The Royal Cover

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Things to discuss

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2. Maximality Principle.
3. Unique Hierarchy Principle.

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Assume $0^\#$ *doesn't exist. Then for every singular cardinal* κ*,* $(\kappa^+)^L = \kappa^+$*.*

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Insight from Woodin: The theory of "the core model" cannot be developed under *ZFC* or in large cardinal extensions of *ZFC*. So what can be done?

Covering with derived models

Suppose κ is a measurable cardinal that is a limit of Woodin cardinals and strong cardinals. Let

$$\mathcal{H}^- = V_{\Theta}^{\mathrm{HOD}}$$

where HOD is the HOD of the derived model at κ and Θ is the Θ of the derived model at κ.

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Here $Lp^\Sigma(\mathcal{H}^-)$ is the union of all sound Σ-mice that project to Θ and whose countable substructures are $< \kappa$-iterable.

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Theorem (S.)

The above conjecture fails inside the minimal mouse with a Woodin cardinal that is a limit of Woodin cardinals.

Covering with derived models

We continue with κ and $\mathcal{H}$.

Conjecture (Covering with derived models)

Assume there is no inner model with a superstrong. There is a hybrid mouse $\mathcal{M}$ extending $\mathcal{H}$ such that

1. $\mathcal{M} \vDash \mathit{ZFC} - \mathit{Powerset}$,
2. $\mathcal{M}$ *has a largest cardinal* η,
3. $\mathcal{M} \vDash \square_\eta$,
4. $\mathrm{cf}(\mathit{Ord} \cap \mathcal{M}) \geq \kappa$ *and*
5. $\mathcal{M}$ *has a* $\square_\eta$ *sequence that is not threadable in* V.

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5. $\mathcal{M}$ *has a* $\square_\eta$ *sequence that is not threadable in* V.

$\mathcal{M}$ has to somehow be defined by "lifting" information from the derived model above $\mathcal{H}$.

Lifting information from the derived model

(Assume no inner model with a superstrong.) There is a formula ϕ independent of κ such that $\phi(\kappa)$ defines a pair of models $(\mathcal{M}_0, \mathcal{M}_1)$ such that letting $g \subseteq Coll(\omega, <\kappa)$ be generic and $L(\Gamma, \mathbb{R}^{V[g]})$ be the derived model at κ,

1. $\mathcal{H}^- \trianglelefteq \mathcal{M}_0 \triangleleft \mathcal{M}_1$,
2. $L(\mathcal{M}_0^\omega, \Gamma, \mathbb{R}^{V[g]}) \vDash AD^+$,
3. letting $\eta = Ord \cap \mathcal{M}_0$, $(\mathcal{M}_1, \eta)$ satisfy the conjecture.

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$\mathcal{M}_0$ is obtained via some “generalized direct limit procedure” and $\mathcal{M}_1$ is obtained as a “stack” over $\mathcal{M}_0$.

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Assume (short) Mouse Capturing holds. Then V_Θ^{HOD} is a hybrid mouse.

(short) Mouse Capturing: MC is the conjunction of the following statements.

1. $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$
2. "no inner model with a superstrong cardinal",
3. for $x, y \in \mathbb{R}$, $x \in OD(y)$ if and only if there is a y-mouse containing x.

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The Mouse Set Conjecture is one of our ways of guaranteeing that the Maximality Prinicple, all the canonical reals appear in mice.

Partial results

We say κ is a *Suslin cardinal* if there is a set A and a tree T on κ such that

1. $A = p[T]$ and
2. for every $\lambda < \kappa$, A cannot be represented as a projection of a tree on λ.

Largest Suslin Axiom: The Largest Suslin Axiom is the conjunction of the following set of axioms.

1. There is a largest Suslin cardinal κ (necessarily less than Θ).
2. For every set A whose Wadge rank is $< \kappa$, there is no OD_A surjection $f : \mathbb{R} \to \kappa$.

A partial result

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Theorem

The following theories are equiconsistent.

1. *The Largest Suslin Axiom.*
2. *There is a cardinal λ that is a limit of Woodin cardinals, the old derived model at λ satisfies $AD_{\mathbb{R}}$ and the new and old derived models at λ are different.*

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(Very good project:) Not much is know about covering with derived models, though we have number of tools.

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They will also, hopefully, clarify the goal of descriptive inner model theory.

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Assuming various strong hypothesis such as

1. existence of large cardinals,
2. forcing axioms,
3. generic ideals,
4. determinacy axioms and etc.

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The theorem is "important" because it proves a "derived model like theorem" from *PFA*. It builds on a work of S., and also of Trevor Wilson.

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1. *PFA*
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3. $\mathrm{HOD}^{L(\Gamma_{uB})}$ *has very large cardinals.*

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4. (A concrete project) Show that the failure of square at a singular strong limit cardinal of cofinality ω implies there is an inner model for $AD_{\mathbb{R}} + DC$.

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7. (A very good project) Show that the Chang Model over the Derived Model satisfies AD^+ assuming supercompacts.

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The Solovay sequence and the Solovay hierarchy

Assume *AD* and define a closed sequence $(\theta_\alpha : \alpha \leq \Omega)$ with supremum Θ as follows:

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By speculating that Ω is large we get a hierarchy of axioms.

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3. (Woodin) Under $AD^+ + AD_{\mathbb{R}}$, Ω is a limit ordinal.
4. (Woodin) Successor θs are Woodin in HOD.
5. (Project:) Determine the large cardinal strength of various axioms in the Solovay hierarchy.

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4. Recently Dominik Adolf and S. found a large cardinal that should correspond to $AD^+ + \theta_{\omega_2+1} = \Theta$. They showed that its derived model satisfies $AD^+ + \theta_{\omega_2+1} = \Theta$. The converse is open, and very doable.

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5. Adolf-S extended their work up to $\theta_\Theta = \Theta$.
6. Not much is known for other axioms, or towards showing that the large cardinals they defined give equiconsistencies.

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Ideals to determinacy

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The following theories are equiconsistent.

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Trevor Wilson obtained lower bounds from the existence of ideals on $\mathcal{P}_{\omega_1}(\mathbb{R})$.

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(Best known upper bound, Gittik) Proper class of strongly compact cardinals.

Generic absoluteness to determinacy

Definition

1. Suppose Γ is a collection of formulas. One step generic absoluteness holds for Γ if for every real x, every formula $\phi \in \Gamma$ and every generic extension $V[g]$, $\phi[x]$ holds in $V[g]$ if and only if $\phi[x]$ holds in V.

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4. $\exists^{\mathbb{R}}(\Pi^2_1)^{uB_\lambda}$ is the collection of formulas of the form

$$\exists u \in \mathbb{R} \forall B \in uB_\lambda (HC, \in, B) \models \phi[u],$$

where ϕ is some formula.

Generic absoluteness to determinacy

Theorem (Wilson)

Suppose λ is a limit of Woodin cardinals and $\delta < \lambda$ is $< \lambda$-strong. If one step $\exists^{\mathbb{R}}(\Pi^2_1)^{uB_\lambda}$ generic absoluteness for Coll(ω, δ^+) holds after forcing with Coll(ω, δ) then:

1. *The derived model at δ satisfies ZF $+$ AD$^+$ $+ \theta_0 < \Theta$.*
2. *The derived model at λ satisfies ZF $+$ AD$^+$ $+ \theta_1 < \Theta$.*

All sets are universally Baire

Theorem (Larson, S., Wilson)

Assume λ is a limit of strong cardinals and Woodin cardinals. Let $g \subseteq Coll(\omega, <\lambda)$ and Γ be the sets of reals of the derived model. Then, in $V(\mathbb{R}^*)$, there is a definable predicate F such that $L(F, \Gamma) \models AD^+ +$ " all sets are universally Baire".

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(Another nice project:) Determine the strength of $AD^+ +$ "all sets are homogenously Souslin".

Varsovian models

Theorem (S.-Schindler)

Suppose $\mathcal{M}$ is the minimal mouse with a strong cardinal and a Woodin cardinal. Let κ be the strong cardinal of $\mathcal{M}$ and $g \subseteq \mathit{Coll}(\omega, (\kappa^+)^{\mathcal{M}})$ be generic. Then the core model of $\mathcal{M}[g]$ is the mantle of $\mathcal{M}$, i.e., is the intersection of all grounds of $\mathcal{M}[g]$.

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(Nice project) Compute the mantle of other mice.

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3. We demonstrated the Unique Hierarchy Principle by stating number of theorems linking various hierarchies together.
4. Of course, there is a lot more to do towards 1 and 2.

How to really learn inner model theory

1. Jensen, Fine structure of L,
2. Schimmerling, ABC of mice (motivational),
3. Jensen, Inner models and large cardinals (motivational),
4. Schinler-Zeman, Fine Structure,
5. Philip Welch, Fine Structure,
6. Mitchell-Steel, Fine structure and iteration trees,
7. Steel, The core model iterability problem,
8. Steel, Outline of Inner Model Theory,
9. Steel-Woodin, HOD is a core model,
10. S., Descriptive inner model theory (motivational).
11. Neeman, Determinacy in $L(\mathbb{R})$.

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6. Mouse!