The Royal holes

Grigor Sargsyan

Department of Mathematics
Rutgers University

July 10-14, 2017
Young Set Theory Meeting, Edinburgh, Scotland

Reflections on Royal weather

1. There are 4,000 holes in the road in Blackburn, Lancashire, or one twenty-sixth of a hole per person, according to a council survey. If Blackburn is typical, there are two million holes in Britain's roads and 300,000 in London (in 1967).

Reflections on Royal weather

1. There are 4,000 holes in the road in Blackburn, Lancashire, or one twenty-sixth of a hole per person, according to a council survey. If Blackburn is typical, there are two million holes in Britain's roads and 300,000 in London (in 1967).
2. So now we know how many holes it takes to fill the Albert Hall.

Reflections on Royal weather

1. There are 4,000 holes in the road in Blackburn, Lancashire, or one twenty-sixth of a hole per person, according to a council survey. If Blackburn is typical, there are two million holes in Britain's roads and 300,000 in London (in 1967).
2. So now we know how many holes it takes to fill the Albert Hall.
3. But we do not know how many it takes to fill the rational line.

Reflections on Royal weather

1. There are 4,000 holes in the road in Blackburn, Lancashire, or one twenty-sixth of a hole per person, according to a council survey. If Blackburn is typical, there are two million holes in Britain's roads and 300,000 in London (in 1967).
2. So now we know how many holes it takes to fill the Albert Hall.
3. But we do not know how many it takes to fill the rational line.
4. So what are these holes?

Reflections on Royal weather

1. There are 4,000 holes in the road in Blackburn, Lancashire, or one twenty-sixth of a hole per person, according to a council survey. If Blackburn is typical, there are two million holes in Britain's roads and 300,000 in London (in 1967).
2. So now we know how many holes it takes to fill the Albert Hall.
3. But we do not know how many it takes to fill the rational line.
4. So what are these holes?
5. Unlike Dedekind, who lived on German soil and felt keenly "the lack of a really scientific foundation for arithmetic", it seems we feel keenly the lack of platonistic foundation for arithmetic,

Reflections on Royal weather

1. There are 4,000 holes in the road in Blackburn, Lancashire, or one twenty-sixth of a hole per person, according to a council survey. If Blackburn is typical, there are two million holes in Britain's roads and 300,000 in London (in 1967).
2. So now we know how many holes it takes to fill the Albert Hall.
3. But we do not know how many it takes to fill the rational line.
4. So what are these holes?
5. Unlike Dedekind, who lived on German soil and felt keenly "the lack of a really scientific foundation for arithmetic", it seems we feel keenly the lack of platonistic foundation for arithmetic, or at least we feel this way while on Royal soil.

What holes there are

1. There are the ones in Blackburn that can be used to fill Albert Hall.

What holes there are

1. There are the ones in Blackburn that can be used to fill Albert Hall.
2. There is always 1.

What holes there are

1. There are the ones in Blackburn that can be used to fill Albert Hall.
2. There is always 1.
3. Reals we force, or holes we force to exist.

What holes there are

1. There are the ones in Blackburn that can be used to fill Albert Hall.
2. There is always 1.
3. Reals we force, or holes we force to exist.
4. (Less known) Inner model theoretic reals.

Generic holes and inner model theoretic holes

Theorem (Jensen, 70s)

The following are equivalent.

1. *There is a singular cardinal* κ *such that* $(\kappa^+)^L < \kappa^+$.
2. $0^\#$ *exists.*

Generic holes and inner model theoretic holes

Theorem (Jensen, 70s)

The following are equivalent.

1. *There is a singular cardinal* κ *such that* $(\kappa^+)^L < \kappa^+$.
2. $0^\#$ *exists.*

1. ($0^\#$) Suppose there a club class of indiscernibles for L. Then $0^\#$ is a real coding the first order theory of the first ω-many of these indiscernbiles (in the appropriate language).

Generic holes and inner model theoretic holes

Theorem (Jensen, 70s)

The following are equivalent.

1. *There is a singular cardinal κ such that $(\kappa^+)^L < \kappa^+$.*
2. $0^\#$ *exists.*

1. ($0^\#$) Suppose there a club class of indiscernibles for L. Then $0^\#$ is a real coding the first order theory of the first ω-many of these indiscernbiles (in the appropriate language).
2. ($0^\#$) Suppose there is $\mathcal{M} = (L_\lambda, \mu)$ where for some L-cardinal κ, $\lambda = (\kappa^+)^L$, μ is a normal κ-complete L-ultrafilter, and all of the iterated ultrapowers of μ are well-founded. Then $0^\#$ is the "natural code" of $\mathcal{M}$ as above for which λ is the least possible.

Generic holes and inner model theoretic holes

1. Inner model theory is essentially the theory of “canonical” inner models.

Generic holes and inner model theoretic holes

1. Inner model theory is essentially the theory of “canonical” inner models.
2. $0^\#$ is an inner model theoretic object and it exists.

Generic holes and inner model theoretic holes

1. Inner model theory is essentially the theory of “canonical” inner models.
2. $0^{\#}$ is an inner model theoretic object and it exists.
3. Cohen reals and other generic reals don’t exist.

Generic holes and inner model theoretic holes

1. Inner model theory is essentially the theory of “canonical” inner models.
2. $0^\#$ is an inner model theoretic object and it exists.
3. Cohen reals and other generic reals don’t exist. These are reals that exists in Boolean ultrapowers.

Generic holes and inner model theoretic holes

1. Inner model theory is essentially the theory of “canonical” inner models.
2. $0^{\#}$ is an inner model theoretic object and it exists.
3. Cohen reals and other generic reals don’t exist. These are reals that exists in Boolean ultrapowers.
4. (The view we take) Generic holes are imaginary, inner model theoretic holes are not.

A simple philosophy

$V \neq L$ because $0^{\#}$ exists. Otherwise, foundational aspects of set theory would really end with L.

A definition of real numbers

Definition

x is a real number if it belongs to the set of reals of some canonical inner model of set theory.

A definition of real numbers

Definition

x is a real number if it belongs to the set of reals of some canonical inner model of set theory.

1. Wait a minute, before we had no clue what the powerset of natural numebrs are, and now we have no clue of what *canonical inner models are*.

A definition of real numbers

Definition

x is a real number if it belongs to the set of reals of some canonical inner model of set theory.

1. Wait a minute, before we had no clue what the powerset of natural numebrs are, and now we have no clue of what *canonical inner models are*.
2. This sequence of talks is about showing that in fact we do know what canonical inner models are, and we know great deal on how to build them and analyze them.

A definition of real numbers

Definition

x is a real number if it belongs to the set of reals of some canonical inner model of set theory.

1. Wait a minute, before we had no clue what the powerset of natural numebrs are, and now we have no clue of what *canonical inner models are*.
2. This sequence of talks is about showing that in fact we do know what canonical inner models are, and we know great deal on how to build them and analyze them.
3. Fortunately for you, we also do not know a great deal about them.

Things to keep in mind

1. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two "canonical" models of set theory. If $\mathbb{R}^{\mathcal{M}}$ is not compatible with $\mathbb{R}^{\mathcal{N}}$, then our definition is bad.

Things to keep in mind

1. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two "canonical" models of set theory. If $\mathbb{R}^{\mathcal{M}}$ is not compatible with $\mathbb{R}^{\mathcal{N}}$, then our definition is bad.
2. (Compatibility Principle:) Suppose $\mathcal{M}$ and $\mathcal{N}$ are two canonical inner models. Then we say $\mathbb{R}^{\mathcal{M}}$ is compatible with $\mathbb{R}^{\mathcal{N}}$ if either $\mathbb{R}^{\mathcal{M}} \subseteq \mathbb{R}^{\mathcal{N}}$ or $\mathbb{R}^{\mathcal{N}} \subseteq \mathbb{R}^{\mathcal{M}}$.

Things to keep in mind

1. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two "canonical" models of set theory. If $\mathbb{R}^{\mathcal{M}}$ is not compatible with $\mathbb{R}^{\mathcal{N}}$, then our definition is bad.
2. (Compatibility Principle:) Suppose $\mathcal{M}$ and $\mathcal{N}$ are two canonical inner models. Then we say $\mathbb{R}^{\mathcal{M}}$ is compatible with $\mathbb{R}^{\mathcal{N}}$ if either $\mathbb{R}^{\mathcal{M}} \subseteq \mathbb{R}^{\mathcal{N}}$ or $\mathbb{R}^{\mathcal{N}} \subseteq \mathbb{R}^{\mathcal{M}}$.
3. All canonical reals have to appear in canonical models of set theory. Otherwise we are again in bad shape.

Things to keep in mind

1. Suppose $\mathcal{M}$ and $\mathcal{N}$ are two "canonical" models of set theory. If $\mathbb{R}^{\mathcal{M}}$ is not compatible with $\mathbb{R}^{\mathcal{N}}$, then our definition is bad.
2. (Compatibility Principle:) Suppose $\mathcal{M}$ and $\mathcal{N}$ are two canonical inner models. Then we say $\mathbb{R}^{\mathcal{M}}$ is compatible with $\mathbb{R}^{\mathcal{N}}$ if either $\mathbb{R}^{\mathcal{M}} \subseteq \mathbb{R}^{\mathcal{N}}$ or $\mathbb{R}^{\mathcal{N}} \subseteq \mathbb{R}^{\mathcal{M}}$.
3. All canonical reals have to appear in canonical models of set theory. Otherwise we are again in bad shape.
4. (Maximality Prinicple:) If x is a canonical real then it appears in a canonical inner model of set theory.

Maximality

1. Jensen's theorem is the way we make sure we have maximality. If we can prove that our canonical model is close to the universe then it would have all the canonical reals.

Maximality

1. Jensen's theorem is the way we make sure we have maximality. If we can prove that our canonical model is close to the universe then it would have all the canonical reals. This is more or less an empirical fact.

Maximality

1. Jensen's theorem is the way we make sure we have maximality. If we can prove that our canonical model is close to the universe then it would have all the canonical reals. This is more or less an empirical fact.
2. One way to measure closeness is via some sort of covering.

Maximality

1. Jensen's theorem is the way we make sure we have maximality. If we can prove that our canonical model is close to the universe then it would have all the canonical reals. This is more or less an empirical fact.
2. One way to measure closeness is via some sort of covering.
3. ((Naive) Covering) Given a transitive model $\mathcal{M}$ of some fragment of *ZFC*, and a cardinal κ such that $\mathcal{M} \subseteq H_{\kappa^+}$, we say $\mathcal{M}$ has (naive) covering at κ if $Ord \cap \mathcal{M} = \kappa^+$ and κ is the largest cardinal of $\mathcal{M}$.

The road to canonical models

1. (Naive approach:) Write down a theory, build its minimal model and declare success.

The road to canonical models

1. (Naive approach:) Write down a theory, build its minimal model and declare success.
2. By Gödel (and also Jensen), this cannot be right.

The road to canonical models

1. (Naive approach:) Write down a theory, build its minimal model and declare success.
2. By Gödel (and also Jensen), this cannot be right.
3. We have to have a process that gradually becomes more and more complex and captures the entire set theoretic strength.

The road to canonical models

1. (Naive approach:) Write down a theory, build its minimal model and declare success.
2. By Gödel (and also Jensen), this cannot be right.
3. We have to have a process that gradually becomes more and more complex and captures the entire set theoretic strength.
4. The large cardinal hierarchy has been the most successful one as far as measuring set theoretic strength goes. So it is natural to try to climb this hierarchy.

The road to canonical models

1. (Naive approach:) Write down a theory, build its minimal model and declare success.
2. By Gödel (and also Jensen), this cannot be right.
3. We have to have a process that gradually becomes more and more complex and captures the entire set theoretic strength.
4. The large cardinal hierarchy has been the most successful one as far as measuring set theoretic strength goes. So it is natural to try to climb this hierarchy.
5. We then have to argue that any other potential hierarchy would essentially produce the same reals.

The road to canonical models

1. (Naive approach:) Write down a theory, build its minimal model and declare success.
2. By Gödel (and also Jensen), this cannot be right.
3. We have to have a process that gradually becomes more and more complex and captures the entire set theoretic strength.
4. The large cardinal hierarchy has been the most successful one as far as measuring set theoretic strength goes. So it is natural to try to climb this hierarchy.
5. We then have to argue that any other potential hierarchy would essentially produce the same reals.
6. (Unique Hierarchy Principle:) All set theoretic hierarchies are inter-translatable.

Things to keep in mind

1. Compatibility Principle.
2. Maximality Principle.
3. Unique Hierarchy Principle.

The constructible universe

Definition

The constructible universe over A, $L[A]$, is the model obtained via the following recursion.

1. $L_0[A] = \emptyset$,
2. $L_{\alpha+1}[A] = \{A \subseteq L_\alpha[A] : A$ is definable over $(L_\alpha[A], A \cap L_\alpha[A], \in)$ from parameters$\}$,
3. for λ a limit, $L_\lambda[A] = \cup_{\alpha<\lambda} L_\alpha[A]$.
4. $L[A] = \cup_{\alpha \in Ord} L_\alpha[A]$.

The constructible universe

Definition

The constructible universe over A, $L[A]$, is the model obtained via the following recursion.

1. $L_0[A] = \emptyset$,
2. $L_{\alpha+1}[A] = \{A \subseteq L_\alpha[A] : A$ is definable over $(L_\alpha[A], A \cap L_\alpha[A], \in)$ from parameters$\}$,
3. for λ a limit, $L_\lambda[A] = \cup_{\alpha<\lambda} L_\alpha[A]$.
4. $L[A] = \cup_{\alpha \in Ord} L_\alpha[A]$.

Set $L = L[\emptyset]$.

L is small

Theorem (Scott)

Assume there is a measurable cardinal. Then $V \neq L$.

L is small

Theorem (Scott)

Assume there is a measurable cardinal. Then $V \neq L$.

Proof.

Assume $V = L$. Suppose κ is the least measurable cardinal and μ is a normal κ-complete ultrafilter on κ. Then $L = Ult(L, \mu)$ and $\pi_\mu : L \to L$. But $\pi_\mu(\kappa) > \kappa$. $\square$

L is small

Theorem (Kunen, Jensen)

The following are equivalent.

1. *There is a non-trivial embedding $j : L \to L$ (a measurable cardinal gives such an embedding).*
2. $0^{\#}$*-exists.*
3. *For some singular κ, $L_{(\kappa^+)^L}$ doesn't have covering at κ.*

L is small

Theorem (Kunen, Jensen)

The following are equivalent.

1. *There is a non-trivial embedding $j : L \to L$ (a measurable cardinal gives such an embedding).*
2. $0^{\#}$*-exists.*
3. *For some singular κ, $L_{(\kappa^+)^L}$ doesn't have covering at κ.*

So L is not the largest canonical inner model.

L is small

Theorem (Kunen, Jensen)

The following are equivalent.

1. *There is a non-trivial embedding $j : L \to L$ (a measurable cardinal gives such an embedding).*
2. $0^{\#}$*-exists.*
3. *For some singular κ, $L_{(\kappa^+)^L}$ doesn't have covering at κ.*

So L is not the largest canonical inner model. Maybe the largest canonical inner model has the form $L[A]$ where A is subset of ordinals? For this to work, large cardinals need to be coded as a set of ordinals.

Extenders

1. Suppose $j : M \to N$ is a non-trivial elementary (or just Σ_1-elementary) embedding where M and N are two transitive models of a fragment of set theory.

Extenders

1. Suppose $j : M \to N$ is a non-trivial elementary (or just Σ_1-elementary) embedding where M and N are two transitive models of a fragment of set theory.
2. Suppose λ is any ordinal in N, and ν is the least such that $j(\nu) \geq \lambda$.
3. We then then say E is the (ν, λ)-extender derived from j if

$$(a, A) \in E \leftrightarrow a \in [\lambda]^{<\omega} \wedge A \subseteq [\nu]^{|a|} \wedge a \in j(A).$$

Extenders

1. Suppose $j : M \to N$ is a non-trivial elementary (or just Σ_1-elementary) embedding where M and N are two transitive models of a fragment of set theory.
2. Suppose λ is any ordinal in N, and ν is the least such that $j(\nu) \geq \lambda$.
3. We then then say E is the (ν, λ)-extender derived from j if
$$(a, A) \in E \leftrightarrow a \in [\lambda]^{<\omega} \wedge A \subseteq [\nu]^{|a|} \wedge a \in j(A).$$
4. We then let $Ult(M, E)$ be the ultrapower of M by E.

Extenders continued

1. Let $D = \{(a, f) : a \in [\lambda]^{<\omega}, f \in M, f : [\nu]^{|a|} \to M\}$.

Extenders continued

1. Let $D = \{(a, f) : a \in [\lambda]^{<\omega}, f \in M, f : [\nu]^{|a|} \to M\}$.
2. Given $(a, f) \in D$ and $b \in [\lambda]^{<\omega}$ such that $a \subseteq b$, $f^b : [\nu]^{|a|} \to M$ is given by
$$f^b(s) = f(s_{i_0}, ..., s_{i_{|a|}})$$
where if $b = (b_0, ..., b_k)$ then $b_{i_j} = a_j$ for $j \leq |a|$.

Extenders continued

1. Let $D = \{(a, f) : a \in [\lambda]^{<\omega}, f \in M, f : [\nu]^{|a|} \to M\}$.
2. Given $(a, f) \in D$ and $b \in [\lambda]^{<\omega}$ such that $a \subseteq b$, $f^b : [\nu]^{|a|} \to M$ is given by
$$f^b(s) = f(s_{i_0}, ..., s_{i_{|a|}})$$
where if $b = (b_0, ..., b_k)$ then $b_{i_j} = a_j$ for $j \leq |a|$.
3. Given $(a, f), (b, g) \in D$, we set $(a, f) =_E (b, g)$ if and only if
$$\{s : f^{b \cup a}(s) = g^{b \cup a}(s)\} \in E_{b \cup a}$$
where $E_c = \{A : (c, A) \in E\}$.

Extenders continued

1. Let $D = \{(a, f) : a \in [\lambda]^{<\omega}, f \in M, f : [\nu]^{|a|} \to M\}$.
2. Given $(a, f) \in D$ and $b \in [\lambda]^{<\omega}$ such that $a \subseteq b$, $f^b : [\nu]^{|a|} \to M$ is given by
$$f^b(s) = f(s_{i_0}, ..., s_{i_{|a|}})$$
where if $b = (b_0, ..., b_k)$ then $b_{i_j} = a_j$ for $j \leq |a|$.
3. Given $(a, f), (b, g) \in D$, we set $(a, f) =_E (b, g)$ if and only if
$$\{s : f^{b\cup a}(s) = g^{b\cup a}(s)\} \in E_{b\cup a}$$
where $E_c = \{A : (c, A) \in E\}$.
4. Similarly we define $\in_E$ on D.

Extenders continued

1. Let $D = \{(a, f) : a \in [\lambda]^{<\omega}, f \in M, f : [\nu]^{|a|} \to M\}$.
2. Given $(a, f) \in D$ and $b \in [\lambda]^{<\omega}$ such that $a \subseteq b$, $f^b : [\nu]^{|a|} \to M$ is given by
$$f^b(s) = f(s_{i_0}, ..., s_{i_{|a|}})$$
where if $b = (b_0, ..., b_k)$ then $b_{i_j} = a_j$ for $j \leq |a|$.
3. Given $(a, f), (b, g) \in D$, we set $(a, f) =_E (b, g)$ if and only if
$$\{s : f^{b\cup a}(s) = g^{b\cup a}(s)\} \in E_{b\cup a}$$
where $E_c = \{A : (c, A) \in E\}$.
4. Similarly we define $\in_E$ on D.
5. Let then $[a, f]_E$ be the $=_E$ equivalence class of (a, f).

Extenders continued

1. Let $D = \{(a, f) : a \in [\lambda]^{<\omega}, f \in M, f : [\nu]^{|a|} \to M\}$.
2. Given $(a, f) \in D$ and $b \in [\lambda]^{<\omega}$ such that $a \subseteq b$, $f^b : [\nu]^{|a|} \to M$ is given by
$$f^b(s) = f(s_{i_0}, ..., s_{i_{|a|}})$$
where if $b = (b_0, ..., b_k)$ then $b_{i_j} = a_j$ for $j \leq |a|$.
3. Given $(a, f), (b, g) \in D$, we set $(a, f) =_E (b, g)$ if and only if
$$\{s : f^{b \cup a}(s) = g^{b \cup a}(s)\} \in E_{b \cup a}$$
where $E_c = \{A : (c, A) \in E\}$.
4. Similarly we define $\in_E$ on D.
5. Let then $[a, f]_E$ be the $=_E$ equivalence class of (a, f).
6. Set
$$\mathbb{D} = \{[a, f]_E : (a, f) \in D\} \text{ and}$$
$$Ult(M, E) = (\mathbb{D}, \in_E).$$

Extenders continued

1. Alternatively, we can let $M_a = Ult(M, E_a)$ (this makes sense as E_a is just an ultrafilter on some $[\xi]^{|a|}$).

Extenders continued

1. Alternatively, we can let $M_a = Ult(M, E_a)$ (this makes sense as E_a is just an ultrafilter on some $[\xi]^{|a|}$).
2. We then have, for $a \subseteq b$, canonical elementary maps $j_{a,b} : M_a \to M_b$ given by $j_{a,b}([a, f]) = [b, f^b]$.

Extenders continued

1. Alternatively, we can let $M_a = Ult(M, E_a)$ (this makes sense as E_a is just an ultrafilter on some $[\xi]^{|a|}$).
2. We then have, for $a \subseteq b$, canonical elementary maps $j_{a,b} : M_a \to M_b$ given by $j_{a,b}([a, f]) = [b, f^b]$.
3. The system $(M_a, j_{a,b} : a \subseteq b)$ is directed, and

Exercise

1. *$Ult(M, E)$ is the direct limit of $(M_a, j_{a,b} : a \subseteq b)$.*

Extenders continued

1. Alternatively, we can let $M_a = Ult(M, E_a)$ (this makes sense as E_a is just an ultrafilter on some $[\xi]^{|a|}$).
2. We then have, for $a \subseteq b$, canonical elementary maps $j_{a,b} : M_a \to M_b$ given by $j_{a,b}([a, f]) = [b, f^b]$.
3. The system $(M_a, j_{a,b} : a \subseteq b)$ is directed, and

Exercise

1. *$Ult(M, E)$ is the direct limit of $(M_a, j_{a,b} : a \subseteq b)$.*
2. *The map $\pi_E : M \to Ult(M, E)$ given by $\pi_E(x) = [\kappa, c_x]$ is elementary (where $\kappa = crit(j)$).*

Extenders continued

1. Alternatively, we can let $M_a = Ult(M, E_a)$ (this makes sense as E_a is just an ultrafilter on some $[\xi]^{|a|}$).
2. We then have, for $a \subseteq b$, canonical elementary maps $j_{a,b} : M_a \to M_b$ given by $j_{a,b}([a, f]) = [b, f^b]$.
3. The system $(M_a, j_{a,b} : a \subseteq b)$ is directed, and

Exercise

1. *$Ult(M, E)$ is the direct limit of $(M_a, j_{a,b} : a \subseteq b)$.*
2. *The map $\pi_E : M \to Ult(M, E)$ given by $\pi_E(x) = [\kappa, c_x]$ is elementary (where $\kappa = crit(j)$).*
3. *The map $\sigma : Ult(M, E) \to N$ given by $\sigma([a, f]) = j(f)(a)$ is elementary and $j = \sigma \circ \pi_E$.*

Extenders continued

Exercise

Develop the notion of extender independently of an associated embedding and show that all extenders in this new sense are derived from embeddings. (Hint: Think of the embedding definition and ultrafilter definition of measurability.)

Extenders continued

Exercise

Develop the notion of extender independently of an associated embedding and show that all extenders in this new sense are derived from embeddings. (Hint: Think of the embedding definition and ultrafilter definition of measurability.)

From now on we assume you have done this!

Parameters associated to extenders

Suppose E is a (κ, λ)-extender.

1. $\mathit{strength}(E) = \sup\{\eta : V_\eta \subseteq \mathit{Ult}(V, E)\}$.
2. $\mathit{lh}(E) = \lambda$ (sometimes $\mathit{sp}(E) = \lambda$),
3. $\mathit{crit}(E) = \kappa$.

Parameters associated to extenders

Suppose E is a (κ, λ)-extender.

1. $\mathit{strength}(E) = \sup\{\eta : V_\eta \subseteq \mathit{Ult}(V, E)\}$.
2. $\mathit{lh}(E) = \lambda$ (sometimes $\mathit{sp}(E) = \lambda$),
3. $\mathit{crit}(E) = \kappa$.

We assume that $\mathit{strength}(E) = \mathit{lh}(E)$ and that $\mathit{lh}(E)$ is an inaccessible cardinal.

Parameters associated to extenders

Suppose E is a (κ, λ)-extender.

1. $\mathit{strength}(E) = \sup\{\eta : V_\eta \subseteq \mathit{Ult}(V, E)\}$.
2. $\mathit{lh}(E) = \lambda$ (sometimes $\mathit{sp}(E) = \lambda$),
3. $\mathit{crit}(E) = \kappa$.

We assume that $\mathit{strength}(E) = \mathit{lh}(E)$ and that $\mathit{lh}(E)$ is an inaccessible cardinal. (This assumption will lead to numerous white lies).

Large cardinals and extenders

All relevant large cardinals say that there is an extender with certain properties.

Large cardinals and extenders

All relevant large cardinals say that there is an extender with certain properties.

Exercise

(Strong cardinal) The following are equivalent for κ and λ.

1. *There is an elementary embedding $j : V \to M$ such that $j(\kappa) > \lambda$ and $V_\lambda \subseteq M$.*
2. *There is a (κ, λ)-extender E such that $V_\lambda \subseteq Ult(V, E)$ and $\kappa = crit(\pi_E)$.*

Large cardinals and extenders

Exercise

(Woodin cardinals) The following are equivalent for an inaccessible cardinal d.

1. *For each $A \subseteq \delta$ there is $\kappa < \delta$ such that for every $\lambda < \delta$ there is an elementary embedding $j : V \to M$ such that $j(\kappa) > \lambda$, $V_\lambda \subseteq M$ and $A \cap \lambda = j(A) \cap \lambda$.*
2. *For each $A \subseteq \delta$ there is $\kappa < \delta$ such that for every $\lambda < \delta$ there is a (κ, λ)-extender E such that $V_\lambda \subseteq Ult(V, E)$, $crit(\pi_E) = \kappa$ and $\pi_E(A) \cap \lambda = A \cap \lambda$.*

Large cardinals and extenders

Exercise

(Superstrong cardinals) The following are equivalent for an inaccessible cardinal k.

1. *There is an elementary embedding $j : V \to M$ such that $\kappa = crit(j)$ and $V_{j(\kappa)} \subseteq M$.*
2. *There is a (κ, λ)-extender E such that $\kappa = crit(\pi_E)$, $\pi_E(\kappa) = \lambda$ and $V_\lambda \subseteq Ult(V, E)$.*

Large cardinals and extenders

Exercise

(Superstrong cardinals) The following are equivalent for an inaccessible cardinal k.

1. *There is an elementary embedding* $j : V \to M$ *such that* $\kappa = crit(j)$ *and* $V_{j(\kappa)} \subseteq M$.
2. *There is a* (κ, λ)*-extender* E *such that* $\kappa = crit(\pi_E)$, $\pi_E(\kappa) = \lambda$ *and* $V_\lambda \subseteq Ult(V, E)$.

Exercise

Suppose $j : V \to M$ *is an elementary embedding such that letting* $crit(j) = \kappa$ *and* $\lambda = \sup\{j(f)(\kappa) : f : \kappa \to V\}$, $V_\lambda \subseteq M$. *Show that* κ *is a superstrong cardinal.*

Short and long extenders

1. A (κ, λ)-extender E is a short extender if $crit(\pi_E) = \kappa$. Otherwise it is a long extender.

Short and long extenders

1. A (κ, λ)-extender E is a short extender if $crit(\pi_E) = \kappa$. Otherwise it is a long extender.
2. All large cardinals in the region of superstrong cardinals can be defined in terms of short extenders.

Short and long extenders

1. A (κ, λ)-extender E is a short extender if $crit(\pi_E) = \kappa$. Otherwise it is a long extender.
2. All large cardinals in the region of superstrong cardinals can be defined in terms of short extenders.
3. A major assumption. In these talks we will always assume extenders are short. The long region is being investigated by Woodin, and at the moment it is not fully understood.

Canonical inner models with large cardinals

1. Since every large cardinal is essentially an extender, what we need to do is develop a theory of models of the form $L[\vec{E}]$, where $\vec{E}$ is a sequence of extenders.

Canonical inner models with large cardinals

1. Since every large cardinal is essentially an extender, what we need to do is develop a theory of models of the form $L[\vec{E}]$, where $\vec{E}$ is a sequence of extenders.
2. Because we want canonical models $\vec{E}$ has to have nice properties.

Canonical inner models with large cardinals

1. Since every large cardinal is essentially an extender, what we need to do is develop a theory of models of the form $L[\vec{E}]$, where $\vec{E}$ is a sequence of extenders.
2. Because we want canonical models $\vec{E}$ has to have nice properties.
3. (Repeating) Suppose $x \subseteq \omega$ and F is an extender. Let $\vec{E}$ be the sequence: $\vec{E}(i) = F \leftrightarrow i \in \omega$. Then $x \in L[\vec{E}]$.

Canonical inner models with large cardinals

1. Since every large cardinal is essentially an extender, what we need to do is develop a theory of models of the form $L[\vec{E}]$, where $\vec{E}$ is a sequence of extenders.
2. Because we want canonical models $\vec{E}$ has to have nice properties.
3. (Repeating) Suppose $x \subseteq \omega$ and F is an extender. Let $\vec{E}$ be the sequence: $\vec{E}(i) = F \leftrightarrow i \in \omega$. Then $x \in L[\vec{E}]$.
4. (Initial segments). Suppose F is a (κ, λ) extender and $(\lambda_i : i < \omega) \subseteq (\kappa, \lambda)$ is an increasing sequence. Let $F_i = \{(a, A) : a \in [\lambda_i]^{<\omega} \wedge (a, A) \in F\}$. Now again if $x \subseteq \omega$ and $\vec{E}$ is given by $\vec{E}(i) = F_i \leftrightarrow i \in \omega$, then $x \in L[\vec{E}]$.

The good extender sequence

1. Repeating is solved by assigning to each extender an index in the sequence. The assignment is one to one, and so no extender can be repeated.

The good extender sequence

1. Repeating is solved by assigning to each extender an index in the sequence. The assignment is one to one, and so no extender can be repeated.
2. (Mitchell-Steel Indexing) Given a (κ, λ)-extender E, index E at $(\lambda^+)^{Ult(V,E)}$.

The good extender sequence

1. Repeating is solved by assigning to each extender an index in the sequence. The assignment is one to one, and so no extender can be repeated.
2. (Mitchell-Steel Indexing) Given a (κ, λ)-extender E, index E at $(\lambda^+)^{Ult(V,E)}$.
3. (Jensen-Friedman Indexing) Given a (κ, λ)-extender E, index E at $(j(\kappa)^+)^{Ult(V,E)}$.

The good extender sequence

1. Repeating is solved by assigning to each extender an index in the sequence. The assignment is one to one, and so no extender can be repeated.
2. (Mitchell-Steel Indexing) Given a (κ, λ)-extender E, index E at $(\lambda^+)^{Ult(V,E)}$.
3. (Jensen-Friedman Indexing) Given a (κ, λ)-extender E, index E at $(j(\kappa)^+)^{Ult(V,E)}$.
4. The two indices coincide if and only if E is a superstrong extender.

The good extender sequence

1. Repeating is solved by assigning to each extender an index in the sequence. The assignment is one to one, and so no extender can be repeated.
2. (Mitchell-Steel Indexing) Given a (κ, λ)-extender E, index E at $(\lambda^+)^{Ult(V,E)}$.
3. (Jensen-Friedman Indexing) Given a (κ, λ)-extender E, index E at $(j(\kappa)^+)^{Ult(V,E)}$.
4. The two indices coincide if and only if E is a superstrong extender.
5. The initial segment problem is solved by requiring that an extender can be added to the sequence if and only if all of its relevant initial segments are already on the sequence.

Coherence

1. Suppose μ is a normal κ-complete ultrafilter, and E_μ is the (κ, κ^+)-extender derived from $\pi_\mu : V \to Ult(V, \mu)$.

Coherence

1. Suppose μ is a normal κ-complete ultrafilter, and E_μ is the (κ, κ^+)-extender derived from $\pi_\mu : V \to Ult(V, \mu)$. Then $L[\mu] = L[E_\mu]$, $E_\mu \cap L[E_\mu], \mu \cap L[\mu] \in L[E_\mu]$ and $L[E_\mu] \vDash$ "κ is a measurable cardinal as witnessed by $\mu \cap L[\mu]$" (Exercise).

Coherence

1. Suppose μ is a normal κ-complete ultrafilter, and E_μ is the (κ, κ^+)-extender derived from $\pi_\mu : V \to Ult(V, \mu)$. Then $L[\mu] = L[E_\mu]$, $E_\mu \cap L[E_\mu], \mu \cap L[\mu] \in L[E_\mu]$ and $L[E_\mu] \models$ "κ is a measurable cardinal as witnessed by $\mu \cap L[\mu]$" (Exercise).
2. (Kunen) Moreover, $L[\mu] \models$ "κ is the only measurable cardinal carrying a single normal κ-complete ultrafilter".

Coherence

1. Suppose μ is a normal κ-complete ultrafilter, and E_μ is the (κ, κ^+)-extender derived from $\pi_\mu : V \to Ult(V, \mu)$. Then $L[\mu] = L[E_\mu]$, $E_\mu \cap L[E_\mu], \mu \cap L[\mu] \in L[E_\mu]$ and $L[E_\mu] \vDash$ "κ is a measurable cardinal as witnessed by $\mu \cap L[\mu]$" (Exercise).
2. (Kunen) Moreover, $L[\mu] \vDash$ "κ is the only measurable cardinal carrying a single normal κ-complete ultrafilter".
3. How can we have two measures on κ?

Coherence

1. Suppose μ is a normal κ-complete ultrafilter, and E_μ is the (κ, κ^+)-extender derived from $\pi_\mu : V \to Ult(V, \mu)$. Then $L[\mu] = L[E_\mu]$, $E_\mu \cap L[E_\mu], \mu \cap L[\mu] \in L[E_\mu]$ and $L[E_\mu] \models$ "κ is a measurable cardinal as witnessed by $\mu \cap L[\mu]$" (Exercise).
2. (Kunen) Moreover, $L[\mu] \models$ "κ is the only measurable cardinal carrying a single normal κ-complete ultrafilter".
3. How can we have two measures on κ?
4. (Naive solution:) Suppose ν is another κ-complete ultrafilter on κ. Perhaps $L[\mu, \nu] \models$ "κ carries two normal measures".

Coherence

1. Suppose μ is a normal κ-complete ultrafilter, and E_μ is the (κ, κ^+)-extender derived from $\pi_\mu : V \to Ult(V, \mu)$. Then $L[\mu] = L[E_\mu]$, $E_\mu \cap L[E_\mu], \mu \cap L[\mu] \in L[E_\mu]$ and $L[E_\mu] \models$ "κ is a measurable cardinal as witnessed by $\mu \cap L[\mu]$" (Exercise).
2. (Kunen) Moreover, $L[\mu] \models$ "κ is the only measurable cardinal carrying a single normal κ-complete ultrafilter".
3. How can we have two measures on κ?
4. (Naive solution:) Suppose ν is another κ-complete ultrafilter on κ. Perhaps $L[\mu, \nu] \models$ "κ carries two normal measures".
5. (Kunen) $L[\mu, \nu] = L[\mu]$.

Coherence

1. Suppose μ is a normal κ-complete ultrafilter, and E_μ is the (κ, κ^+)-extender derived from $\pi_\mu : V \to \mathit{Ult}(V, \mu)$. Then $L[\mu] = L[E_\mu]$, $E_\mu \cap L[E_\mu], \mu \cap L[\mu] \in L[E_\mu]$ and $L[E_\mu] \models$ "κ is a measurable cardinal as witnessed by $\mu \cap L[\mu]$" (Exercise).
2. (Kunen) Moreover, $L[\mu] \models$ "κ is the only measurable cardinal carrying a single normal κ-complete ultrafilter".
3. How can we have two measures on κ?
4. (Naive solution:) Suppose ν is another κ-complete ultrafilter on κ. Perhaps $L[\mu, \nu] \models$ "κ carries two normal measures".
5. (Kunen) $L[\mu, \nu] = L[\mu]$.
6. So how do we get two measures on κ?

Mitchell order

Suppose κ is a measurable cardinal and μ is a normal κ-complete ultrafilter on κ.

1. We say μ has Mitchell order 0 if $Ult(V, \mu) \models$ "κ is not measurable".

Mitchell order

Suppose κ is a measurable cardinal and μ is a normal κ-complete ultrafilter on κ.

1. We say μ has Mitchell order 0 if $Ult(V, \mu) \models$ "κ is not measurable".
2. We say μ has Mitchell order α if $Ult(V, \mu) \models$ "κ has MItchell order $< \alpha$".

Coherence continued: just the idea

1. Suppose κ is a measurable cardinal such that it carries a normal measure μ that concentrates on measurable cardinals. Let *E* be the (κ, κ^+)-extender derived from it.

Coherence continued: just the idea

1. Suppose κ is a measurable cardinal such that it carries a normal measure μ that concentrates on measurable cardinals. Let E be the (κ, κ^+)-extender derived from it.
2. Let now $\vec{F} \in Ult(V, E)$ be the good extender sequence constructed by adding extenders of Mitchell order 1 at all measurables $\leq \kappa$ indexed according to Mitchell-Steel indexing.

Coherence continued: just the idea

1. Suppose κ is a measurable cardinal such that it carries a normal measure μ that concentrates on measurable cardinals. Let E be the (κ, κ^+)-extender derived from it.
2. Let now $\vec{F} \in Ult(V, E)$ be the good extender sequence constructed by adding extenders of Mitchell order 1 at all measurables $\leq \kappa$ indexed according to Mitchell-Steel indexing.
3. Let $\vec{F}^\frown E$ be the result of indexing E after F.

Coherence continued: just the idea

1. Suppose κ is a measurable cardinal such that it carries a normal measure μ that concentrates on measurable cardinals. Let E be the (κ, κ^+)-extender derived from it.
2. Let now $\vec{F} \in Ult(V, E)$ be the good extender sequence constructed by adding extenders of Mitchell order 1 at all measurables $\leq \kappa$ indexed according to Mitchell-Steel indexing.
3. Let $\vec{F}^\frown E$ be the result of indexing E after F. Then $L[\vec{F}^\frown E] \models$ "κ has a normal measure concentrating on measurable" (Exercise).

Coherence continued: just the idea

1. To add genuinely new extenders to the sequence we need coherence.

Coherence continued: just the idea

1. To add genuinely new extenders to the sequence we need coherence.
2. Suppose $\mathcal{M} = L_\alpha[\vec{E}]$. We say $\mathcal{M}$ satisfies coherence if for each $F \in \vec{E}$, letting β be the index of F, $\pi_F(\vec{E})|\beta = \vec{E}|\beta$.

Coherence continued: just the idea

1. To add genuinely new extenders to the sequence we need coherence.
2. Suppose $\mathcal{M} = L_\alpha[\vec{E}]$. We say $\mathcal{M}$ satisfies coherence if for each $F \in \vec{E}$, letting β be the index of F, $\pi_F(\vec{E})|\beta = \vec{E}|\beta$.

Exercise

Show that $\pi_F(\vec{E})(\beta) = \emptyset$ (hint: show that indices of extenders are cardinals in the corresponding ultrapowers but not inside the model).

Premice

Definition

We say $\mathcal{M}$ is a premouse if for some $\vec{E}$ and some ordinal α, $\mathcal{M} = L_\alpha[\vec{E}]$ and

1. $\vec{E}$ is indexed according to Mitchell-Steel indexing scheme,
2. $\vec{E}$ has initial segment condition, and
3. $\mathcal{M}$ satisfies coherence.

Definitions don't solve problems

1. Goal was to define canonical models.

Definitions don't solve problems

1. Goal was to define canonical models.
2. We had two ways of coding information into $L[\vec{E}]$ and blocked both.

Definitions don’t solve problems

1. Goal was to define canonical models.
2. We had two ways of coding information into $L[\vec{E}]$ and blocked both.
3. What guarantees that there is no other way?

Definitions don't solve problems

1. Goal was to define canonical models.
2. We had two ways of coding information into $L[\vec{E}]$ and blocked both.
3. What guarantees that there is no other way?
4. We had issues with reaching large cardinals. What guarantees that we reach them.

Definitions don't solve problems

1. Goal was to define canonical models.
2. We had two ways of coding information into $L[\vec{E}]$ and blocked both.
3. What guarantees that there is no other way?
4. We had issues with reaching large cardinals. What guarantees that we reach them.

Well, there is always next time.