The Royal Mice

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Last time: Definitions don't solve problems

1. The goal was to define canonical models.
2. We had two ways of coding information into $L[\vec{E}]$ and blocked both.
3. Why is there no other way of coding info?
4. We had issues with reaching large cardinals. Do we reach them?

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Iterability: the easy case

Suppose $\mathcal{M}$ is a premouse of the form $L_\alpha[E]$ satisfying the following properties:

1. For some $\mathcal{M}$-cardinal κ, $\mathcal{M} \models$ "E is a (κ, κ^+)-extender".
2. For each n, $(\kappa^{+n})^{\mathcal{M}}$ exists.
3. $Ord \cap \mathcal{M} = \sup_{n \in \omega} (\kappa^{+n})^{\mathcal{M}}$.

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We say $(\mathcal{M}_\alpha, E_\alpha, \pi_{\alpha,\beta} : \alpha < \beta < \eta)$ is the putative length η iteration of $\mathcal{M}$ if

1. $\mathcal{M}_0 = \mathcal{M}$, $E_0 = E$ and $\pi_{0,1} = \pi_{E_0}$.

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2. $E_\alpha = \pi_{0,\alpha}(E_0)$, $\mathcal{M}_{\alpha+1} = Ult(\mathcal{M}_\alpha, E_\alpha)$ and $\pi_{\alpha,\alpha+1} = \pi_{E_\alpha}$.

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3. $\pi_{\alpha,\beta} : \mathcal{M}_\alpha \to \mathcal{M}_\beta$ is the composition of the relevant embeddings.
4. For $\lambda < \eta$ a limit ordinal, $\mathcal{M}_\lambda$ is the direct limit of $(\mathcal{M}_\alpha, \pi_{\alpha,\beta} : \alpha < \beta < \lambda)$ and $\pi_{\alpha,\lambda} : \mathcal{M}_\alpha \to \mathcal{M}_\lambda$ is the direct limit embedding.

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Suppose $\mathcal{M}$ is a countable mouse. Then it is fully iterable.

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Suppose $\mathcal{M}$ is a countable mouse. Then it is fully iterable.

We say $\mathcal{P}$ is an iterate of $\mathcal{M}$ if for some ξ, $\mathcal{P}$ is a model appearing in the length ξ iteration of $\mathcal{M}$.

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2. $\mathcal{M} \subseteq \mathcal{N}$,
3. $E = F \cap \mathcal{M}$.

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Theorem (Comparison, Kunen)

Suppose $\mathcal{M}, \mathcal{N}$ are mice. Then there are $\mathcal{P}$ and $\mathcal{Q}$ that are iterates of $\mathcal{M}$ and $\mathcal{N}$ respectively and, either

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1. We do the proof for countable $\mathcal{M}$ and $\mathcal{N}$.
2. Let $(\mathcal{M}_\alpha, E_\alpha, \pi_{\alpha,\beta} : \alpha < \beta < \omega_1 + 1)$ and $(\mathcal{N}_\alpha, F_\alpha, \sigma_{\alpha,\beta} : \alpha < \beta < \omega_1 + 1)$ be the two length ω_1 iterations.

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3. Suppose that for $\alpha < \omega_1$, neither $\mathcal{M}_\alpha \trianglelefteq \mathcal{N}_\alpha$ nor $\mathcal{N}_\alpha \trianglelefteq \mathcal{M}_\alpha$.
4. Let $k : H \to H_{\omega_2}$ be an elementary countable hull such that both iterations are in the range of k. Let $\kappa = \omega_1^H (= \mathit{crit}(k))$.

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Exercise

Show the following equalities.

1. *crit*$(\pi_{\kappa,\omega_1}) =$ *crit*$(\sigma_{\kappa,\omega_1}) = \kappa$.

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3. $\mathit{crit}(\pi_{\kappa+1,\omega_1}) > \kappa$ *and* $\mathit{crit}(\sigma_{\kappa+1,\omega_1}) > \kappa$.

It follows that if $\lambda = (\kappa^+)^{\mathcal{M}_\kappa}$ then

$$\mathcal{M}_\kappa \models \text{“}E_\kappa \text{ is a } (\kappa, \lambda)\text{-extender”.}$$

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Similarly for $\mathcal{N}$ and $\eta =_{def} (\kappa^+)^{\mathcal{N}_\kappa}$.

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We now have that, for $A \in \mathcal{P}([\kappa]^a) \cap \mathcal{M}_\kappa \cap \mathcal{N}_\kappa$ and $a \in [\lambda]^{<\omega} \cap [\eta]^{<\omega}$,

$$\begin{aligned}
(a, A) \in E_\kappa &\leftrightarrow a \in \pi_{\kappa,\kappa+1}(A) \\
&\leftrightarrow \pi_{\kappa+1,\omega_1}(a) \in \pi_{\kappa+1,\omega_1}(\pi_{\kappa,\kappa+1}(A)) \\
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Exercise

Show that either $\mathcal{M}_\kappa \trianglelefteq \mathcal{N}_\kappa$ or $\mathcal{N}_\kappa \trianglelefteq \mathcal{M}_\kappa$.

Compatability

Corollary

Suppose $\mathcal{M}$ and $\mathcal{N}$ are two mice. Then $\mathbb{R}^{\mathcal{M}}$ and $\mathbb{R}^{\mathcal{N}}$ are compatible.

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    2. $\pi_{\kappa,\omega_1} \restriction (\mathcal{M}_k \cap \mathcal{N}_\kappa) = k \restriction (\mathcal{M}_\kappa \cap \mathcal{N}_\kappa) = \sigma_{\kappa,\omega_1} \restriction (\mathcal{M}_\kappa \cap \mathcal{N}_\kappa)$.

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   3. $crit(\pi_{\kappa+1,\omega_1}), crit(\pi_{\kappa+1,\omega_1}) > \kappa$.
3. The first two equalities are easy to obtain, almost any reasonable way of iterating mice has these properties.

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3. Then $E_\kappa = \{(a, A) : a \in [\lambda]^{<\omega}....$
4. So now, to conclude that E_κ and F_κ are compatible we need to know that
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5. We could have this equality for as long as there are no strong cardinals in the interval (κ, λ).
6. But what if there are? The solution is quite simple, apply the extender not to the model you choose it from but to the earliest model you can.

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3. According to the above rule, we want to apply it to V and not to $Ult(V, E)$.
4. The resulting iteration will have a structure of a tree.

Iteration trees

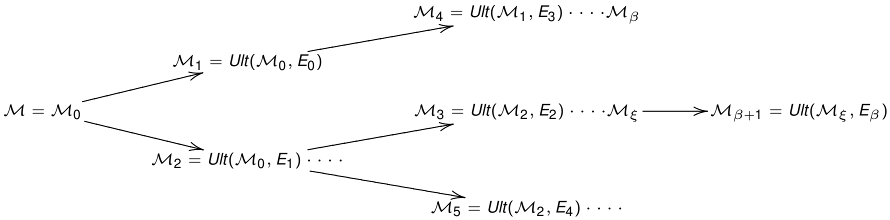

Figure: An iteration tree

Strict tree order

T is said to be a *strict tree order* on α if T is a strict partial order on α such that

1. $\beta T \gamma \Rightarrow \beta < \gamma$,
2. for every γ, $\{\beta : \beta T \gamma\}$ is wellordered by T,
3. γ is a successor ordinal $\Leftrightarrow \gamma$ is T-successor, and
4. γ is a limit ordinal $\Rightarrow \{\beta : \beta T \gamma\}$ is closed and $\in$-cofinal in γ.

If T is a strict partial order on α then we let $pred_T$ be the predecessor function on T.

Iteration game, Martin-Steel, Mitchell

Suppose $\mathcal{M}$ is a premouse. The iteration game of length η on $\mathcal{M}$ is a two player game played as follows.

1. The players produce a strict tree order T on η, a sequence of models $\langle \mathcal{M}_\alpha : \alpha < \eta \rangle$, and a sequence of extenders $\langle E_\alpha : \alpha < \eta \rangle$ such that E_α is an extender appearing on the sequence of extenders of $\mathcal{M}_\alpha$.

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1. Suppose at stage $\gamma + 1$ the players have constructed $\langle \mathcal{M}_\alpha, E_\alpha : \alpha \leq \gamma \rangle$ and $T \restriction \gamma + 1$. Then *I* chooses an extender E_γ from the extender sequence of $\mathcal{M}_\gamma$ such that for every $\alpha < \gamma$, $lh(E_\alpha) < lh(E_\gamma)$.

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2. Let $\beta \leq \gamma$ be the least ξ such that
$$crit(E_\gamma) \in [crit(E_\xi), lh(E_\xi)).$$
It makes sense to apply E_γ to $\mathcal{M}_\beta$. Player *I* then sets
$$\beta = pred_T(\gamma + 1) \text{ and } \mathcal{M}_{\gamma+1} = Ult(\mathcal{M}_\gamma, E_\gamma).$$

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The game lasts η-steps and *II* wins if for all $\alpha < \eta$, $\mathcal{M}_\alpha$ is wellfounded.

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Definition

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Comparison

Given two mice $\mathcal{M}$ and $\mathcal{N}$, we say $\mathcal{M} \trianglelefteq \mathcal{N}$ if for some β, $\mathcal{M} = L_\beta[\vec{E}^{\mathcal{N}}|\beta]$.

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Theorem (Mitchell-Steel)

Suppose $\mathcal{M}$ and $\mathcal{N}$ are two countable mice. Let Σ and Λ be their respective strategies. Then there is a Σ-iterate $\mathcal{P}$ of $\mathcal{M}$ and Λ-iterate $\mathcal{Q}$ of $\mathcal{N}$ such that either

1. $\mathcal{P} \trianglelefteq \mathcal{Q}$ *or*
2. $\mathcal{Q} \trianglelefteq \mathcal{P}$.

The proof

The proof follows the same line of thought as before. We construct an iteration tree $\mathcal{T}$ on $\mathcal{P}$ and $\mathcal{U}$ on $\mathcal{Q}$ as follows. Suppose we have $\mathcal{T} \upharpoonright \gamma$ and $\mathcal{U} \upharpoonright \gamma$.

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4. Let E_α and F_α be this extenders. Player I extends $\mathcal{T}$ and $\mathcal{U}$ by playing E_α and F_α.

A Royal exercise

Exercise

Suppose $\mathcal{T}$ is an iteration tree on $\mathcal{M}$. Show that for $\alpha < \beta < lh(\mathcal{T})$,

1. $\mathcal{M}_\alpha | lh(E_\alpha) = \mathcal{M}_\beta | lh(E_\alpha)$ *and*
2. $\mathcal{M}_\beta | lh(E_\alpha) + 1 \subset \mathcal{M}_\alpha | lh(E_\alpha) + 1$.

The proof

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3. Let $k : H \to H_{\omega_2}$ be such that $\mathcal{M}, \mathcal{N}, \Sigma, \Lambda \in rng(k)$.
4. Let $\kappa = \omega_1^H$.

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$$k^{-1}(b) = b \cap \kappa \text{ and } k^{-1}(c) = c \cap \kappa.$$

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$$k^{-1}(b) = b \cap \kappa \text{ and } k^{-1}(c) = c \cap \kappa.$$
6. So $\kappa \in b \cap c$.

The proof

1. Let $\mathcal{P} = \mathcal{M}^{\mathcal{T}}_{\kappa}$ and $\mathcal{Q} = \mathcal{M}^{\mathcal{T}}_{\kappa}$.
2. Let *E* and *F* be the extenders used in *b* and *c* respectively to get the next model in *b* and *c*.

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1. Let $\mathcal{P} = \mathcal{M}_\kappa^\mathcal{T}$ and $\mathcal{Q} = \mathcal{M}_\kappa^\mathcal{T}$.
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4. As before we have that
    1. $crit(\pi^\mathcal{T}_{\kappa,\omega_1}) = \kappa = crit(\pi^\mathcal{U}_{\kappa,\omega_1})$.
    2. Fox $X \in \mathcal{P} \cap \mathcal{Q}$, $\pi^\mathcal{T}_{\kappa,\omega_1}(X) = k(X) = \pi^\mathcal{U}_{\kappa,\omega_1}(X)$.
    3. (Crucial) $crit(\pi^\mathcal{T}_{\alpha+1,\omega_1}) > \lambda$ and $crit(\pi^\mathcal{U}_{\beta+1,\omega_1}) > \nu$.

The proof: crucial fact

Lemma

crit$(\pi^{\mathcal{T}}_{\alpha+1,\omega_1}) > \lambda$ *and crit*$(\pi^{\mathcal{U}}_{\beta+1,\omega_1}) > \nu$.

We now finish as before.

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crit$(\pi^{\mathcal{T}}_{\alpha+1,\omega_1}) > \lambda$ *and crit*$(\pi^{\mathcal{U}}_{\beta+1,\omega_1}) > \nu$.

We now finish as before.

Corollary

Suppose $\mathcal{M}$ *and* $\mathcal{N}$ *are two mice. Then* $\mathbb{R}^{\mathcal{M}} \subseteq \mathbb{R}^{\mathcal{N}}$ *or* $\mathbb{R}^{\mathcal{N}} \subseteq \mathbb{R}^{\mathcal{M}}$.

Woodin's genericity iteration

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4. Let $[\phi] \leq_T [\psi]$ if $T \vdash \phi \rightarrow \psi$.
5. Set $\mathbb{B}_T = (\{[\phi]\}, \leq_T)$.

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Theorem (Woodin)

$\mathcal{M} \models$ "$\mathbb{B}_\delta$ *has* δ*-cc and hence, it is a complete Boolean algebra".*

Woodin's genericity iteration

Theorem (Woodin)

Suppose $\mathcal{M}$ is an $\omega_1 + 1$-ietarble mouse via Σ and $x \subseteq \omega$. Suppose δ is a Woodin cardinal of $\mathcal{M}$. There is then a Σ-iterate $\mathcal{N}$ of $\mathcal{M}$ via some tree $\mathcal{T}$ such that x is generic over $\mathcal{N}$ for $\pi^{\mathcal{T}}(\mathbb{B}_\delta)$.

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Exercise

Suppose $\mathcal{M}$ is the minimal class size mouse with a Woodin cardinal. Assume the Woodin of $\mathcal{M}$ is countable and x is a real such that $\mathcal{M} \in L[x]$. Show that in $L[x]$, $\mathcal{M}$ is ω_1-iterable but not $\omega_1 + 1$-iterable.