$

That F is a function is a closed condition, that it is order preserving is a closed condition, and for each σ, $\sigma \in S \Rightarrow F(\sigma) \in T$ is a clopen condition.

For (2), let $R = \{(S, T) \mid (\exists f \in 2^{\omega^{<\omega}\times\omega^{<\omega}})(\exists n \in \omega)(f$ is order preserving from S to $T(\langle n \rangle))\}$. By the second part of Lemma 111, R is as desired. $\dashv$

Theorem 117 (Boundedness Theorem) *Suppose $A \subseteq \mathcal{WF}$ is $\mathbf{\Sigma}^1_1$. Then there is an $\alpha < \omega_1$ such that $A \subseteq \mathcal{WF}_\alpha$.*

$\vdash$ Suppose not. Then S is well founded if and only if $(\exists T \in A)(\rho(S) \leq \rho(T))$. By the previous lemma, the set of well founded trees is a $\mathbf{\Sigma}^1_1$ subset of $\mathcal{T}rees$. Since the set of well founded trees is $\mathbf{\Pi}^1_1$, this shows they form a Borel set, a contradiction. $\dashv$

Exercise 118 *Show that for $A \subseteq \mathcal{LO}$, if $A \subseteq \mathcal{WO}$ is analytic then there is an $\alpha \in \omega_1$ such that for all $L \in A$ the order-type of L is less than α.*

Corollary 119 *Suppose that $R \subseteq \omega^\omega \times \omega^\omega$ is $\mathbf{\Sigma}^1_1$ and R is well-founded. Then R has height less than ω_1.*

$\vdash$ We start with an abstract claim:

Claim: Suppose that R is an arbitrary well-founded relation on a set X of height at least ω_1. Then for all countable linear orderings $I = (B, <_I)$:

$<_I$ is a well-ordering

iff

there is a countable set $S \subseteq X$ and a surjection $f : S \to B$ such that:

for all $s \in S$ and for all $b <_I f(s)$ there is a $t \in S$ with tRs and $b \leq_I f(t)$.

To see the claim: Suppose that $<_I$ is a well-ordering of length $\alpha < \omega_1$. Let $x_0 \in X$ be such that $ht_R(x_0) = \alpha$. Build a sequence of countable sets $\langle S_n : n \in \omega \rangle$ by induction so that:

1. $S_0 = \{x_0\}$,
2. if $y \in S_n$ there are $\{z_i : i \in \omega\} \subseteq S_{n+1}$ such that for all i, $z_i R y$ and $\sup\{ht_R(z_i) + 1 : i \in \omega\} = ht_R(y)$.

Let $S = \bigcup_n S_n$. Then $\{ht_{R \upharpoonright S}(s) : s \in S\} = \alpha$. Let $f : S \to B$ be defined by setting

$$f(z) = m \text{ iff } ht_R(z) = ht_I(m).$$

For the other direction: towards a contradiction, suppose that $\langle b_n : n \in \omega \rangle$ is a $<_I$-decreasing sequence and f is a function with the stated property. Inductively choose a sequence $\langle s_n : n \in \omega \rangle$ such that

$$s_{n+1} R s_n \text{ and } b_{n+1} \leq f(s_{n+1}).$$

The existence of such a sequence contradicts the well-foundedness of R.

Using the claim we can give a $\mathbf{\Sigma}^1_1$ definition of $\mathcal{WO}$. If $I \in \mathcal{LO}$, we have:

$$I \in \mathcal{WO} \text{ iff } (\exists \langle z^i_n : i, n \in \omega\rangle \in (\omega^\omega)^{\omega\times\omega})$$
$$(\forall m)(\forall n)(m <_I n \implies (\forall i)(\exists k)(\exists j)(z^j_k R z^i_n \wedge m \leq_I k)).$$

⊣

4.4 Norms

In this section we formalize the properties we have shown about the well-founded sets.

Definition 120

- *A norm on a set $A \subseteq X$ is a function $\varphi : X \to \mathcal{ON} \cup \{\infty\}$, the class of ordinals such that $x \in A$ iff $\varphi(x) \neq \infty$. The set $\{(x, y) : \varphi(x) \leq \varphi(y)\}$ is called a* pre-well-ordering *of A. We treat "∞" as a symbol that is bigger than all of the ordinals. Since A is a set, we could also take ∞ to be any particular ordinal larger than* $\sup \varphi[A]$.

  It is an easy set-theoretic exercise to show that for every norm φ there is a norm ψ such that φ and ψ have the same pre-well-orderings and there is an ordinal β such that $\psi : A \to \beta$ is surjective.

- *Let $A \in \mathbf{\Pi}^1_1(X)$, and $\varphi : A \to \mathcal{ON}$ be a norm. Then φ is a $\mathbf{\Pi}^1_1$-norm if and only if there are relations $\leq^{\mathbf{\Pi}^1_1} \in \mathbf{\Pi}^1_1(X \times X)$ and $\leq^{\mathbf{\Sigma}^1_1} \in \mathbf{\Sigma}^1_1(X \times X)$ such that for all $y \in A$*

$$\begin{aligned}(x \in A) \wedge (\varphi(x) \leq \varphi(y)) &\Leftrightarrow x \leq^{\mathbf{\Pi}^1_1} y \\ &\Leftrightarrow x \leq^{\mathbf{\Sigma}^1_1} y\end{aligned}$$

Here are an equivalent forms of this definition:

Proposition 121 *Let X be Polish, A be a $\mathbf{\Pi}^1_1$-set and $\varphi : X \to \mathcal{ON} \cup \{\infty\}$ be such that $x \in A$ iff $\varphi(x) < \infty$. Then φ is a $\mathbf{\Pi}^1_1$-norm iff the relations $\leq^*_\varphi$ and $<^*_\varphi$ are both $\mathbf{\Pi}^1_1$ where:*

1. *$x \leq^*_\varphi y$ iff $x \in A \wedge (\varphi(x) \leq \varphi(y))$,*

2. $x <^*_\varphi y$ *iff* $x \in A \wedge \varphi(x) < \varphi(y)$*.*

Theorem 122 *The function* $\rho : \mathcal{WF} \to \omega_1$ *is a* $\mathbf{\Pi}^1_1$*-norm.*

$\vdash$ The relation $\leq^{\mathbf{\Sigma}^1_1}$ is the one in part (1) of Lemma 116.

Let R be the relation given in part (2) of this same Lemma. Then

$$x \leq^{\mathbf{\Pi}^1_1} y \Leftrightarrow (y, x) \notin R \wedge x \in \mathcal{WF}$$

so that $\leq^{\mathbf{\Pi}^1_1}$ is the intersection of two $\mathbf{\Pi}^1_1$ sets.

$\dashv$

Corollary 123 *Let* $A \in \mathbf{\Pi}^1_1(X)$*, and* $f : X \to \mathcal{T}rees$ *be a Borel reduction of* A *to the well founded trees. Then* $\rho \circ f : A \to \omega_1$ *is a* $\mathbf{\Pi}^1_1$*-norm.*

$\vdash$ Let $\leq^{\mathbf{\Pi}^1_1}_{\mathcal{T}rees}$ and $\leq^{\mathbf{\Sigma}^1_1}_{\mathcal{T}rees}$ be witnesses to ρ being a $\mathbf{\Pi}^1_1$-norm on well founded trees. Define $\leq^{\mathbf{\Pi}^1_1} = \{(x, y) \mid f(x) \leq^{\mathbf{\Pi}^1_1}_{\mathcal{T}rees} f(y)\}$ and $\leq^{\mathbf{\Sigma}^1_1} = \{(x, y) \mid f(x) \leq^{\mathbf{\Pi}^1_1}_{\mathcal{T}rees} f(y)\}$ which are the witnesses for $\rho \circ f : A \to \omega_1$ being a $\mathbf{\Pi}^1_1$-norm. $\dashv$

Proposition 124 *Suppose that* $A \subseteq X$ *is a* $\mathbf{\Pi}^1_1$ *set, with* X *Polish and that* $\varphi : A \to \omega_1 \cup \{\infty\}$ *is a* $\mathbf{\Pi}^1_1$*-norm. Then* A *is Borel iff there is a countable ordinal* α *such that* $\varphi[A] \subseteq \alpha$*.*

$\vdash$ Suppose that A is Borel. Since A is $\mathbf{\Sigma}^1_1$ the following relation is $\mathbf{\Sigma}^1_1$:

$$xRy \text{ iff } x \in A \wedge y \in A \wedge x \leq^{\mathbf{\Sigma}^1_1} y.$$

By Corollary 119 we know that the height of R is countable. Hence there is an α such that the range of φ is a subset of α.

For the other direction: suppose that there is a countable ordinal α such that the range of φ is a subset of α. Then we can find a sequence $\langle z_i : i \in \omega\rangle$ of elements of A such that $\{\varphi(z_i) : i \in \omega\}$ is cofinal in $\sup \varphi[A]$. Then A is defined by:

$$x \in A \text{ iff } (\exists i)(x \leq^{\mathbf{\Sigma}^1_1} z_i).$$

Hence A is $\mathbf{\Sigma}^1_1$. $\dashv$

4.5 Reduction and Separation

Definition 125 *A class* Γ *of sets has the* reduction property *if and only if for any Polish space* X *and for all* $A, B \in \Gamma(X)$*, there are* $A_0 \subseteq A$*,* $B_0 \subseteq B$ *with:*

- $A_0, B_0 \in \Gamma(X)$
- $A_0 \cap B_0 = \emptyset$
- $A_0 \cup B_0 = A \cup B$

Theorem 126 *Let* X *be a perfect Polish space. Then* $\mathbf{\Pi}^1_1(X)$ *has the reduction property.*

$\vdash$ Let f, g be reductions of A, B to $\mathcal{WF}$. Let $A_0 = \{x \in A \mid \rho(f(x)) \leq \rho(g(x))\}$ and $B_0 = \{x \in B \mid \rho(f(x)) > \rho(g(x))\}$. It is clear that B_0 is $\mathbf{\Pi}^1_1$ as it is the complement of a $\mathbf{\Sigma}^1_1$ set by Lemma 116. For every $x \in A$, we have that $f(x)$ is well founded, so that $A = \{x \in A \mid \rho(f(x)) \leq^{\mathbf{\Pi}^1_1} \rho(g(x))\}$ is also $\mathbf{\Pi}^1_1$.

Finally, notice that if $x \in A \setminus A_0$, then in fact $x \in B_0$ (the reverse is also true). Hence $A_0 \cup B_0 = A \cup B$.

$\dashv$

Notice that if C and D are analytic, we can apply the reduction property to their complements to obtain a $\mathbf{\Delta}^1_1$ (hence Borel) set S separating C and D. In general, the reduction property for Γ implies the separation for its dual class.

Proposition 127 $\mathbf{\Pi}^1_1$ *does not have the separation property.*

$\vdash$ Let $U \in \mathbf{\Pi}^1_1(\omega^\omega \times \omega^\omega)$ be universal for $\mathbf{\Pi}^1_1(\omega^\omega)$. Let $h : \omega^\omega \to \omega^\omega \times \omega^\omega$ be a homeomorphism. Denote $h(x) = (x_0, x_1)$.

Let $P = \{x \mid (x_{even}, x) \in U\}$ and $Q = \{x \mid (x_{odd}, x) \in U\}$. Using $\mathbf{\Pi}^1_1$ reduction, produce P_0 and Q_0 that reduce P and Q.

Let us assume, for a contradiction, that there is a Borel set B with $B \cap Q_0 = \emptyset$ and $P_0 \subseteq B$. By universality of U, there are $a, b \in \omega^\omega$ such that $B = U_a$ and $\omega^\omega \setminus B = U_b$. Let $x = h^{-1}(b, a)$. Note that if $x \in Q \cap B$ then $x \notin Q_0$ so $x \in Q \cap P$.

$$\begin{aligned} x \in B &\Leftrightarrow (a, x) \in U \\ &\Leftrightarrow x \in Q \cap P \\ &\Leftrightarrow (b, x) \in U \\ &\Leftrightarrow x \in U_b \\ &\Leftrightarrow x \in \omega^\omega \setminus B \end{aligned}$$

This contradiction proves the proposition. $\dashv$

5 Uniformization

Definition 128 *Let X, Y be Polish, $A \subseteq X \times Y$. We say that a set B uniformizes A if and only if:*

1. $\Pi_X(B) = \Pi_X(A)$
2. *for every $x \in \Pi_X(A)$ there is a unique $y \in Y$ such that $(x, y) \in B$*

Clearly B can be viewed as a choice function defined on $\Pi_X(A)$ and thus the axiom of choice guarantees the existence of a uniformization for any set. However, we are only interested in "nice" uniformizations.

We make this into a definition:

Definition 129 *Let X, Y be Polish and $A \subseteq X \times Y$. A function $f : A \to Y$* uniformizes *A iff B =graph(f) uniformizes A.*

5.1 The bad news

Example 130 *There is a closed set $C \subseteq \omega^\omega \times \omega^\omega$ that cannot be uniformized by any analytic set.*

$\vdash$ Let $A_0, A_1 \in \mathbf{\Pi}^1_1(\omega^\omega)$ be disjoint and inseparable by any Borel set. Then there are closed sets C_0, C_1 such that $\omega^\omega \setminus A_i = \{x : (\exists y)(x, y) \in C_i\}$. Without loss of generality we can assume that $C_i \subseteq \omega^\omega \times [\langle i \rangle]$, where $[\langle i \rangle]$ consists of those elements of ω^ω such that $y(0) = i$. (Exercise: why can we assume this?)

Let $C = C_0 \cup C_1$. Then C is a closed subset of ω^ω, and since $A_0 \cap A_1 = \emptyset$, $\Pi_X(C) = \omega^\omega$. Suppose that C can be uniformized by an analytic set A. Since the projection of C is ω^ω, the analytic set A is the graph of an analytic function $f : \omega^\omega \to \omega^\omega \times \omega^\omega$. Hence, by Corollary 89, f is Borel measurable.

Let $B_i = f^{-1}(\omega^\omega \times [\langle i \rangle])$ for $i = 1, 2$. Then $B_i \subseteq \omega^\omega \setminus A_i$, $B_0 \cap B_1 = \emptyset$, and $B_0 \cup B_1 = \omega^\omega$. Hence $B_i \supseteq A_{1-i}$ and thus B_i separates A_{1-i} from A_i, a contradiction. $\dashv$

5.2 Another digression on trees

Suppose that $T \subseteq A^{<\omega}$ is an ill-founded tree and $<$ is a well-ordering of A. Then the *left-most branch* through T is the function $b : \omega \to A$ with the property that for all $n \in \omega, b(n)$ is the $<$-least element of A among $\{c(n) : c \restriction n = b \restriction n \wedge c \in [T]\}$.

We note that there is always a unique left most branch and the map "$T \mapsto$ the leftmost branch of T", is a continuous map from the closed set $\{T \subseteq A^{<\omega} : T$ is well-pruned$\} \subseteq \mathcal{T}rees$ to A^ω.

5.3 Jankov-von Neumann uniformization

The easiest (and most commonly used) uniformization theorem is often called the "Jankov-von Neumann" uniformization theorem. Let $\sigma(\mathbf{\Sigma}^1_1)$ be the σ-algebra generated by the $\mathbf{\Sigma}^1_1$-sets. Since every $\mathbf{\Sigma}^1_1$-set is universally measurable (Theorem 85) every function that is measurable with respect to the σ-algebra $\sigma(\mathbf{\Sigma}^1_1)$ is universally measurable, and hence the next theorem shows that every analytic set can be uniformized by a function that is universally measurable.

Theorem 131 *Suppose that X, Y are uncountable Polish spaces and $A \subseteq X \times Y$ is analytic. Then there is a function f that uniformizes A that is measurable with respect to $\sigma(\mathbf{\Sigma}^1_1)$.*

$\vdash$ Since this theorem only refers to the Borel structure of X and Y we can Exercise 81 to see that it holds for all standard Borel spaces. Moreover it suffices to prove the result in the case that $X = Y = \omega^\omega$.

Let $T \subseteq (\omega \times \omega \times \omega)^{<\omega}$ be a well-pruned tree such that $A = \{(x, y) : T^{x,y}$ is ill-founded$\}$. Let $A' = [T]$. Then A' is a closed subset of $\omega^\omega \times \omega^\omega \times \omega^\omega$. Call the three copies of ω^ω by X, Y and Z.

Suppose that we can uniformize $A' \subseteq X \times (Y \times Z)$ by a function f that is measurable with respect to $\sigma(\mathbf{\Sigma}^1_1)$. Let $\Pi_Y : Y \times Z \to Y$ be the projection map. Then Π_Y is continuous and hence $\Pi_Y \circ f : X \to Y$ is measurable with respect to $\sigma(\mathbf{\Sigma}^1_1)$ and hence satisfies the theorem.

Thus we are reduced to finding the function f. Note that $Y \times Z \cong \omega^\omega$ so we can assume that we are trying to uniformize a closed subset A of $\omega^\omega \times \omega^\omega$.

Now let $T \subseteq (\omega^\omega \times \omega^\omega)^{<\omega}$ be a well-pruned tree such that $A = [T]$. For each $x \in \Pi_X(A)$, let $f(x)$ be the left-most branch of $[T^x]$. We claim that $f : A \to \omega^\omega$ is $\sigma(\mathbf{\Sigma}^1_1)$-measurable; i.e. for all $s \in \omega^{<\omega}$, the set $f^{-1}([s]) \in \sigma(\mathbf{\Sigma}^1_1)$. We show this by induction on the length of s.

Since A is $\mathbf{\Sigma}^1_1$, $\Pi_X(A)$ is $\mathbf{\Sigma}^1_1$ and hence this holds if $s = \emptyset$. Suppose that $f^{-1}([s]) \in \sigma(\mathbf{\Sigma}^1_1)$, we show it for $f^{-1}([s^\frown n])$. But $x \in f^{-1}([s^\frown n])$ iff $x \in f^{-1}([s])$ and

$$(\exists y \in [s^\frown n])((x, y) \in T) \wedge (\forall m < n)(T^x(s^\frown m) \text{ is well-founded}). \tag{5.4}$$

This set is the intersection of $f^{-s}([s])$ with a $\mathbf{\Sigma}^1_1$ set and an intersection of countably many $\mathbf{\Pi}^1_1$ sets. Hence it is in $\sigma(\mathbf{\Sigma}^1_1)$. $\dashv$

Exercise 132 *Show that there is a function f that uniformizes P that is measurable with respect to the σ-algebra of sets with the property of Baire.*

References

[1] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[2] http://www.math.uic.edu/∼marker/math512/

[3] Basic Set Theory, by Azriel Levy. Dover, 2002. Paperback, 398 pp.