The Royal Strategy

Grigor Sargsyan

Department of Mathematics
Rutgers University

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2. We gave ideas behind the comparison theorem. As a corollary, we obtained compatibility, that reals appearing in mice are compatible.
3. Also, as a corollary, we obtain that mice only have the information they ought to have, no other way of coding info except the two we have discussed.
4. We have not discussed the construction of mice with large cardinals, such as Woodin cardinals and beyond. This is what we will do here.

Things to keep in mind

1. Compatibility Principle (We are more or less done with this).
2. Maximality Principle.
3. Unique Hierarchy Principle.

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3. Suppose $(\mathcal{M}_\alpha, E_\alpha : \alpha < \beta)$ has been defined and β is a limit model. Then let $\mathcal{M}^*_\beta = \cup_{\alpha<\beta}\mathcal{M}_\alpha$. Suppose there is a (κ, λ)-extender E such that
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$$\pi_E((\mathcal{M}_\alpha, E_\alpha : \alpha < \beta))|\beta = (\mathcal{M}_\alpha, E_\alpha : \alpha < \beta).$$
4. Let $F = E \cap \mathcal{M}^*_\beta$. If $(\mathcal{M}^*_\beta, F)$ is a mouse then set $\mathcal{M}_\beta = (\mathcal{M}^*_\beta, F)$. Otherwise stop the construction. If there is no such extender then let $\mathcal{M}_\beta = \mathcal{M}^*_\beta$.

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4. $\mathcal{M}_{\beta+1}$ is not $(\mathcal{M}^*_\beta, F)$ but its *fine structural* Skolem hull (*the core*).
5. We can only make sense of this Skolem hull under iterability hypothesis.

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4. This approach is sometimes referred to as coarse inner model theory. All extenders on the sequence are V-extenders.

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2. It has major shortcomings. We don't for instance have a reasonable comparison theory, and cannot prove that these models satisfy GCH.
3. This points to a major shortcoming: we do not know that unwanted information doesn't get into coarse inner models.
4. This line of thought is more or less hopeless from the current point of view.

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3. However, it creates several difficulties, as now we are forced to take ultrapowers of very weak fragemnts of ZFC.
4. Fine structure theory is the theory behind such ultrapowers.
5. Unfortunately, one cannot avoid lying if one wants to avoid fine structure.

A few words about the fine structure

Suppose $\mathcal{M}$ is a premouse. Let ρ be the least ordinal such that there is a $\Sigma_1^{\mathcal{M}}$-definable subset A of ρ not in $\mathcal{M}$. It can be shown that there is a finite sequence of ordinals $p \in (Ord(\mathcal{M}) - \rho)^{<\omega}$ such that A is definable from ordinals below ρ and p.

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These questions and their brothers and sisters can be answered by any logician. Fine structure is not hard, but it is tedious, it is long and it is necessary.

Iterability conjecture

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Assume there is a Woodin cardinal and $L[\vec{E}]$ converges. Then $L[\vec{E}]$ has a Woodin cardinal.

Theorem (Neeman)

Assume there is a Woodin cardinal that is a limit of Woodin cardinals. Then $L[\vec{E}]$ doesn't break down before reaching a Woodin cardinal that is a limit of Woodin cardinals.

The end of the classical approach

Up until now we have been describing what may be called the classical inner model theory.

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4. What does nice mean?

Universally Baire sets

1. Suppose T and S are two trees on some $\eta \times \omega$. We say T and S are κ-complementing if whenever g is a $< \kappa$-generic, $V[g] \models p[T] = \mathbb{R} - p[S]$.

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6. We think of A_g as the extension or lift-up of A to $V[g]$.

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4. Suppose T, S witness that $Code(\Sigma)$ is ω_2-universally Biare, and let $\mathcal{T}$ be a tree according to Σ of length ω_1. Let $g \subseteq Coll(\omega, \omega_1)$. Then $\Sigma_g(\mathcal{T})$ is defined and has a symmetric name. So $\Sigma_g(\mathcal{T}) \in V$.

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5. A and A^c have the same degree in this relation.
6. (Woodin, Martin-Steel) Assume proper class of Woodin cardinals. Then every set in Γ_{uB} is determined.

The Derived Model Theorem

Theorem (Woodin, New DMT)

Suppose λ *is a limit of Woodin cardinals and* $g \subseteq Coll(\omega, < \lambda)$ *is generic. Let* $\mathbb{R}^* = \cup_{\alpha<\lambda} \mathbb{R}^{V[g \cap Coll(\omega,\alpha)]}$*. In* $V(\mathbb{R}^*)$*, define* $\Gamma = \{A \subseteq \mathbb{R}^* : L(A, \mathbb{R}^*) \models AD\}$*. Then* $L(\Gamma, \mathbb{R}^*) \models AD^+$*.*

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4. (Martin-Steel) $L(\mathbb{R}) \vDash \Delta_S = \mathcal{P}(\mathbb{R}) \cap L_\delta(\mathbb{R})$.

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4. This is one way of thinking of descriptive inner model theory.

The central problem of descriptive inner model theory

Conjecture (Strategic Representation Hypothesis)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$*. Then every set of reals is Wadge reducible to an iteration strategy of a hod mouse.*

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When a set of reals A is reducible to an iteration strategy of a hod mouse then we say that A is generated.

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5. This early work jump-started the project of computing HOD of models of AD^+, representing them as hod mice.

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7. There are some hard problems to solve. Steel recently solved them.

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3. Full normalization is a property of Σ that essentially says that if we play iteration game first on $\mathcal{P}$ and get $\mathcal{Q}$ then play it on $\mathcal{Q}$ and get $\mathcal{R}$, the we could re-arrange this two games into a single iteration game according to Σ that goes from $\mathcal{P}$ to $\mathcal{Q}$.

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4. Under full normalization, we can do this for arbitrary number of stacks.

Comparison theory for hod mice

Theorem (Steel)

Assume AD^+ and suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs. Then there is a Σ-iterate $\mathcal{R}$ of $\mathcal{P}$ and a Λ-iterate $\mathcal{S}$ of $\mathcal{Q}$ such that either

1. *$\mathcal{R} \trianglelefteq \mathcal{S}$ and $\Sigma = \Lambda_{\mathcal{R}}$, or*
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Building then on earlier results of S., Steel and Woodin,

Theorem (Steel)

Assume $AD^+ + AD_{\mathbb{R}}$ and that every set of reals is generated. Then $V_{\Theta}^{\mathrm{HOD}}$ is a hod mouse obtained as a direct limit of hod mice.

Direct limit of hod mice

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3. $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ is the direct limit of the resulting system.
4. Because of comparison, if $(\mathcal{Q}, \Lambda)$ is another hod pair then either $\mathcal{M}_\infty(\mathcal{P}, \Sigma) \trianglelefteq \mathcal{M}_\infty(\mathcal{Q}, \Lambda)$ or $\mathcal{M}_\infty(\mathcal{Q}, \Lambda) \trianglelefteq \mathcal{M}_\infty(\mathcal{P}, \Sigma)$.

Computation of HOD

1. What was known before is that if one proves the Strategic Representation Conjecture and develop a comparison theory for hod mice then one obtains that under $AD_{\mathbb{R}}$, $V_{\Theta}^{\mathrm{HOD}} = \cup\mathcal{M}_{\infty}(\mathcal{P}, \Sigma)$ where the union is taken over all hod pairs.

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2. Steel proved comparison.
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4. Solving the Strategic Representation Conjecture is what needs to be done.

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6. What we need to do is show that the model has certain form, namely that it is a hod mouse.

The goal of descriptive inner model theory.

Assuming various strong hypothesis such as

1. existence of large cardinals,
2. forcing axioms,
3. generic ideals,
4. determinacy axioms and etc.

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Conjecture (Strategic Representation Hypothesis)

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2. Maximality Principle.
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