Splitting Numbers

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Overview

Estimates in ZFC

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Increasing the splitting number by forcing

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The splitting number at regular uncountable cardinals

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The classical splitting number

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(1) If $\mathscr{S}$ does not split X and $X' \subseteq^* X$, then $\mathscr{S}$ does not split X'.

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We state two very easy properties:
(1) If $\mathscr{S}$ does not split X and $X' \subseteq^* X$, then $\mathscr{S}$ does not split X'.
(2) If $\mathscr{S}'$ is a set of infinite sets of natural numbers and $|\mathscr{S}'| < \mathfrak{s}$, then given any infinite set X, we find an $X' \subseteq X$ that is not split by $\mathscr{S}'$.

Proposition

$\mathfrak{s} \leq \mathrm{unif}(\mathcal{M}), \mathrm{unif}(\mathcal{N}), \mathfrak{d}$.

We recall the definitions:

Definition

If $\mathcal{I} \subseteq \omega^\omega$ is an ideal, we let $\mathrm{unif}(\mathcal{I})$ be the smallest size of a set of reals that is not in $\mathcal{I}$. We apply this to the ideal of $\mathcal{M}$ of meager sets and the ideal $\mathcal{N}$ of Lebesgue null sets.

Proof: Suppose $\mathscr{S}$ is not a splitting family and this is witnessed by X. Then

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For each k and each infinite set Y, the set $\{\chi_A \in 2^\omega \mid A \cup [0, k) \supseteq Y\} \subseteq 2^\omega$ and the set $\{\chi_A \mid A \cap [k, \infty) \subseteq Y\} \subseteq 2^\omega$ are both nowhere dense in 2^ω and both have measure 0.

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For each increasing function $f \in \omega^\omega$ we define an interval partition $\mathcal{I}_f = \{I_n \mid n \in \omega\}$ such that for any $k \in I_n$, $f(k) \in I_n \cup I_{n+1}$. Then we let $S_f = \{I_{4n} \cup I_{4n+1} \mid n \in \omega\}$. We let $\mathrm{next}_X(n) = \min(X \cap [n,\infty))$. Now if $\mathrm{next}_X \leq^* f$ then S_f splits X. We let $\mathscr{S} = \{S_f \mid f \in \mathscr{D}\}$, where $\mathscr{D}$ is a $\leq^*$-dominating family.

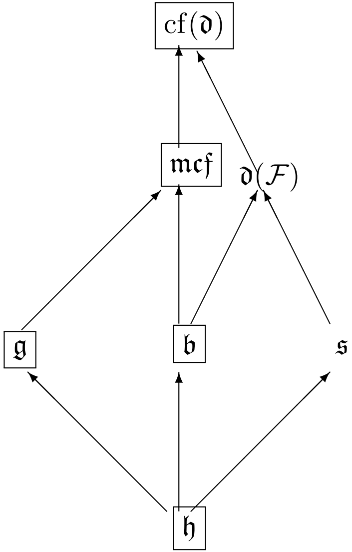
Definition

A family $\mathscr{D} \subseteq [\omega]^\omega$ is called dense iff for any $X \in [\omega]^\omega$ there is an $D \in \mathscr{D}$ such that $D \subseteq^* X$.

A family $\mathscr{D} \subseteq [\omega]^\omega$ is called open, iff it is closed under almost subsets.

$\mathfrak{h}$, the distributivity number, is the smallest size κ of a family $\{\mathscr{D}_i \mid i < \kappa\}$ of open dense sets whose intersection is empty.

A Hasse diagramme



A lower bound

Proposition

$\mathrm{cf}(\mathfrak{s}) \geq \mathfrak{h}$.

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Proof: We assume for a contradiction that $\mathrm{cf}(\mathfrak{s}) < \mathfrak{h}$. Then there is a splitting family $\mathscr{S} = \bigcup\{\mathscr{S}_\alpha \mid \alpha < \mathrm{cf}(\mathfrak{s})\}$ such that $|\mathscr{S}_\alpha| < \mathfrak{s}$. We let $\mathscr{D}_\alpha$ be the set of sets that are not split by any member of $\mathscr{S}_\alpha$. Using (1), (2), we see that $\mathscr{D}_\alpha$ is open and dense. Since $\mathrm{cf}(\mathfrak{s}) < \mathfrak{h}$, there is an infinite set $X \in \bigcap\{\mathscr{D}_\alpha \mid \alpha < \mathrm{cf}(\mathfrak{s})\}$. The infinite set X is not split by any element of $\mathscr{S}$. Contradiction.

A sharper upper bound

Theorem (M.)

$\mathfrak{s} \leq \mathrm{cf}(\mathfrak{d})$.

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Lemma (Blass, M.)

Suppose that the filter $\mathcal{F}$ is not nearly ultra, and let ω be partitioned into finite intervals I_n. Then there are sets $D, D' \subseteq \omega$ with the properties:

1. *Every set in $\mathcal{F}$ intersects both D and D'.*
2. *Each of D and D' is a union of intervals I_n.*
3. *If $I_n \subseteq D$ then I_n and its neighbours $I_{n\pm1}$ are disjoint from D'.*

Proof on the blackboard.

Reduced products $(\omega, <)^\omega/\mathcal{F}$

Definition

Let $\mathcal{F}$ be a non-principal filter over ω. Then we have the directed partial order $(\omega^\omega/\mathcal{F}, \leq_\mathcal{F})$ by letting

1. $[f]_\mathcal{F} = \{g \mid \{n \mid g(n) = f(n)\} \in \mathcal{F}\}$,
2. $\omega^\omega/\mathcal{F} = \{[f]_\mathcal{F} \mid f \in \omega^\omega\}$
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Definition

We let $\mathfrak{d}(\mathcal{F})$ be the smallest size of a $\leq_\mathcal{F}$-dominating set (this need not be a dominating chain). It is also called the cofinality.

Remarks on $\mathfrak{d}(\mathcal{F})$

However, $\mathfrak{d}(\mathcal{F})$ is not the true cofinality of $\leq_{\mathcal{F}}$ in the sense of pcf theory.

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- $\mathfrak{d}(\mathcal{F} \cap \mathcal{F}') = \max(\mathfrak{d}(\mathcal{F}), \mathfrak{d}(\mathcal{F}'))$.
- If $\mathfrak{u} < \mathfrak{d}$, then $\mathfrak{d}$ is regular.

An abstract upper bound

Theorem (Blass, M.)

If $\mathcal{F}$ is not nearly ultra then $\mathfrak{s} \leq \mathfrak{d}(\mathcal{F})$.

Proof on the blackboard.

There is such a filter

Theorem (M.)

There is a filter that is not nearly ultra that has $\mathfrak{d}(\mathcal{F}) = \mathrm{cf}(\mathfrak{d})$.

Proof on the blackboard.

Corollary

$\mathfrak{s} \leq \mathrm{cf}(\mathfrak{d})$.

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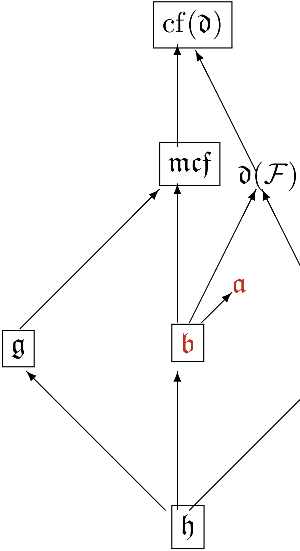
Focus on $\mathfrak{b}$ and on $\mathfrak{s}$



A matrix forcing

We remark that $\aleph_1 = \mathfrak{s} < \mathfrak{b}$ is consistent: Blow up the continuum by many Hechler reals and then add $\aleph_1$ random reals. The latter form a small splitting family.

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Also $\mathfrak{b} = \aleph_1 < \mathfrak{s}$ is relatively easy: The matrix of Blass and Shelah can be used. For larger $\mathfrak{b}$ we have to take extra care that there are no small unbounded families.
For arbitrary regular values now explain Blass and Shelah's work and a part of Brendle's and Fischer's work, focussing on $\mathfrak{b} = \kappa << \mathfrak{s} = \lambda$ for two regular uncountable cardinals (and leaving out the part on mad families).

Diagonalising many filters

First some heuristics: Suppose $|\mathscr{S}| < \kappa$ and we want to show that $\mathscr{S}$ is not a splitting family. We have to show that there is an X such that for any $S \in \mathscr{S}$, $X \subseteq^* S$ or $X \subseteq^* S^c$. So any X that diagonalises a filter $\mathcal{F}$ that contains for each $S \in \mathscr{S}$, S or its complement, would serve as such an X. So a strategy is to show that any small family $\mathscr{S}$ has an filter $\mathcal{F}$ containing for each S, S or S^c and that $\mathcal{F}$ lies "early" in the construction so that at a later time $\mathcal{F}$ is diagonalised by forcing. If we do this in a linear iteration, most likely we will end with a model of $\mathfrak{p} = \mathfrak{c} = \mathfrak{s} = \mathfrak{b} = \mathfrak{d} = \mathfrak{u}$.

Definition

(Mathias forcing) Let $\mathcal{F}$ be a non-principal filter. Conditions in $\mathbb{M}_{\mathcal{F}}$ are pairs (s, A), s a finite subset of ω and $A \in \mathcal{F}$ and $\max(s) < \min(A)$. $(t, B) \leq (s, A)$ iff
$t \supseteq s$ and
$t \setminus s \subseteq A$ and
$B \subseteq A$.
We say t is permitted by (s, A) iff $s \subseteq t \subseteq s \cup A$.

Complete suborders

Definition

We write $\mathbb{P} \subseteq \mathbb{Q}$ iff $\mathbb{P} \subseteq \mathbb{Q}$, $p \leq p'$ implies $q \leq q'$.
Note that this implies that $p \perp_{\mathbb{Q}} p'$ implies $p \perp_{\mathbb{P}} p'$ for any $p, p' \in \mathbb{P}$.

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We say $\mathbb{P}$ is a complete suborder of $\mathbb{Q}$ (short $\mathbb{P} \lessdot \mathbb{Q}$) iff $\mathbb{P} \subseteq_{ic} \mathbb{Q}$ and every maximal antichain of $\mathbb{P}$ is maximal in $\mathbb{Q}$.

$\mathbb{P} \lessdot \mathbb{Q}$ iff there is a reduction function from $\mathbb{Q}$ to $\mathbb{P}$ such that for each $q \in \mathbb{Q}$, $p = \mathrm{red}_{\mathbb{Q},\mathbb{P}}(q)$ is a reduction of q (with respect to $\mathbb{P}$, $\mathbb{Q}$) iff

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A pair of models with a witness g for unboundedness in der upper model

Definition

Let $M \subseteq N$ be models of set theory and $g \in N \cap \omega^\omega$ is such that for all $f \in M \cap \omega^\omega$, $N \models g \not\leq^* f$, we say that $(\star M, N, g)$ holds.

Lemma (Blass and Shelah)

Let $M \subseteq N$ be models of set theory and $g \in \omega^\omega \cap N$ such that $(\star M, N, g)$. In addition, let $\mathcal{U}$ be an ultrafilter in M. Then there is an ultrafilter $\mathcal{V} \supseteq \mathcal{U}$ in N such that

(1) every maximal antichain of $\mathbb{M}_\mathcal{U}$ which belongs to M is a maximal antichain of $\mathbb{M}_\mathcal{V}$ in N, we write $\mathbb{M}_\mathcal{U} \leq_M \mathbb{M}_\mathcal{V}$,

(2) $(\star M[G], N[G], g)$ holds for any $\mathbb{M}_\mathcal{V}$-generic G over N.

What can go wrong in the choice of $\mathcal{V}$?

We say r is permitted by (s, X) if $s \subseteq r \subseteq s \cup X$.

A violation of $\mathbb{M}_{\mathcal{U}} \leq_M \mathbb{M}_{\mathcal{V}}$: A maximal antichain C of $\mathbb{M}_{\mathcal{U}}$ and a condition $(t, A) \in \mathbb{M}_{\mathcal{V}}$ such that for any $c \in C$, no finite set is permitted by (t, A) and c. We say A is forbidden by t and C.

A violation of $(\star M[G], N[G], g)$: An $\mathbb{M}_{\mathcal{U}}$-name $\underset{\sim}{f} = \langle (W_n, f_n) \mid n \in \omega \rangle$ (meaning: $p \in W_n$ forces $\underset{\sim}{f}(n) = f_n(p)$) and a condition $(t, B) \in \mathbb{M}_{\mathcal{V}}$ such that for any $n \in \omega$, for any $p \in W_n$, if $f_n(p) < g(n)$, then no finite set is permitted by (t, B) and p. We say B is forbidden by t and $\underset{\sim}{f}$.

A compactness argument

Show that no $Z \in \mathcal{U}$ is covered by forbidden sets A_i, B_i, $i < k$, with witnesses a_i, C_i and b_i, $\underset{\sim}{f}$.

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Claim

(to the lemma) For every $n \in \omega$ there exists $h(n) \in \omega$ such that $h(n) > n$ and whenever the interval $Z \cap [n, h(n))$ of Z is partitioned into $2k$ pieces then at least one of the pieces P has both of the following properties:

(i) For each $i < k$ there is a finite $e \subseteq P$ such that $a_i \cup e$ is permitted by C_i,

(ii) For each $i < k$ there is a finite set $e \subseteq P$ such that $b_i \cup e$ is permitted by some $p \in W_n$ such that $f_n(p) \leq g(n)$.

Proof on the blackboard. Then back to the proof of the lemma.

More harmless forcings for $(\star M, N, g)$

Lemma (Brendle and Fischer)

Let $M \subseteq N$ be models of set theory $\mathbb{P} \in M$ be a poset that that $\mathbb{P} \subseteq M$ and let G be a $\mathbb{P}$-generic filter over N (and hence over M). If $g \in N$ is such that $(\star M, N, g)$ holds then $(\star M[G], N[G], g)$ holds.

Instructive to take $\mathbb{P} = \mathbb{D}^M$, Hechler forcing in M.

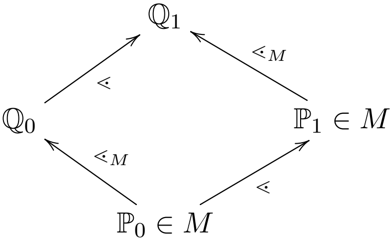
Direct limits along rows and columns

Lemma

Let $\langle \mathbb{P}_{\ell,\eta}, \mathbb{Q}_{\ell,\eta} \mid \eta < \xi \rangle$, $\ell = 0, 1$ *be finite support iterations such that* $\mathbb{P}_{0,\eta}$ *is a complete suborder of* $\mathbb{P}_{1,\eta}$ *for all* $\eta < \xi$*. Then* $\mathbb{P}_{0,\xi}$ *is a complete suborder of* $\mathbb{P}_{1,\xi}$*.*

This is an instance of correctness preserving. Let us introduce a basic rectangle (or lozenge) and recall the notion of correctness (Brendle, Mejía):

Correct diagrammes



Definition

For $i = 0, 1$ let $\mathbb{P}_i$ and $\mathbb{Q}_i$ be posets When $\mathbb{P}_i \lessdot \mathbb{Q}_i$ for $i = 0, 1$ and $\mathbb{P}_0 \leq \mathbb{P}_1$ and $\mathbb{Q}_0 \lessdot \mathbb{Q}_1$ we say that the diagramme $\langle \mathbb{P}_0, \mathbb{P}_1, \mathbb{Q}_0, \mathbb{Q}_1 \rangle$ is correct if for each $q \in \mathbb{Q}_0$ and $p_1 \in \mathbb{P}_1$ if both have a common reduction in $\mathbb{P}_0$ then they are compatible in $\mathbb{Q}_1$.
An equivalent formulation is: Whenever $p_0 \in \mathbb{P}_0$ is a reduction of $p_1 \in \mathbb{P}_1$ in the $\mathbb{P}_0$, $\mathbb{P}_1$-sense, then p_0 is a reduction of p_1 w.r.t. $\mathbb{Q}_0$, $\mathbb{Q}_1$.

A successor step in a pattern of correct rectangles

Lemma (Brendle and Fischer)

Let $\mathbb{P}$, $\mathbb{Q}$ be partial orders such that $\mathbb{P}$ is completely embedded into $\mathbb{Q}$. Let $\underset{\sim}{\mathbb{A}}$ be a $\mathbb{P}$-name of a forcing notion, $\underset{\sim}{\mathbb{B}}$ be a $\mathbb{Q}$-name for a forcing notion such that $\mathbb{Q} \Vdash \underset{\sim}{\mathbb{A}} \subseteq_{ic} \underset{\sim}{\mathbb{B}}$ and every maximal antichain of $\underset{\sim}{\mathbb{A}}$ in $V^{\mathbb{P}}$ is a maximal antichain of $\underset{\sim}{\mathbb{B}}$ in $V^{\mathbb{Q}}$, i.e. $\mathbb{Q} \Vdash \underset{\sim}{\mathbb{A}} \lessdot_{V^{\mathbb{P}}} \underset{\sim}{\mathbb{B}}$. Then $\mathbb{P} * \underset{\sim}{\mathbb{A}} \lessdot \mathbb{Q} * \underset{\sim}{\mathbb{B}}$ and $\langle \mathbb{P}, \mathbb{P} * \underset{\sim}{\mathbb{A}}, \mathbb{Q}, \mathbb{Q} * \underset{\sim}{\mathbb{B}} \rangle$ is a correct diagramme.

Definition (Blass and Shelah)

A matrix iteration of ccc posets is given by $\langle \mathbb{P}_{\alpha,\xi}, \underset{\sim}{\mathbb{Q}}_{\alpha,\xi} \mid \xi < (\leq)\lambda, \alpha \leq \kappa \rangle$ with the following conditions.

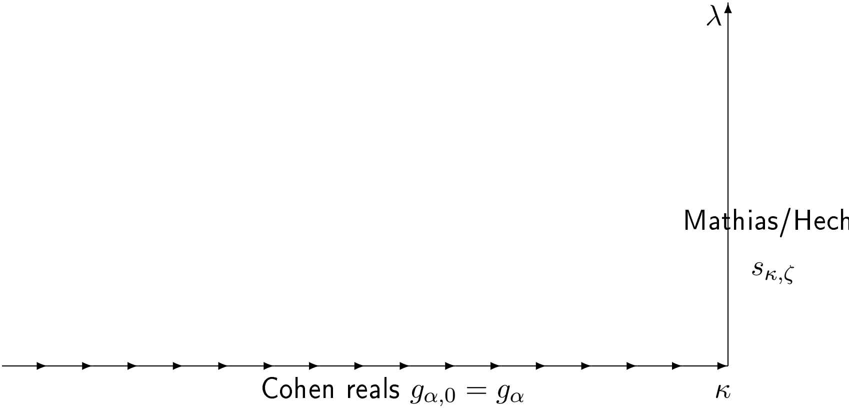
(1) The ground row (ξ-coordinate is 0): $\mathbb{P}_{\kappa,0} = \mathrm{fslimit}\langle \mathbb{P}_{\alpha,0}, \underset{\sim}{\mathbb{R}}_\alpha \mid \alpha < \lambda \rangle$, and the sequence is a finite support iteration,

(2) The α-th solid column for $\alpha \leq \kappa$: $\mathbb{P}_{\alpha,\lambda} = \mathrm{fslimit}\langle \mathbb{P}_{\alpha,\xi}, \underset{\sim}{\mathbb{Q}}_{\alpha,\xi} \mid \xi < \lambda \rangle$, and the sequence is a finite support iteration, (solid column means: spans from 0 to α in the horizontal direction in the picture).

(3) Each rectangle of height 1 is correct: For all $\xi < \lambda$ and $\alpha < \beta \leq \kappa$ $\mathbb{P}_{\beta,\xi} \Vdash, \underset{\sim}{\mathbb{Q}}_{\alpha,\xi} \lessdot_{V^{\mathbb{P}_{\alpha,\xi}}} \underset{\sim}{\mathbb{Q}}_{\beta,\xi}$,

(4) For each $\xi < \lambda$, for each limit $\beta \leq \kappa$, $\mathbb{P}_{\beta,\xi}$ is the direct limit of $\mathbb{P}_{\beta',\xi}$, $\beta' < \beta$.

An iteration with a rectangular structure



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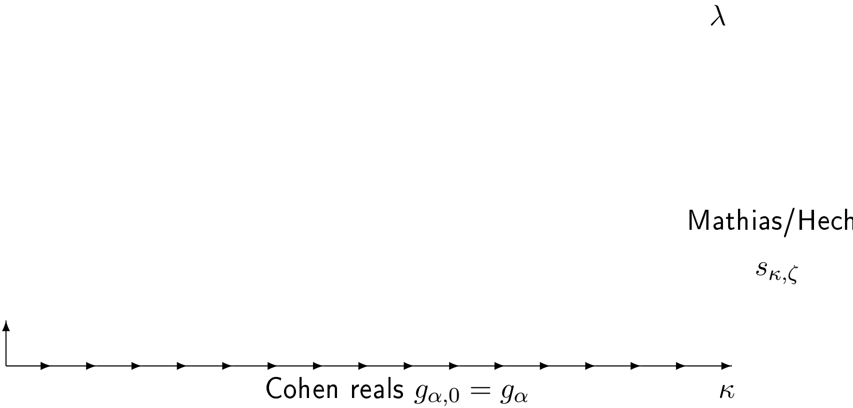

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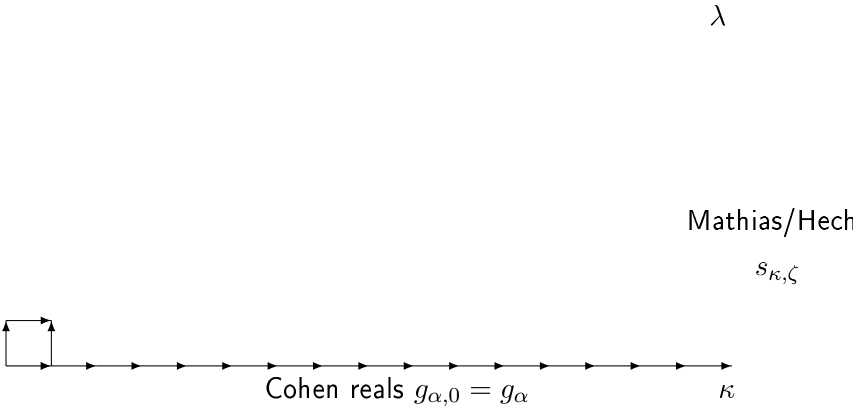

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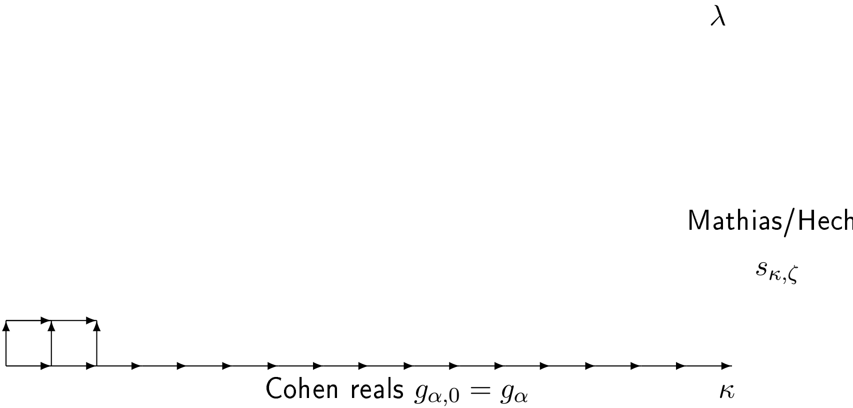
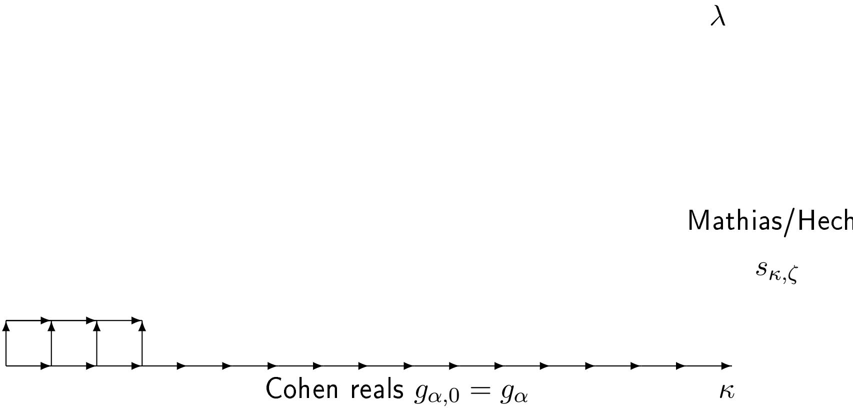
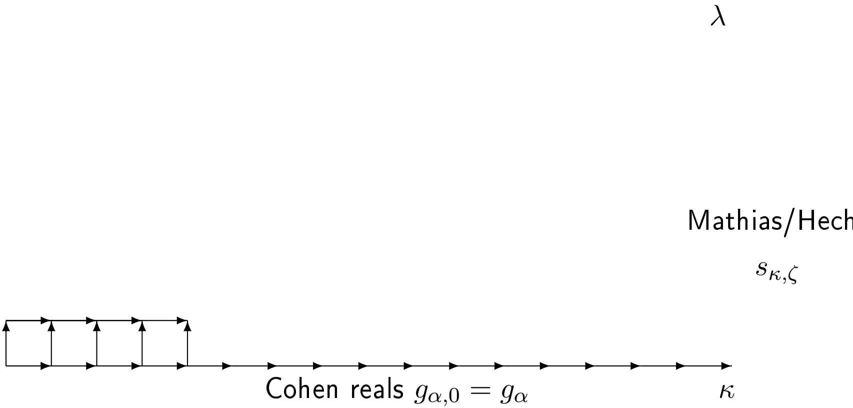


An iteration with a rectangular structure



An iteration with a rectangular structure



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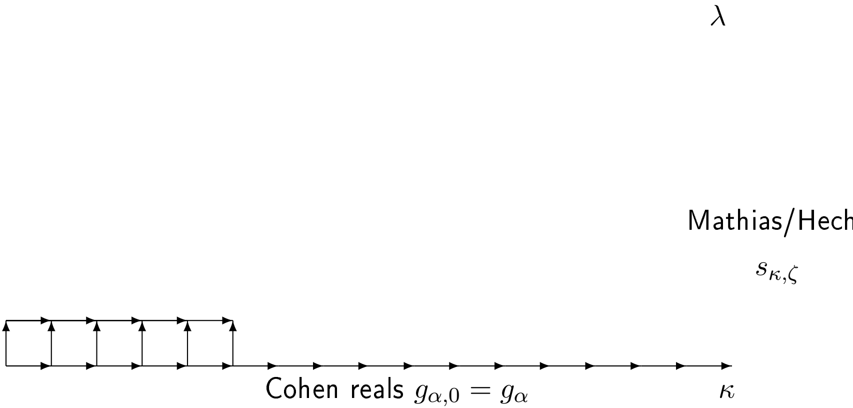

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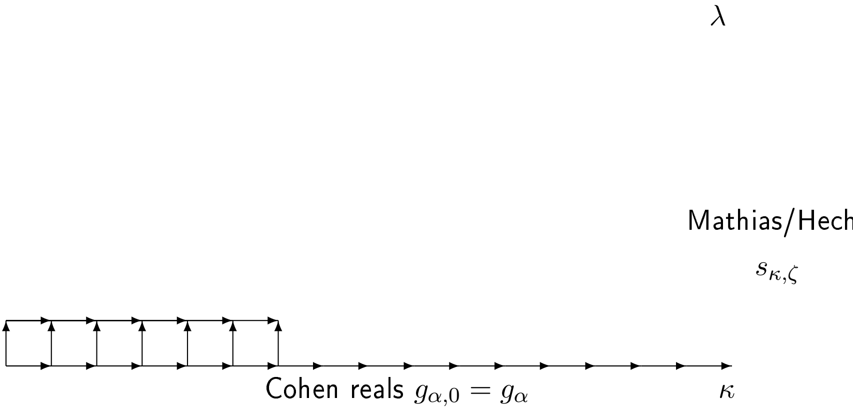

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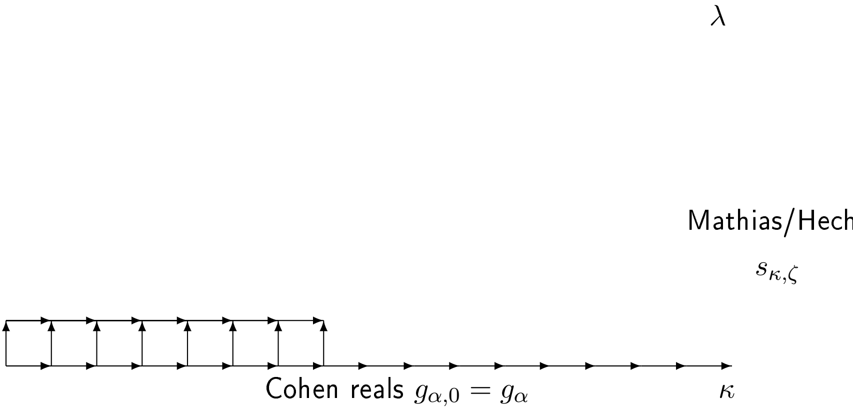

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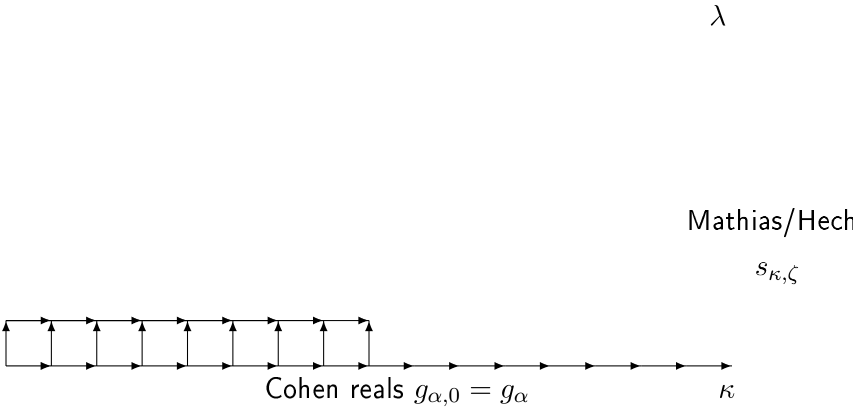

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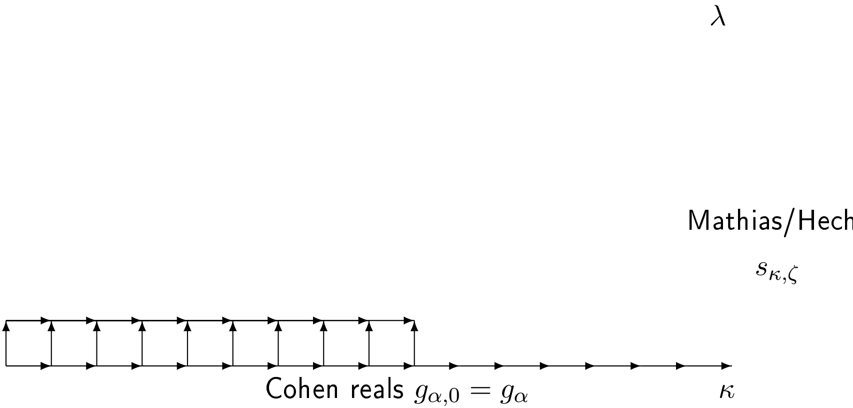
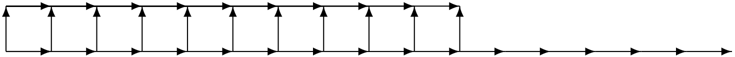


An iteration with a rectangular structure

λ

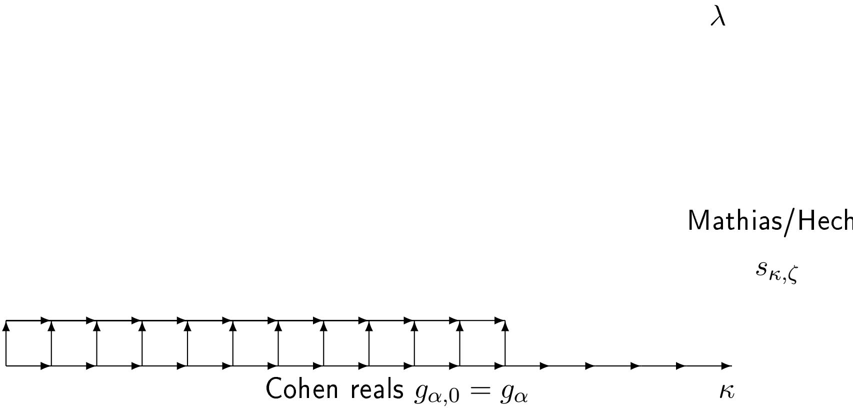
Mathias/Hech

$s_{\kappa,\zeta}$

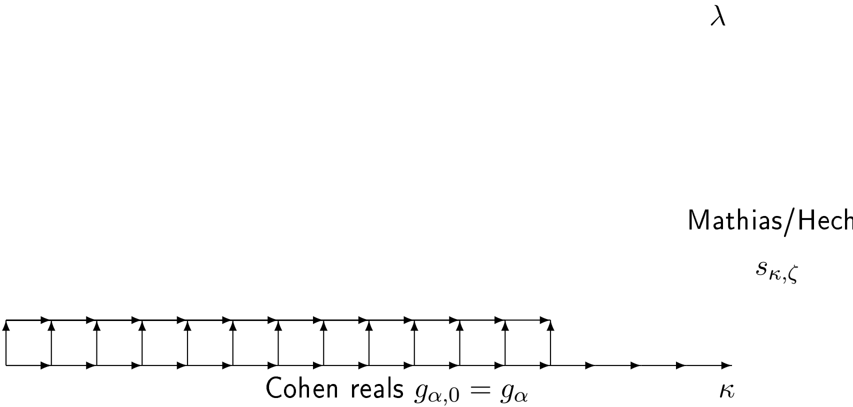


Cohen reals $g_{\alpha,0} = g_\alpha$ κ

An iteration with a rectangular structure



An iteration with a rectangular structure



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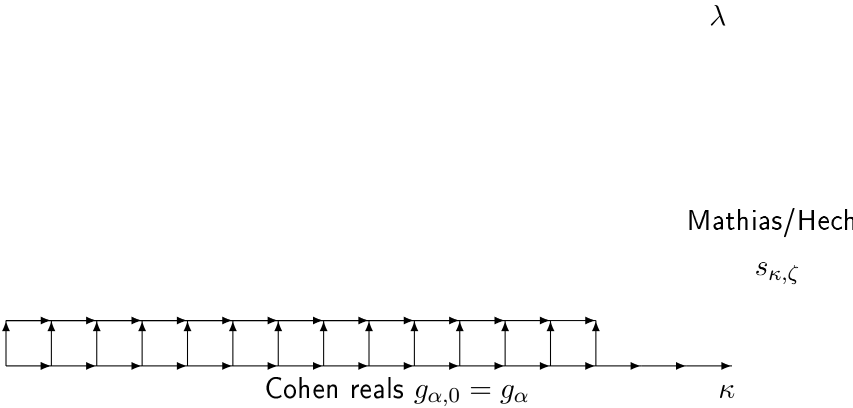

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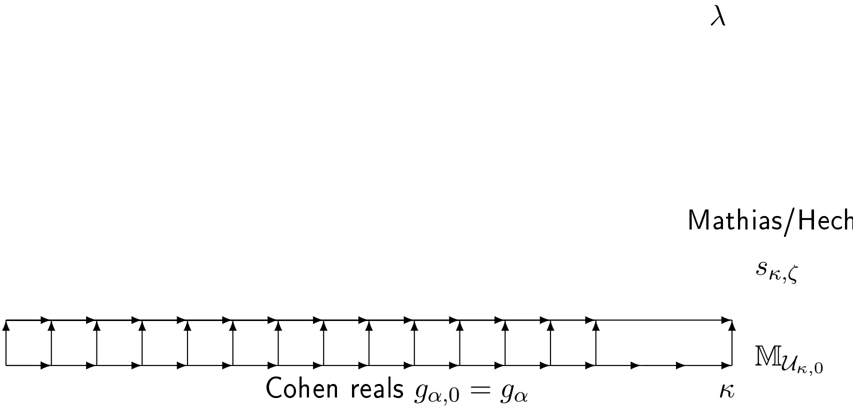
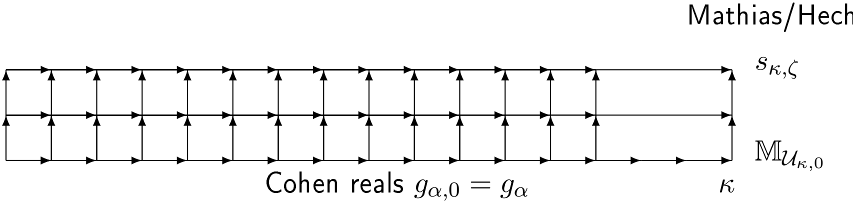
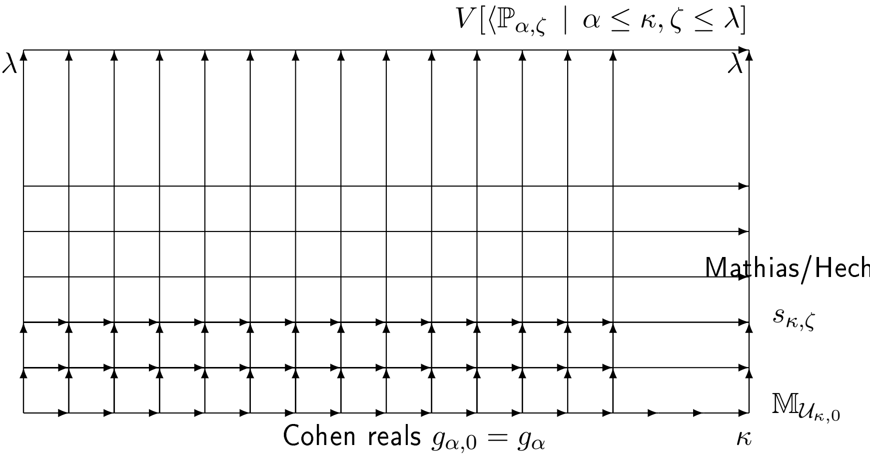


An iteration with a rectangular structure

λ



An iteration with a rectangular structure



Upwards limits and limits to the right

Consequence:
Any lower part limit solid column $\mathbb{P}_{\alpha,\lambda'}$, $\alpha \leq \kappa$ limit, $\lambda' \leq \lambda$ is a direct limit of the lower parts of the solid columns $\mathbb{P}_{\alpha',\lambda'}$, $\alpha' < \alpha$.
and turned around.

Upwards limits and preservation of the unboundedness of g

Lemma (Brendle and Fischer)

Let $\langle \mathbb{P}_{\ell,\eta}, \mathbb{Q}_{\ell,\eta} \mid \eta < \xi \rangle$, $\ell = 0, 1$ *be finite support iterations such that* $\mathbb{P}_{0,\eta}$ *is a complete suborder of* $\mathbb{Q}_{\ell,\eta}$ *for all* $\eta < \xi$. *Let* ξ *be a limit ordinal. If* $g \in V^{\mathbb{P}_{1,0}} \cap \omega^\omega$ *and* $(\star V^{\mathbb{P}_{0,\eta}}, V^{\mathbb{P}_{1,\eta}}, g)$ *holds for all* $\eta < \xi$ *then* $(\star V^{\mathbb{P}_{0,\xi}}, V^{\mathbb{P}_{1,\xi}}, g)$.

An upwards limit, a diagramme

$$
\begin{array}{ccc}
\mathbb{P}_{0,\eta} & \xrightarrow[\lessdot]{\hspace{6em}} & g, \mathbb{P}_{1,\eta} \\
\vdots & & \vdots \\
\uparrow{\scriptstyle \lessdot} & & \uparrow{\scriptstyle \lessdot} \\
\mathbb{P}_{0,\xi+2} = \mathbb{P}_{1,\xi} * \mathbb{D}^{V^{\mathbb{P}_{0,\xi+1}}} & \xrightarrow[\lessdot]{} & g, \mathbb{P}_{1,\xi+2} = \mathbb{P}_{1,\xi} * \mathbb{D}^{V^{\mathbb{P}_{0,\xi+1}}} \\
\uparrow{\scriptstyle \lessdot} & & \uparrow{\scriptstyle \lessdot} \\
\mathbb{P}_{0,\xi+1} = \mathbb{P}_{0,\xi} * \mathbb{M}_{\mathcal{U}_{0,\xi}} & \xrightarrow[\lessdot]{} & g, \mathbb{P}_{1,\xi+1} = \mathbb{P}_{1,\xi} * \mathbb{M}_{\mathcal{U}_{1,\xi}} \\
\uparrow{\scriptstyle \lessdot} & & \uparrow{\scriptstyle \lessdot} \\
\mathbb{P}_{0,\xi} & \xrightarrow[\lessdot]{\hspace{6em}} & g, \mathbb{P}_{1,\xi}
\end{array}
$$

Let $f\colon \{\eta < \lambda \mid \eta \text{ even}\} \to \kappa$ be a surjection such that for each $\alpha < \kappa$, $f^{-1}(\alpha)$ is cofinal in λ. We define a matrix

$$\langle\langle \mathbb{P}_{\alpha,\zeta} \mid \alpha \leq \kappa, \zeta \leq \lambda\rangle, \langle \underset{\sim}{\mathbb{Q}}_{\alpha,\zeta} \mid \alpha \leq \kappa, \zeta < \lambda\rangle\rangle$$

as follows by induction on ζ (and for a fixed ζ, by induction on α:

Induction on $\zeta \leq \lambda$ and $\alpha \leq \kappa$

(1) $\mathbb{P}_{\alpha,0} = \mathrm{Fn}_{<\omega}(\alpha \times \omega, \omega)$ adding a Cohen real g_β for $\beta < \alpha$.

Induction on $\zeta \leq \lambda$ and $\alpha \leq \kappa$

(1) $\mathbb{P}_{\alpha,0} = \mathrm{Fn}_{<\omega}(\alpha \times \omega, \omega)$ adding a Cohen real g_β for $\beta < \alpha$.

(2) if $\zeta = \eta + 1$ and ζ is odd then $\mathbb{P}_{\alpha,\eta} \Vdash \underset{\sim}{\mathbb{Q}}_{\alpha,\eta} = \mathbb{M}_{\underset{\sim}{\mathcal{U}}_{\alpha,\eta}}$ and for all $\alpha < \beta \leq \kappa$, $\mathbb{P}_{\beta,\eta} \Vdash \underset{\sim}{\mathcal{U}}_{\alpha,\eta} \subseteq \underset{\sim}{\mathcal{U}}_{\beta,\eta}$ and this is done and in the main lemma that that for any $\beta < \alpha$, $(\star V^{\mathbb{P}_{\beta,\zeta}}, V^{\mathbb{P}_{\alpha,\zeta}}, g_\beta)$, $\mathbb{P}_{\delta,\zeta}$ is the direct limit of $\mathbb{P}_{\alpha,\zeta}$, for limit δ.

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(3) if $\zeta = \eta + 1$ and ζ is even and $\alpha \leq f(\eta)$ then $\underset{\sim}{\mathbb{Q}}_{\alpha,\eta}$ is the one point forcing notion if $\alpha > f(\eta)$ then then

$$\mathbb{P}_{\alpha,\eta} \Vdash \underset{\sim}{\mathbb{Q}}_{\alpha,\eta} = \text{Hechler forcing in } V^{\mathbb{P}_{f(\eta),\eta}}.$$

Same clause at limits.

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Same clause at limits.

(4) If $\zeta \leq \lambda$ is a limit then for all $\alpha \leq \kappa$, $\mathbb{P}_{\alpha,\zeta}$ is the finite support iteration of $\langle \mathbb{P}_{\alpha,\eta}, \underset{\sim}{\mathbb{Q}}_{\alpha,\eta} \mid \eta \leq \zeta \rangle$. For each $\xi < \lambda$, for each limit $\beta \leq \kappa$, $\mathbb{P}_{\beta,\xi}$ is the direct limit of $\mathbb{P}_{\beta',\xi}$, $\beta' < \beta$.

Good properties

Along the induction on ζ we prove:

(a) For $\zeta \leq \lambda$, $\forall \alpha < \beta \leq \kappa$, $\mathbb{P}_{\alpha,\zeta} \lessdot \mathbb{P}_{\beta,\zeta}$.

(b) $\forall \zeta \leq \lambda$, $\forall \alpha < \kappa$, $(\star V^{\mathbb{P}_{\alpha,\zeta}}, V^{\mathbb{P}_{\alpha+1,\zeta}}, g_\alpha)$ holds.

(c) every $p \in \mathbb{P}_{\kappa,\zeta}$ there is an $\alpha < \kappa$ such that $p \in \mathbb{P}_{\alpha,\zeta}$.

(d) for every $\mathbb{P}_{\kappa,\zeta}$-name for a real $\underset{\sim}{f}$ there is $\alpha < \kappa$ such that $\underset{\sim}{f}$ is a $\mathbb{P}_{\alpha,\zeta}$-name.

Outline

Generalisation

Let κ be a regular uncountable cardinal.

Definition

$\mathfrak{s}(\kappa)$ is the smallest size of a splitting family of subsets of κ. Here splitting is meant in the κ-sense: S splits X iff $X \in [\kappa]^\kappa$ and $S \cap X$ and $X \setminus S$ both have cardinality κ.

Consistency strength beyond ZFC

Remark

$\mathfrak{s}(\kappa) \leq \mathfrak{s}(\mathrm{cf}(\kappa))$.

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Theorem (Suzuki)

If $\mathfrak{s}(\kappa) \geq \kappa$ *then* κ *is strongly inaccessible.*

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Theorem (Suzuki)

If $\mathfrak{s}(\kappa) \geq \kappa$ then κ is strongly inaccessible.

Theorem (Suzuki)

Let $\kappa > \omega$ be a regular cardinal. $\mathfrak{s}_\kappa > \kappa$ iff κ is weakly compact.

Proof on the blackboard.

A result in ZFC

Theorem (Raghavan, Shelah)

Let κ be a regular uncountable cardinal. $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$.

Definition

Let $\kappa > \omega$ be regular and suppose that there exists a cardinal λ such that $\kappa < \lambda < \mathfrak{s}_\kappa$. Fix a sufficiently large regular cardinal θ ($\theta = \left(2^{2^{\mathfrak{s}_\kappa}}\right)^+$ will suffice). Let $M \prec H(\theta)$ be such that $\lambda \subset M$ and $|M| = \lambda$. $M \cap \mathcal{P}(\kappa)$ is not a splitting family. So there exists $A_* \in [\kappa]^\kappa$ such that for all $x \in M \cap \mathcal{P}(\kappa)$ either $A_* \subset^* (\kappa \setminus x)$ or $A_* \subset^* x$. Define D to be $\{x \in \mathcal{P}(\kappa) : A_* \subset^* x\}$.

A linear order

$L = \{[f]_D \mid f \in {}^\kappa\kappa \cap M\}$.

Lemma

The structure $(L, <_D)$ *is a linear order. Moreover* $\{[c_\alpha]_D \mid \alpha < \kappa\}$ *has a least upper bound in* L.

A small unbounded family

Definition

Fix a function $f_* \in M \cap \kappa^\kappa$ such that $[f_*]_D \in L$ is a least upper bound of $\{[c_\alpha]_D \mid \alpha < \kappa\}$.

Lemma

If $C \in M$ is a club in κ, then $f_*^{-1}(C) \in D$.

Lemma

$M \cap \kappa^\kappa$ is bounded.