The perfect set property for universally Baire sets of reals

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Definition

A set of reals B has the perfect set property (PSP) if it is countable or it has a perfect subset.

Theorem (Suslin)

Every analytic set of reals has PSP.

Theorem (Bernstein)

There is a set of reals without PSP, assuming AC.

Remark

Between analytic sets of reals and arbitrary sets of reals, interesting things happen.

Theorem (Solovay, Gödel, Specker)

The following statements are equiconsistent modulo ZFC.

1. There is an inaccessible cardinal.
2. Every coanalytic set of reals has PSP.
3. Every set of reals in $L(\mathbb{R})$ has PSP.

Remark

- If (1) is witnessed by κ, then (3) holds in $V^{\mathrm{Col}(\omega,<\kappa)}$.
- If (3) holds, then (2) holds.
- If (2) holds, then (1) holds in L, witnessed by ω_1^V.

What about PSP for other sets of reals?
Which other sets of reals might possibly be well-behaved?

Definition

- A tree on a set X is a subset of $X^{<\omega}$ that is closed under initial segments.
- $[T] \subset X^\omega$ is the set of all branches of T.
- If T is a tree on a Cartesian product $X \times Y$, then $\mathrm{p}[T]$ is the projection of T onto X^ω.

Example

- Every analytic set of reals is the projection of some tree on $\omega \times \omega$, and vice versa.
- Every coanalytic set of reals is the projection of some tree on $\omega \times \omega_1$.
- Under AC, every set of reals is the projection of some tree on $\omega \times \mathfrak{c}$.

Definition (Feng–Magidor–Woodin)

- For a poset $\mathbb{P}$, a pair of trees $(T, \tilde{T})$ on $\omega \times \gamma$ ($\gamma \in \mathrm{Ord}$) is called $\mathbb{P}$-absolutely complementing if $\mathrm{p}[T] = \omega^\omega \setminus \mathrm{p}[\tilde{T}]$ in every generic extension by $\mathbb{P}$.
- A set of reals A is universally Baire if, for every poset $\mathbb{P}$, there is a $\mathbb{P}$-absolutely complementing pair of trees $(T, \tilde{T})$ such that $A = \mathrm{p}[T]$ $(= \omega^\omega \setminus \mathrm{p}[\tilde{T}])$.

Example

- Every analytic set of reals is universally Baire.
- Every coanalytic set of reals is universally Baire.
- Universally Baire sets of reals have some regularity properties, but not necessarily PSP.

Question

What is the consistency strength of ZFC + "every universally Baire set of reals has the perfect set property"?

Upper bound

If there is a Woodin cardinal, then by results of Woodin (or Neeman) every universally Baire set of reals has PSP, using

- the stationary tower,
- weak homogeneity and the unfolded perfect set game,
- Neeman's genericity iterations, or
- determinacy and the perfect set game.

Lower bound

If every universally Baire set of reals has PSP, then every coanalytic set of reals has PSP, so ω_1^V is inaccessible in L.

Main theorem (Schindler-W.)

The following statements are equiconsistent modulo ZFC.

1. There is a cardinal that is "Shelah for remarkability."
2. Every universally Baire set of reals has PSP.
3. Every set of reals in $L(\mathbb{R}, \mathrm{uB})$ has PSP, where uB is the collection of all universally Baire sets of reals.

Remark

- If (1) is witnessed by κ, then (3) holds in $V^{\mathrm{Col}(\omega,<\kappa)}$.
- If (3) holds, then (2) holds.
- If (2) holds, then (1) holds in L, witnessed by ω_1^V.

Before defining "Shelah for remarkability," we review some familiar large cardinals.

Some large cardinals are critical points of embeddings from V into models containing prescribed rank initial segments of V:

Definition

A cardinal κ is strong if for every $\lambda > \kappa$ there is a transitive class M and an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$ and $V_\lambda \subset M$.

Instead of ordinals $\lambda > \kappa$, we can use functions $f : \kappa \to \kappa$:

Definition

A cardinal κ is Shelah if for every function $f : \kappa \to \kappa$ there is a transitive class M and an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $V_{j(f)(\kappa)} \subset M$.

Some stronger large cardinal axioms require the codomain M to have a prescribed degree of closure under sequences:

Definition

A cardinal κ is supercompact if for every $\lambda > \kappa$ there is a transitive class M and an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \lambda$ and $M^\lambda \subset M$.

Next we consider a "Shelah-ized" version of supercompactness (or is it a "supercompactified" version of Shelah-ness?)

Definition (N. Perlmutter)

A cardinal κ is Shelah for supercompactness if for every function $f : \kappa \to \kappa$ there is a transitive class M and an e.e. $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $M^{j(f)(\kappa)} \subset M$.

It is convenient to reformulate these axioms to make κ the *image* of the critical point instead of the critical point:

Theorem (Magidor)

A cardinal κ is supercompact if and only if for every $\lambda > \kappa$ there is a $\bar{\lambda} < \kappa$ and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ such that $j(\mathrm{crit}(j)) = \kappa$.

Remark

A cardinal κ is Shelah for supercompactness if and only if
for every function $f : \kappa \to \kappa$
there are $\lambda > \kappa$ and $\bar{\lambda} < \kappa$
and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$
such that $j(\mathrm{crit}(j)) = \kappa$ and $\bar{\lambda} \geq f(\mathrm{crit}(j))$ and $f \in \mathrm{ran}(j)$.

Large cardinals can be weakened to "virtual" large cardinals having elementary embeddings in generic extensions:

Recall

A cardinal κ is supercompact if and only if for every $\lambda > \kappa$ there is a $\bar{\lambda} < \kappa$ and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ such that $j(\mathrm{crit}(j)) = \kappa$.

Definition (Schindler)

A cardinal κ is remarkable if for every $\lambda > \kappa$ there is a $\bar{\lambda} < \kappa$ and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ in $V^{\mathrm{Col}(\omega, V_{\bar{\lambda}})}$ such that $j(\mathrm{crit}(j)) = \kappa$.

Similarly, we can virtualize "Shelah for supercompactness."

Recall

A cardinal κ is Shelah for supercompactness if and only if
for every function $f : \kappa \to \kappa$
there are $\lambda > \kappa$ and $\bar{\lambda} < \kappa$
and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$
such that $j(\mathrm{crit}(j)) = \kappa$ and $\bar{\lambda} \geq f(\mathrm{crit}(j))$ and $f \in \mathrm{ran}(j)$.

Definition (Schindler–W.)

A cardinal κ is Shelah for remarkability if
for every function $f : \kappa \to \kappa$
there are $\lambda > \kappa$ and $\bar{\lambda} < \kappa$
and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ in $V^{\mathrm{Col}(\omega, V_{\bar{\lambda}})}$
such that $j(\mathrm{crit}(j)) = \kappa$ and $\bar{\lambda} \geq f(\mathrm{crit}(j))$ and $f \in \mathrm{ran}(j)$.

The virtual forms of the axioms are weaker:

Theorem

- If κ is strong, then κ is remarkable, i.e. virtually supercompact. (Schindler)
- If κ is Shelah, then κ is Shelah for remarkability, i.e. virtually Shelah for supercompactness.

Proof idea

For $j : V \to M$, even if M does not contain $j \upharpoonright V_\lambda$ for the relevant λ because $M^{V_\lambda} \not\subset M$, in $M^{\mathrm{Col}(\omega, V_\lambda)}$ there is an elementary embedding with the same domain and range as $j \upharpoonright V_\lambda$ by an absoluteness argument. Reflect this downward.

In terms of consistency strength, these virtual large cardinal properties are even weaker than measurability.

Theorem

If $j : V \to M$ witnesses measurability of κ, then

- κ is remarkable in $j(V_\kappa)$, and
- κ is Shelah for remarkability in $j(V_\kappa)$.

Proof idea

In $j(j(V_\kappa))$, show that $j(\kappa)$ is remarkable and Shelah for remarkability using embeddings that resemble restrictions of j.

In fact, their consistency strengths are weaker than $0^\sharp$.
Moreover, they are downward absolute to L.

Theorem (Schindler)

- If κ is remarkable, then κ is remarkable in L.
- If $0^\sharp$ exists, every Silver indiscernible is remarkable in L.

Remark

By a similar argument,

- If κ is Shelah for remarkability, then κ is Shelah for remarkability in L.
- If $0^\sharp$ exists, every Silver indiscernible is Shelah for remarkability in L.

We now outline the proof of the main theorem.

Forcing direction

If κ is Shelah for remarkability and $G \subset \mathrm{Col}(\omega, {<}\kappa)$ is V-generic, then every set of reals in $L(\mathbb{R}, \mathrm{uB})^{V[G]}$ has PSP.

Review of definition

A cardinal κ is *Shelah for remarkability* if
for every function $f : \kappa \to \kappa$
there are $\lambda > \kappa$ and $\bar{\lambda} < \kappa$
and an elementary embedding $j : V_{\bar{\lambda}} \to V_\lambda$ in $V^{\mathrm{Col}(\omega, V_{\bar{\lambda}})}$
such that $j(\mathrm{crit}(j)) = \kappa$ and $\bar{\lambda} \geq f(\mathrm{crit}(j))$ and $f \in \mathrm{ran}(j)$.

Reflection lemma

Let κ be Shelah for remarkability and let $G \subset \mathrm{Col}(\omega, {<}\kappa)$ be a V-generic filter. Let $A \in \mathrm{uB}^{V[G]}$. Then there is $\alpha < \kappa$ such that for all $\beta < \kappa$ there are $\mathrm{Col}(\omega, {<}\beta)$-absolutely complementing trees $T_\beta, \tilde{T}_\beta \in V[G \restriction \alpha]$ such that $\mathrm{p}[T_\beta]^{V[G \restriction \beta]} = A \cap V[G \restriction \beta]$.

Proof idea

If not, then for all $\alpha < \kappa$ there is $\beta < \kappa$ for which no trees exist as above. Define $f(\alpha) = \beta + \omega$ where β is least such. Take j witnessing the fact that κ is Shelah for remarkability with respect to f. Let $\alpha = \mathrm{crit}(j)$. Then $j(\alpha) = \kappa$ and $\mathrm{dom}(j)$ contains some β for which no trees exist as above. Reflect $\mathrm{Col}(\omega, {<}j(\beta))$-absolutely complementing trees for A downward to get trees $T_\beta, \tilde{T}_\beta \in V[G \restriction \alpha]$ as above, a contradiction.

The pairs of trees $(T_\beta, \tilde{T}_\beta)$ for various $\beta < \kappa$ all agree, and we can amalgamate them into a single pair $(T, \tilde{T})$:

Reflection lemma (improved)

Let κ be Shelah for remarkability and let $G \subset \mathrm{Col}(\omega, {<}\kappa)$ be a V-generic filter. Let $A \in \mathrm{uB}^{V[G]}$. Then there is $\alpha < \kappa$ and $\mathrm{Col}(\omega, {<}\kappa)$-absolutely complementing trees $T, \tilde{T} \in V[G \restriction \alpha]$ with $\mathrm{p}[T]^{V[G]} = A$.

Proof of forcing direction

Let κ be Shelah for remarkability and let $G \subset \mathrm{Col}(\omega, {<}\kappa)$ be a V-generic filter. Let $B \in L(\mathbb{R}, \mathrm{uB})^{V[G]}$. Then B is OD from a set of reals $A \in \mathrm{uB}^{V[G]}$. By the lemma, A is definable from an element of $V[G \restriction \alpha]$, so B is definable from an element of $V[G \restriction \alpha]$. By Solovay, B has the perfect set property.

Inner model direction

If every universally Baire set of reals has the perfect set property, then ω_1^V is Shelah for remarkability in L.

Special case

Assume $\kappa = \omega_1^V$ is not inaccessible in L. Then $\kappa = \omega_1^{L[z]}$ for some real z. For each $\alpha < \kappa$, take $x_\alpha \in \mathbb{R}$ coding the least level of $L[z]$ containing α and projecting to ω. Then the set $\{x_\alpha : \alpha < \kappa\}$ is coanalytic but has no perfect subset.

Remaining case

Assume $\kappa = \omega_1^V$ is inaccessible in L but $f : \kappa \to \kappa$ witnesses that κ is not Shelah for remarkability in L. Use AC to choose, for each $\alpha < \kappa$, some $x_\alpha \in \mathbb{R}$ coding the least level L_β of L containing $f(\alpha)$ such that $L_\beta = (V_\beta)^L$. Then one can show that the set $\{x_\alpha : \alpha < \kappa\}$ is uB but has no perfect subset.

Other regularity properties for sets in $L(\mathbb{R}, \mathrm{uB})$:

Theorem (Schindler-W.)

1. If κ is Shelah for remarkability and $G \subset \mathrm{Col}(\omega, {<}\kappa)$ is V-generic, then every set of reals in $L(\mathbb{R}, \mathrm{uB})^{V[G]}$ is Lebesgue measurable and has the property of Baire.
2. If every set of reals in $L(\mathbb{R}, \mathrm{uB})$ is Lebesgue measurable, then ω_1^V is Shelah for remarkability in L.

(1) follows from the proof of PSP. For (2), if not, then there is a uB set without PSP. From this, one can show that $\omega_1 \leq \mathbb{R}$ in $L(\mathbb{R}, \mathrm{uB})$, which implies by Shelah that some set of reals in $L(\mathbb{R}, \mathrm{uB})$ is not Lebesgue measurable.

Question

What is the consistency strength of ZFC + "every set of reals in $L(\mathbb{R}, \mathrm{uB})$ has the property of Baire"?

Determinacy for uB sets:

Theorem

The following statements are equiconsistent modulo ZFC.

1. There is a Woodin cardinal.
2. Every universally Baire set of reals is determined.
3. Every Δ^1_2 set of reals is determined.
4. Every ordinal-definable set of reals is determined.

Proof.

(1) $\implies$ (2) by Neeman. If (2) holds, then we have winning strategies for all uB Δ^1_2 sets, and we can force to make all the non-uB Δ^1_2 sets no longer Δ^1_2. Winningness of strategies is preserved, so the generic extension satisfies (3).
Con(3) $\implies$ Con(4) by Kechris and Solovay.
Con(4) $\implies$ Con(1) by Woodin. $\square$