Combinatorics, Cardinal Characteristics of the Continuum, and the Colouring Calculus

03E05, 03E17 & 03E02

Thilo Weinert

Ben-Gurion-University of the Negev
Joint work with William Chen and Chris Lambie-Hanson

10th Young Set Theory Workshop, Edinburgh,
Thursday, 13th July 2017, 11:30-12:30

Definition (Cantor, 1878)

CH

Problem (Souslin, 1920)

Is every dense complete linear order without endpoints, for which every collection of disjoint open intervals is countable, order-isomorphic to the real line?

Definition

Souslin's Hypothesis(SH): Yes.

Definition (Jensen, 1972)

$\diamondsuit$ says that there is a sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ with $S_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$ and $\forall S \subseteq \omega_1 (\{\alpha < \omega_1 \mid S_\alpha = S \cap \alpha\}$ is stationary.)

Theorem (Jensen, 1972)

$L \models \diamondsuit$.

Corollary

ZFC $+ \diamondsuit$ *is consistent.*

Theorem (Jensen, 1972)

$\neg(\diamondsuit \wedge \mathrm{SH})$.

Theorem (Jensen, 1974)

ZFC + CH + SH *is consistent.*

Corollary

ZFC + CH + $\neg\diamondsuit$ *is consistent.*

Definition (Ostaszewski, 1976)

$\clubsuit$ says that there is a sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ with $S_\alpha \subseteq \alpha$ for all $\alpha < \omega_1$ and $\forall S \subseteq \omega_1 \exists \alpha < \omega_1 (S_\alpha \subseteq S)$.

Definition (Broverman, Ginsburg, Kunen & Tall, 1978)

$\P$ says that there is a family $F \in [[\omega_1]^\omega]^{\omega_1}$ such that $\forall S \subseteq \omega_1 \exists X \in F(X \subseteq S)$

Exercise

$\diamondsuit \implies \clubsuit \implies$ ⫯, *and* $\diamondsuit \implies \mathrm{CH} \implies$ ⫯.

Theorem (Devlin, 1979)

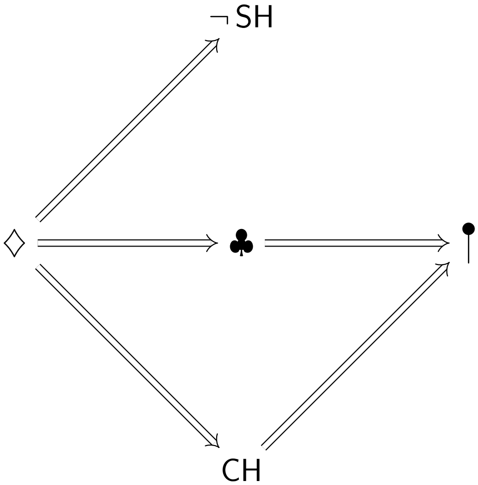
$\diamondsuit \iff \mathrm{CH} \wedge \clubsuit$

Corollary

$\mathrm{ZFC} + \mathrm{CH} + \neg\clubsuit$ *is consistent.*

Theorem (Shelah, 1980)

$\mathrm{ZFC} + \clubsuit + \neg\,\mathrm{CH}$ *is consistent.*


¬ SH
◇
♣
CH

For a cardinal κ and functions $f, g \in {}^{\kappa}\kappa$, we write $f \geqslant^* g$ to express that $\{\alpha < \kappa \mid f(\alpha) < g(\alpha)\}$ is bounded.

Definition

$\mathfrak{b}_\kappa := \min\{|F| \mid F \subseteq {}^{\kappa}\kappa \wedge \forall g \in {}^{\kappa}\kappa \exists f \in F(g \not\geqslant^* f)\}$.
$\mathfrak{b} := \mathfrak{b}_\omega$. $\mathfrak{b}$ is called the unbounding number.

Definition

$\mathfrak{d}_\kappa := \min\{|F| \mid F \subseteq {}^{\kappa}\kappa \wedge \forall g \in {}^{\kappa}\kappa \exists f \in F(f \geqslant^* g)\}$.
$\mathfrak{d} := \mathfrak{d}_\omega$. $\mathfrak{d}$ is called the dominating number.

For a cardinal κ and $x, y \subseteq \kappa$, we say that x splits y if both $y \cap x$ and $y \setminus x$ have cardinality κ.

Definition

The splitting number $\mathfrak{s}_\kappa$ is the minimal size of a subfamily F of $\wp(\kappa)$ having the property that for every $y \in \wp(\kappa)$ there is an $x \in F$ splitting y.
$\mathfrak{s} := \mathfrak{s}_\omega$.

Definition

The reaping(refining(unsplitting)) number $\mathfrak{r}_\kappa$ is the minimal size of a subfamily F of $\wp(\kappa)$ for which there is no single $x \in \wp(\kappa)$ splitting all $y \in F$.
$\mathfrak{r} := \mathfrak{r}_\omega$.

A tower is a sequence $\langle F_\alpha \mid \alpha < \gamma \rangle$ of infinite sets of natural numbers such that for $\alpha < \beta < \gamma$ the set $F_\beta \setminus F_\alpha$ is finite. It is extendible if there is an infinite set R of natural numbers such that for all $\alpha < \gamma$ the set $R \setminus F_\alpha$ is finite.

Definition

$\mathfrak{t}$ is the minimal length of an unextendible tower.

Definition

For an ideal I on a set X,

- the covering number, $\mathrm{cov}(I)$ denotes the minimal number of sets in I needed to cover all of X.
- the additivity number, $\mathrm{add}(I)$ denotes the minimal number of sets in I whose union lies outside of I.

Theorem (Truss, 1983)

$\Pin \implies \min(\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})) = \aleph_1$.

Theorem (Miyamoto, 1994, unpublished)

$\Pin \wedge \mathrm{SH} \implies \mathrm{cov}(\mathcal{M}) = \aleph_1$.

Definition (Fuchino, Shelah & Soukup, 1997)

$\Pin$ is the minimal size of a family $F \subseteq [\omega_1]^\omega$ such that for every $S \in [\omega_1]^{\omega_1}$ there is an $X \in F$ with $X \subseteq S$.

Theorem (Juhasz unpublished; Fuchino, Shelah & Soukup, 1997)

$\mathsf{ZFC} + \clubsuit + \mathrm{cov}(\mathcal{M}) = \aleph_2$ *is consistent.*

Theorem (Brendle, Malliaris & Shelah, 2006 & 2013)

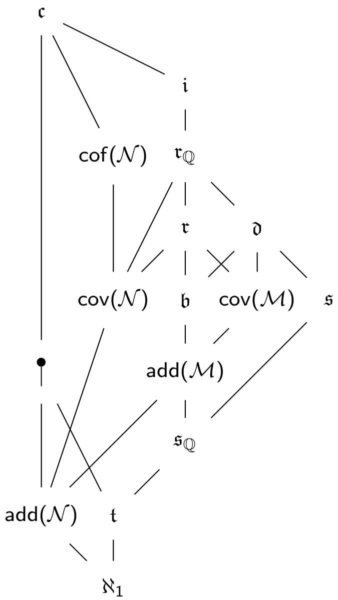
$\mathfrak{t} \leqslant \Pin$.

Theorem (Brendle, 2006)

$\mathrm{add}(\mathcal{N}) \leqslant \Pin$.

Theorem (Brendle, 2006)

$\mathsf{ZFC} + \clubsuit + \mathrm{cov}(\mathcal{N}) = \aleph_2$ *is consistent.*


$\mathfrak{c}$
$\mathfrak{i}$
$\mathrm{cof}(\mathcal{N})$
$\mathfrak{r}_{\mathbb{Q}}$
$\mathfrak{r}$
$\mathfrak{d}$
$\mathrm{cov}(\mathcal{N})$
$\mathfrak{b}$
$\mathrm{cov}(\mathcal{M})$
$\mathfrak{s}$
$\mathrm{add}(\mathcal{M})$
$\mathfrak{s}_{\mathbb{Q}}$
$\mathrm{add}(\mathcal{N})$
$\mathfrak{t}$
$\aleph_1$

Notation (Erdős & Rado, 1956)

$$\alpha \longrightarrow (\beta_0, \ldots, \beta_k)^n$$

says that for every colouring χ *of the n-tuples of a set* X *of size* α *with* $\operatorname{ran}(\chi) = k+1$*, there is an* $i \leqslant k$ *and a subset* $H \subseteq X$ *of size* β_i *such that* $\chi\big[[H]^n\big] = \{i\}$.

Notation

$\alpha \longrightarrow (\beta)^n_k$ *abbreviates* $\alpha \longrightarrow (\underbrace{\beta, \ldots, \beta}_{k \text{ times}})^n$.

Theorem (Sierpinski, 1933)

$\omega_1 \not\to (\omega_1)^2_2$.

Theorem (Erdős & Rado, 1956)

$r(\omega m, n) = \omega r(I_m, L_n)$ *for all natural numbers m and n.*

Theorem (Hajnal, 1960)

$\mathrm{CH} \implies \omega_1 \not\to (\omega_1, \omega + 2)^2$.

Theorem (Hajnal, 1971)

$\mathrm{GCH} \implies (\kappa^+)^2 \not\to ((\kappa^+)^2, 3)^2$ *for regular* κ.

Theorem (Erdős & Hajnal, 1971)

GCH $\implies \kappa^+\kappa \not\rightarrow (\kappa^+\kappa, 3)^2$ *for cardinals* κ.

Theorem (Erdős & Hajnal, 1971)

$(\kappa^+)^2 \longrightarrow (\alpha, 3)^2$ *for all cardinals* κ *and all* $\alpha < (\kappa^+)^2$.

Theorem (Baumgartner, 1972)

$r(\kappa m, n) = \kappa r(I_m, L_n)$ *for all cardinals* κ *and all* $m, n < \omega$.

Theorem (Baumgartner & Hajnal, 1973)

$\forall n < \omega \forall \alpha < \omega_1 (\omega_1 \longrightarrow (\alpha)^2_n)$.

Theorem (Baumgartner, 1975)

$\omega_1^2 \longrightarrow (\omega_1^2, 3)^2 \implies \mathsf{SH}$.

Theorem (Todorcevic, 1983)

$\forall \alpha < \omega_1 : \omega_1 \longrightarrow (\omega_1, \alpha)^2$ *is consistent.*

Theorem (Erdős, Hajnal, Mate & Rado, 1984)

If κ is regular and uncountable, then $\kappa \longrightarrow (\kappa, \omega + 1)^2$.

Theorem (Takahashi, 1987)

$⫯ = \aleph_1 \implies \omega_1^2 \not\longrightarrow (\omega_1^2, 3)^2$.

Theorem (Takahashi, 1987)

$\stick = \aleph_1 = \mathfrak{d} \implies \omega_1\omega \not\rightarrow (\omega_1\omega, 3)^2$.

Theorem (Baumgartner & Hajnal, 1987)

If κ *is regular and* $2^\kappa = \kappa^+$, *then* $(\kappa^+)^2 \not\rightarrow (\kappa^+\kappa, 4)^2$.

Theorem (Baumgartner & Hajnal, 1987)

$\omega_1^2 \longrightarrow (\omega_1\omega, 3, 3)^2$.

Theorem (Baumgartner, 1989)

$\mathrm{MA}_{\aleph_1} \implies \forall n < \omega : \omega_1\omega \longrightarrow (\omega_1\omega, n)^2$

Theorem (Todorcevic, 1989)

$\mathfrak{b} = \aleph_1 \implies \omega_1 \not\rightarrow (\omega_1, \omega + 2)^2$.

Theorem (J. Larson, 1998)

If κ is regular and $\mathfrak{d}(\kappa) = \kappa^+$, then $\kappa^+\kappa \not\rightarrow (\kappa^+\kappa, 3)^2$.

Theorem (J. Larson, 1998)

If κ is regular and $\mathfrak{d}(\kappa) = \kappa^+$, then $(\kappa^+)^2 \not\rightarrow ((\kappa^+)^2, 3)^2$.

Theorem (Raghavan & Todorcevic, 2016)

$\forall \alpha < \omega_1 (\mathfrak{b} \longrightarrow (\mathfrak{b}, \alpha)^2)$ *is consistent, relative to the existence of a measurable cardinal.*

Notation

$$\binom{\alpha}{\beta} \longrightarrow \binom{\gamma}{\delta}$$

says that for all sets A of size α, B of size β and every colouring $\chi : A \times B \longrightarrow 2$ there are $C \subseteq A$ of size γ and $D \subseteq B$ of size δ such that $|\chi[C \times D]| = 1$.

Theorem (Garti & Shelah, 2014)

If κ *is regular and* $\kappa < \lambda < \mathfrak{s}_\kappa$, *then*

$$\binom{\lambda}{\kappa} \longrightarrow \binom{\lambda}{\kappa} \iff \mathrm{cf}(\lambda) \neq \kappa.$$

Theorem (Garti & Shelah, 2014)

If κ *is regular and* $\mathfrak{r}_\kappa < \mathrm{cf}(\lambda) < 2^\kappa$, *then*

$$\binom{\lambda}{\kappa} \longrightarrow \binom{\lambda}{\kappa}.$$

Theorem (Lambie-Hanson & W., 2016)

$\mathfrak{b}_\kappa = \kappa^+ = \text{⫯}_\kappa \implies \kappa^+\kappa \not\to (\kappa^+\kappa, 3)^2$ *for regular* κ.

Theorem (Chen & W., 2016)

$\text{⫯}_\kappa = \kappa^+ \implies \kappa^+ \not\to (\kappa^+, \omega + 2)^2$.

Statement (J. Larson, 1998)

It would be interesting to know if the hypothesis $\big($*for* $(\kappa^+)^2 \not\to (\kappa^+\kappa, 4)^2\big)$ *can be weakened to the existence of a short scale in* ${}^{\kappa^+}\kappa^+$.

Statement (J. Larson, 1998)

It would be interesting to know if the hypothesis $\big($*for* $(\kappa^+)^2 \not\to (\kappa^+\kappa, 4)^2\big)$ *can be weakened to the existence of a short scale in* ${}^{\kappa^+}\kappa^+$.

Theorem (Chen & W., 2017)

$\mathfrak{d}_\kappa = \kappa^+ \implies (\kappa^+)^2 \not\to (\kappa^+\kappa, 4)^2$ *for regular* κ.

Statement (J. Larson, 1998)

It would be interesting to know if the hypothesis $\left(\text{for } (\kappa^+)^2 \not\to (\kappa^+\kappa, 4)^2\right)$ *can be weakened to the existence of a short scale in* ${}^{\kappa^+}\kappa^+$.

Theorem (Chen & W., 2017)

$\mathfrak{d}_\kappa = \kappa^+ \implies (\kappa^+)^2 \not\to (\kappa^+\kappa, 4)^2$ *for regular* κ.

Theorem (Chen & W., 2017)

$\mathfrak{b}_\kappa = \kappa^+ = \text{⫯}_\kappa \implies (\kappa^+)^2 \not\to (\kappa^+\kappa, 4)^2$ *for regular* κ.

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960			
$\mathfrak{d} = \aleph_1 =$ ⫯				
$\mathfrak{d} = \aleph_1$				
$\mathfrak{b} = \aleph_1 =$ ⫯				
⫯ $= \aleph_1$				
$\mathfrak{b} = \aleph_1$				
ZFC				

	$\omega_1 \not\rightarrow (\omega_1, \omega+2)^2$	$\omega_1^2 \not\rightarrow (\omega_1^2, 3)^2$	$\omega_1\omega \not\rightarrow (\omega_1\omega, 3)^2$	$\omega_1^2 \not\rightarrow (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	
$\mathfrak{d} = \aleph_1 =$ ⫯				
$\mathfrak{d} = \aleph_1$				
$\mathfrak{b} = \aleph_1 =$ ⫯				
⫯ $= \aleph_1$				
$\mathfrak{b} = \aleph_1$				
ZFC				

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	
$\mathfrak{d} = \aleph_1 = ⫯$				
$\mathfrak{d} = \aleph_1$				
$\mathfrak{b} = \aleph_1 = ⫯$				
$⫯ = \aleph_1$				
$\mathfrak{b} = \aleph_1$				
ZFC	Todorcevic 1983			

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 = \overset{\bullet}{|}$				
$\mathfrak{d} = \aleph_1$				
$\mathfrak{b} = \aleph_1 = \overset{\bullet}{|}$				
$\overset{\bullet}{|} = \aleph_1$				
$\mathfrak{b} = \aleph_1$				
ZFC	Todorcevic 1983			

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 =$ ⫯			Takahashi 1987	
$\mathfrak{d} = \aleph_1$				
$\mathfrak{b} = \aleph_1 =$ ⫯				
⫯ $= \aleph_1$				
$\mathfrak{b} = \aleph_1$				
ZFC	Todorcevic 1983			

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 =$ ⫯		O	Takahashi 1987	
$\mathfrak{d} = \aleph_1$				
$\mathfrak{b} = \aleph_1 =$ ⫯		O		
⫯ $= \aleph_1$		Takahashi 1987		
$\mathfrak{b} = \aleph_1$				
ZFC	Todorcevic 1983			

	$\omega_1 \not\rightarrow (\omega_1, \omega+2)^2$	$\omega_1^2 \not\rightarrow (\omega_1^2, 3)^2$	$\omega_1\omega \not\rightarrow (\omega_1\omega, 3)^2$	$\omega_1^2 \not\rightarrow (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 = \mathrm{stick}$	O	O	Takahashi 1987	
$\mathfrak{d} = \aleph_1$	O			
$\mathfrak{b} = \aleph_1 = \mathrm{stick}$	O	O		
$\mathrm{stick} = \aleph_1$		Takahashi 1987		
$\mathfrak{b} = \aleph_1$	Todorcevic 1987			
ZFC	Todorcevic 1983			

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 = \mathrm{pin}$	O	O	Takahashi 1987	
$\mathfrak{d} = \aleph_1$	O			
$\mathfrak{b} = \aleph_1 = \mathrm{pin}$	O	O		
$\mathrm{pin} = \aleph_1$		Takahashi 1987		
$\mathfrak{b} = \aleph_1$	Todorcevic 1987			
ZFC	Todorcevic 1983		Baumgartner 1989	Baumgartner 1989

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 = \text{⫯}$	O	O	Takahashi 1987	
$\mathfrak{d} = \aleph_1$	O	Larson 1998	Larson 1998	
$\mathfrak{b} = \aleph_1 = \text{⫯}$	O	O		
$\text{⫯} = \aleph_1$		Takahashi 1987		
$\mathfrak{b} = \aleph_1$	Todorcevic 1987			
ZFC	Todorcevic 1983		Baumgartner 1989	Baumgartner 1989

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 = \stick$	O	O	Takahashi 1987	
$\mathfrak{d} = \aleph_1$	O	Larson 1998	Larson 1998	
$\mathfrak{b} = \aleph_1 = \stick$	O	O	Lambie-Hanson W. 2016	
$\stick = \aleph_1$		Takahashi 1987		
$\mathfrak{b} = \aleph_1$	Todorcevic 1987			
ZFC	Todorcevic 1983		Baumgartner 1989	Baumgartner 1989

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 =$ ⫯	O	O	Takahashi 1987	
$\mathfrak{d} = \aleph_1$	O	Larson 1998	Larson 1998	
$\mathfrak{b} = \aleph_1 =$ ⫯	O	O	Lambie-Hanson W. 2016	
⫯ $= \aleph_1$	Chen W. 2016	Takahashi 1987		
$\mathfrak{b} = \aleph_1$	Todorcevic 1987			
ZFC	Todorcevic 1983		Baumgartner 1989	Baumgartner 1989

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 =$ ⫯	O	O	Takahashi 1987	O
$\mathfrak{d} = \aleph_1$	O	Larson 1998	Larson 1998	Chen W. 2017
$\mathfrak{b} = \aleph_1 =$ ⫯	O	O	Lambie-Hanson W. 2016	
⫯ $= \aleph_1$	Chen W. 2016	Takahashi 1987		
$\mathfrak{b} = \aleph_1$	Todorcevic 1987			
ZFC	Todorcevic 1983		Baumgartner 1989	Baumgartner 1989

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 = \mathrm{pin}$	O	O	Takahashi 1987	O
$\mathfrak{d} = \aleph_1$	O	Larson 1998	Larson 1998	Chen W. 2017
$\mathfrak{b} = \aleph_1 = \mathrm{pin}$	O	O	Lambie-Hanson W. 2016	*Chen* W. 2017
$\mathrm{pin} = \aleph_1$	Chen W. 2016	Takahashi 1987		
$\mathfrak{b} = \aleph_1$	Todorcevic 1987			
ZFC	Todorcevic 1983		Baumgartner 1989	Baumgartner 1989

	$\omega_1 \not\to (\omega_1, \omega+2)^2$	$\omega_1^2 \not\to (\omega_1^2, 3)^2$	$\omega_1\omega \not\to (\omega_1\omega, 3)^2$	$\omega_1^2 \not\to (\omega_1\omega, 4)^2$
CH	Hajnal 1960	Hajnal 1971	Erdős Hajnal 1971	Baumgartner Hajnal 1987
$\mathfrak{d} = \aleph_1 =$ ⫯	O	O	Takahashi 1987	O
$\mathfrak{d} = \aleph_1$	O	Larson 1998	Larson 1998	Chen W. 2017
$\mathfrak{b} = \aleph_1 =$ ⫯	O	O	Lambie-Hanson W. 2016	*Chen* W. 2017
⫯ $= \aleph_1$	Chen W. 2016	Takahashi 1987	?	?
$\mathfrak{b} = \aleph_1$	Todorcevic 1987	?	?	?
ZFC	Todorcevic 1983	?	Baumgartner 1989	Baumgartner 1989

Question

Is ZFC $+\exists\alpha < \omega_1^2(\omega_1^2 \not\rightarrow (\alpha, 3, 3)^2)$ *consistent?*

Question (Baumgartner & Hajnal, 1987)

Is ZFC $+\omega_1^2 \not\rightarrow (\omega_1\omega, 3, 3, 3)^2$ *consistent?*

Question (Jean Larson)

Is ZFC $+\omega_1^2 \longrightarrow (\omega_1^2, 3)^2$ *consistent?*

Question

Is ZFC $+\mathfrak{b} = \aleph_1 + \omega_1^2 \longrightarrow (\omega_1^2, 3)^2$ *consistent?*

Question

Is ZFC $+\mathfrak{b} = \aleph_1 + \omega_1\omega \longrightarrow (\omega_1\omega, 3)^2$ *consistent?*

Question

Is ZFC $+\text{⫯} = \aleph_1 + \omega_1\omega \longrightarrow (\omega_1\omega, 3)^2$ *consistent?*

Question

Is $\mathsf{ZFC} + \mathfrak{b} = \aleph_1 + \omega_1^2 \longrightarrow (\omega_1\omega, 4)^2$ *consistent?*

Question

Is $\mathsf{ZFC} + \text{stick} = \aleph_1 + \omega_1^2 \longrightarrow (\omega_1\omega, 4)^2$ *consistent?*

Question

Is $\mathsf{ZFC} + \exists\kappa(\kappa$ *is regular,* $\mathfrak{b}_\kappa = \kappa^+$ *and* $\kappa^+ \longrightarrow (\kappa^+, \kappa+2)^2)$ *consistent?*

Question

Is $\mathrm{ZFC} + \mathrm{MA}_{\aleph_1} + \exists\alpha < \omega_1 : \omega_1 \not\to (\omega_1, \alpha)^2$ *consistent?*

Question (Brendle, 2006)

Is $\mathrm{ZFC} + \clubsuit + \mathfrak{s} > \aleph_1$ *consistent?*

Question

Is $\mathrm{ZFC} + \clubsuit + \mathfrak{s}_{\mathbb{Q}} > \aleph_1$ *consistent?*

Question (Juhász, ?)

Is $\mathrm{ZFC} + \clubsuit + \mathrm{SH}$ *consistent?*

Thank you for listening!