Playing with equivalent forms of CH

Silvia Steila
on-going work with Alessandro Andretta and Raphaël Carroy

Universität Bern

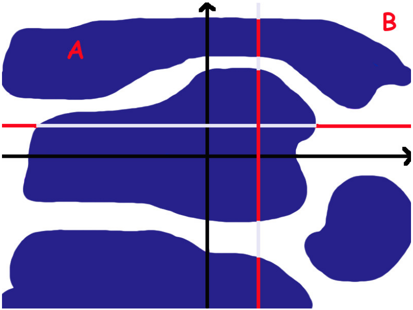
New directions in the higher infinite - 10 YSTW

Edinburgh
July 14, 2017

Equivalent forms of CH ($2^{\aleph_0} = \aleph_1$)

Theorem (Sierpiński 1965)

CH holds iff there are two sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A and all horizontal sections of B are countable.



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Proof sketch.

$\Rightarrow$. Let $f : \mathbb{R} \to \omega_1$ be a bijection. Put

$$A := \{(x, y) : f(y) < f(x)\} \qquad B := \{(x, y) : f(y) \geq f(x)\}$$

$\Leftarrow$. Let $i : \omega_1 \to \mathbb{R}$ any injection. We claim that

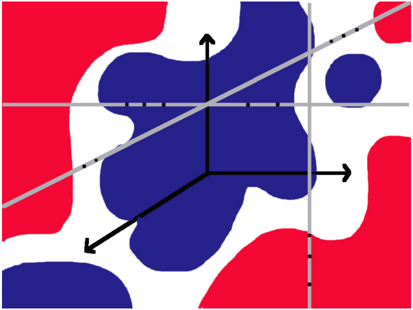
$$\mathbb{R} = \bigcup_{\alpha \in \omega_1} A(i(\alpha), \bullet).$$

If not there exists a such that $\forall \alpha \in \omega_1 . B(i(\alpha), a)$. Thus $|B(\bullet, a)| \geq \omega_1$.

Equivalent forms of CH ($2^{\aleph_0} = \aleph_1$)

Theorem (Sierpiński 1965)

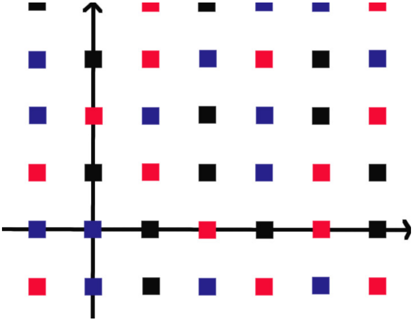
CH holds iff there are sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.



Equivalent forms of CH ($2^{\aleph_0} = \aleph_1$)

Theorem (Komjáth and Totik 2006)

CH holds iff there exists a coloring $f : \mathbb{R} \times \mathbb{R} \to \omega$ such that there are no two sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $f \restriction C \times D$ is monochromatic.



From CH-equivalences to $(\mathbb{R} \subseteq L)$-equivalences

In 2012 Törnquist and Weiss studied many Σ^1_2 definable version of some statements equivalent to CH.

CH $\iff$ there exist some **objects** such that **something happens**.

They proved that these Σ^1_2 counterparts become equivalent to the statement "all reals are constructible".

$\mathbb{R} \subseteq L$ $\iff$ there exist some Σ^1_2 **objects** such that **something happens**.

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Theorem (Törnquist and Weiss 2012)

$\mathbb{R} \subseteq L$ iff there are Σ^1_2 sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

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Theorem (Komjáth and Totik 2006)

CH holds iff there exists $g : \mathbb{R} \times \mathbb{R} \to \omega$ such that there are no two sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $g \restriction C \times D$ is monochromatic.

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$\mathbb{R} \subseteq L$ iff there exists a Σ^1_2-definable function $g : \mathbb{R} \times \mathbb{R} \to \omega$ such that there are no sets $C, D \subseteq \mathbb{R}$ such that $|C| = |D| = 2$ and $g \restriction C \times D$ is monochromatic.

From CH-equivalences to ($\mathbb{R} \subseteq L$)-equivalences

CH $\iff$ there exist some **objects** such that **something happens**.

$\mathbb{R} \subseteq L$ $\iff$ there exist some Σ^1_2 **objects** such that **something happens**.

From CH-equivalences to ($\mathbb{R} \subseteq L^{2k+2}$)-equivalences

We are currently working on the corresponding version for Σ^1_{2k+2}.

CH $\iff$ there exist some **objects** such that **something happens**.

$\mathbb{R} \subseteq L^{2k+2}$ $\iff$ there exist some Σ^1_{2k+2} **objects** such that **something happens**.

From CH-equivalences to ($\mathbb{R} \subseteq L^{2k+2}$)-equivalences

The "right" model seems to be L^{2k+2}, as introduced by Kechris and Moschovakis in the '70.

Let $\mathcal{C}_{2k+2}$ be the smallest Σ^1_{2k+2} set which contains all thin Σ^1_{2k+2} sets. Let $L^{2k+2} = L(\mathcal{C}_{2k+2})$.

Conjecture

Assume $\mathrm{Det}(\Delta^1_{2k})$. $\mathbb{R} \subseteq L^{2k+2}$ holds iff there are two sets Σ^1_{2k+2} $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A are countable and all horizontal sections of B are countable.

...working on this gave us the idea to look for another direction...

Countable, finite

- (Sierpiński) CH holds iff there are two sets $A, B \subseteq \mathbb{R}^2$ with $A \cup B = \mathbb{R}^2$ such that all vertical sections of A and all horizontal sections of B are countable.

- (Sierpiński) CH holds iff there are sets $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$, and every line in the direction of the x_i-axis meets A_i in finitely many points.

Ideals on $\mathbb{R}$

- $\mathcal{C}$ the ideal of countable sets, σ-closure of the ideal of finite sets.

- $\mathcal{M}$ the ideal of meager sets, σ-closure of the ideal of nowhere dense.

  Meager set: union of countably many nowhere dense sets.
  Nowhere dense: if for each nonempty open set U in $\mathbb{R}$, there is a nonempty open set V contained in U such that V and A are disjoint.

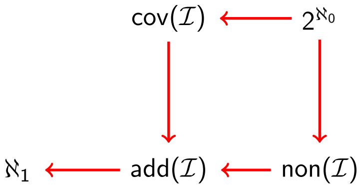
- $\mathcal{N}$ the ideal of null sets, σ-closure of the ideal of negligible sets.

  Null set: $\forall \varepsilon > 0 \exists U \in {}^{\omega}\mathcal{J}(A \subseteq \bigcup_{n\in\omega} U(n) \wedge \sum_{n\in\omega} |U(n)| < \varepsilon)$
  where $\mathcal{J}$ is the set of intervals and $|X|$ is the length of X.
  Negligible set:
  $\forall \varepsilon > 0 \exists U \in {}^{<\omega}\mathcal{J}(A \subseteq \bigcup_{n\in|U|} U(n) \wedge \sum_{n\in|U|} |U(n)| < \varepsilon)$.

Cardinal invariants

Given an ideal of $\mathcal{I}$ of $\mathbb{R}$, recall standard cardinal invariants:

- $\mathrm{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\}$;
- $\mathrm{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = \mathbb{R}\}$;
- $\mathrm{non}(\mathcal{I}) = \min\{|Y| : Y \subseteq \mathbb{R} \wedge Y \notin \mathcal{I}\}$.



Look once more at the proof by Sierpiński

Theorem (Sierpiński 1965)

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Proof sketch.

$\Rightarrow$. Let $f : \mathbb{R} \to \omega_1$ be a bijection. Put

$$A := \{(x, y) : f(y) < f(x)\} \qquad B := \{(x, y) : f(y) \geq f(x)\}\,;$$

$\Leftarrow$. Let $i : \omega_1 \to \mathbb{R}$ any injection. We claim that

$$\mathbb{R} = \bigcup_{\alpha \in \omega_1} A(i(\alpha), \bullet).$$

$\mathrm{cov}(\mathcal{C}) \leq \omega_1$

If not there exists a such that $\forall \alpha \in \omega_1 . B(i(\alpha), a)$. Thus $|B(\bullet, a)| \geq \omega_1$.

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Proof sketch.

$\mathrm{cov}(\mathcal{C}) = \mathrm{add}(\mathcal{C}) = \mathrm{non}(\mathcal{C}) = \omega_1$

$\Rightarrow$. Let $f : \mathbb{R} \to \omega_1$ be a bijection. Put

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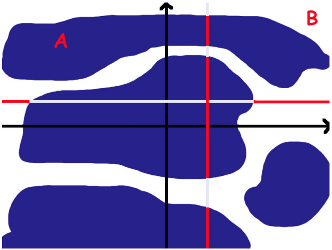
If not there exists a such that $\forall \alpha \in \omega_1 . B(i(\alpha), a)$. Thus $|B(\bullet, a)| \geq \omega_1$.

Replacing the ideal of the countable sets

Theorem (Andretta Carroy S. 2017)

Let $\mathcal{I}$ be any σ-ideal on $\mathbb{R}$.

- If $\text{cov}(\mathcal{I}) \leq \text{add}(\mathcal{I})$ then there exist $A, B \subseteq \mathbb{R}^2$ such that $A \cup B = \mathbb{R}^2$ and for any $x \in \mathbb{R}$, $A(x, \bullet)$ and $B(\bullet, x)$ are both in $\mathcal{I}$.
- Assume that there exist $A, B \subseteq \mathbb{R}^2$ such that $A \cup B = \mathbb{R}^2$ and for any $x \in \mathbb{R}$, $A(x, \bullet)$ and $B(\bullet, x)$ are both in $\mathcal{I}$. Then $\text{cov}(\mathcal{I}) \leq \text{non}(\mathcal{I})$.



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Proof sketch.

$\Rightarrow$. Let $\mathbb{R} = \bigcup \{S_\alpha : \alpha \in \mathrm{add}(\mathcal{I}), S_\alpha \in \mathcal{I}\}$.
Define $h : \mathbb{R} \to \mathrm{add}(\mathcal{I})$ such that $h(x) = \mu\alpha(x \in S_\alpha)$.

$$A := \{(x, y) : h(y) < h(x)\} \qquad B := \{(x, y) : h(y) \geq h(x)\}.$$

Replacing the ideal of the countable sets

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Proof sketch.

$\Leftarrow$. Let $X \notin \mathcal{I}$ be such that $|X| = \text{non}(\mathcal{I})$. We claim that

$$\mathbb{R} = \bigcup_{x \in X} A(x, \bullet).$$

If not there exists a such that $\forall x \in X . B(x, a)$. Thus $B(\bullet, a) \supseteq X$.

Replacing the ideal of the countable sets

Theorem (Andretta Carroy S. 2017)

Let $\mathcal{I} = \sigma\mathcal{J}$ be some ideal on $\mathbb{R}$.

- If $\mathrm{cov}(\mathcal{I}) \leq \mathrm{add}(\mathcal{I})$, then there exist $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$ and any line in the direction of the x_i-axis meets A_i in some set in $\mathcal{J}$.
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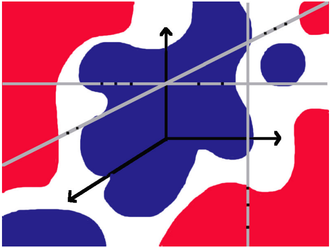

Replacing the ideal of the countable sets

Corollary

Let $\mathcal{I} = \sigma\mathcal{J}$ be some ideal on $\mathbb{R}$. If $\text{add}(\mathcal{I}) = \text{non}(\mathcal{I})$, then:

- $\text{cov}(\mathcal{I}) \leq \text{add}(\mathcal{I})$ iff there exist $A, B \subseteq \mathbb{R}^2$ such that $A \cup B = \mathbb{R}^2$ and for any $x \in \mathbb{R}$, $A(x, \bullet)$ and $B(\bullet, x)$ are both in $\mathcal{I}$.
- $\text{cov}(\mathcal{I}) \leq \text{add}(\mathcal{I})$ iff there exist $A_1, A_2, A_3 \subseteq \mathbb{R}^3$ such that $A_1 \cup A_2 \cup A_3 = \mathbb{R}^3$ and any line in the direction of the x_i-axis meets A_i in some set in $\mathcal{J}$.

...And if $\text{add}(\mathcal{I}) < \text{non}(\mathcal{I})$?

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- Assume given a model such that $\text{add}(\mathcal{I}) < \text{cov}(\mathcal{I}) < \text{non}(\mathcal{I})$.
- If the model satisfies: there exist $A, B \subseteq \mathbb{R}^2$ such that $A \cup B = \mathbb{R}^2$ and for any $x \in \mathbb{R}$, $A(x, \bullet)$ and $B(\bullet, x)$ belong to $\mathcal{I}$. Then this does not imply $\text{cov}(\mathcal{I}) \leq \text{add}(\mathcal{I})$!
- If the model does not satisfies: there exist $A, B \subseteq \mathbb{R}^2$ such that $A \cup B = \mathbb{R}^2$ and for any $x \in \mathbb{R}$, $A(x, \bullet)$ and $B(\bullet, x)$ belong to $\mathcal{I}$. Then $\text{cov}(\mathcal{I}) \leq \text{non}(\mathcal{I})$ does not imply it!

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...Looking for models...

And for Komjáth and Totik?

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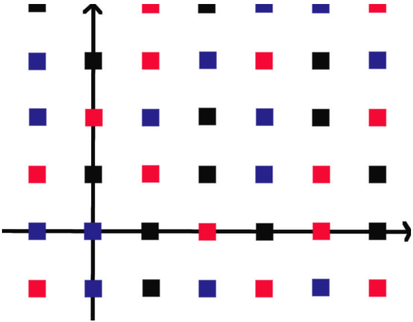

And for Komjáth and Totik?

Theorem (Andretta Carroy S. 2017)

Let $\mathcal{I} = \sigma\mathcal{J}$ be some ideal on $\mathbb{R}$ such that $\text{add}(\mathcal{I})$ is a successor cardinal.

- If $\text{cov}(\mathcal{I}) \leq \text{add}(\mathcal{I})$, then there exists $f : \mathbb{R} \times \mathbb{R} \to \text{add}(\mathcal{I})^-$ such that for any $x \in \mathbb{R}$, for all but $\mathcal{J}$-many $y \in \mathbb{R}$ and for all $D \notin \mathcal{J}$, $f \restriction (\{x, y\} \times D)$ is not monochromatic.
- Assume that there exists $f : \mathbb{R} \times \mathbb{R} \to \text{add}(\mathcal{I})^-$ such that for any $x \in \mathbb{R}$, for all but $\mathcal{J}$-many $y \in \mathbb{R}$ and for all $D \notin \mathcal{J}$, $f \restriction (\{x, y\} \times D)$ is not monochromatic. Then $\text{cov}(\mathcal{I}) \leq \text{non}(\mathcal{I})$.

From which follows...

Corollary

CH holds iff there exists $f : \mathbb{R} \times \mathbb{R} \to \omega$ such that for any $x \in \mathbb{R}$, for all but countably many $y \in \mathbb{R}$ and for all infinite $D \subseteq \mathbb{R}$, $f \restriction (\{x, y\} \times D)$ is not monochromatic.

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CH holds iff there exists $f : \mathbb{R} \times \mathbb{R} \to \omega$ such that for any $x \in \mathbb{R}$, for all but countably many $y \in \mathbb{R}$ and for all infinite $D \subseteq \mathbb{R}$, $f \restriction (\{x, y\} \times D)$ is not monochromatic.

Thank you!