Transversal of full outer measure

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Avoiding rational distances

Question (Komjáth, 1994)

Let $X \subseteq \mathbb{R}^n$. Is there a subset Y of X such that X and Y have the same Lebesgue outer measure and the distance between any two distinct points of Y is irrational?

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- (Kumar, 2012) Yes in dimension one: Every subset of $\mathbb{R}$ has a subset of the same outer measure that avoids rational distances.

In plane

Theorem (Erdös-Hajnal, 1969)

There is a well ordering $\prec$ *of* $\mathbb{R}^2$ *such that for every* $x \in \mathbb{R}^2$,
$\{y : y \prec x \wedge ||x - y|| \in \mathbb{Q}\}$ *is finite.*

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Definition

A graph $G = (V, E)$ is said to have countable coloring number if there is a well ordering $\prec$ of V such that for every $x \in V$, $\{y : y \prec x \wedge yEx\}$ is finite.

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Question (Komjáth)

Suppose $X \subseteq \mathbb{R}^n$ and $G = (X, E)$ is a graph of countable coloring number. Must there exist $Y \subseteq X$ such that Y is E-independent and X, Y have the same Lebesgue outer measure?

The transversal theorem

Theorem (Kumar-Shelah, 2015)

Suppose $X \subseteq \mathbb{R}^n$ *and* $\langle X_i : i \in S \rangle$ *is a partition of* X *into countable sets. Then there exists* $Y \subseteq X$ *such that* X, Y *have the same Lebesgue outer measure and for every* $i \in S$, $|Y \cap X_i| = 1$.

Sometimes ZFC is not enough

Fact

Assume CH. Then for every $X \subseteq \mathbb{R}^2$*, there exists* $Y \subseteq X$ *such that no three points in* Y *are collinear and* X, Y *have the same (planar) Lebesgue outer measure.*

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Fact

Assume CH. Then for every $X \subseteq \mathbb{R}^2$, there exists $Y \subseteq X$ such that no three points in Y are collinear and X, Y have the same (planar) Lebesgue outer measure.

Theorem (Kumar-Shelah, 2017)

It is consistent that there is a subset X of plane such that X does not have zero area and for every $Y \subseteq X$, if Y does not have zero area, then Y contains three collinear points.

Proof ideas I

Towards a contradiction, suppose $X \subseteq \mathbb{R}^n$ and $\langle X_i : i \in S \rangle$ is a partition of X which has no transversal of the same outer measure as X. Then for some non null subset $Y \subseteq X$, letting $\mathbb{P} = \mathcal{P}(Y)/\mathcal{I}$ where $\mathcal{I}$ is the sigma ideal of null subsets of Y, the following holds.

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(3) There exists $\langle (p_n, \mathbb{Q}_n) : n < \omega \rangle$ such that
   (a) $p_n \in \mathbb{P}$
   (b) $\mathbb{Q}_n \lessdot \mathbb{P}_{\leq p_n} = \{p \in \mathbb{P} : p \leq p_n\}$
   (c) $\mathbb{Q}_n$ is isomorphic to random forcing
   (d) $\bigcup \{\mathbb{Q}_n : n < \omega\}$ is dense in $\mathbb{P}$

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Two examples of such forcings are $\mathbb{P} = \text{Random} \times \text{Cohen}$ and an ω-length finite support iteration of random forcing.

Proof ideas II

Let $\mathbb{P}$ be any forcing satisfying (3) above. Suppose V has a non meager set of size κ. Then $V^{\mathbb{P}}$ has a non meager set of size κ.

Proof ideas III

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Proof ideas III

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- Either $\mathbb{Q}$ adds an infinitely often equal real $x \in {}^{\omega}\omega$ which means that $(\forall y \in {}^{\omega}\omega \cap V)(\exists^{\infty} n)(x(n) = y(n))$
- Or for every new real $x \in V^{\mathbb{Q}} \setminus V$, $V[x]$ is a random real extension of V.

Proof ideas IV

Let $\mathcal{I}$ be a sigma ideal on Y such that for some $\kappa \geq \omega_1$, the additivity of $\mathcal{I} \restriction A$ is κ for every $A \in \mathcal{I}^+$. Put $\mathbb{P} = \mathcal{P}(Y)/\mathcal{I}$ and towards a contradiction suppose that $\mathbb{P}$ satisfies $(1) - (3)$ above.

Proof ideas IV

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Proof ideas V

Case 1: $\mathbb{Q}$ adds an infinitely often equal real x: Note that $x \in N$ and hence V has a non meager set of size κ. Since M contains a random real over V, every set of reals of size $< \kappa$ in V is meager. It follows that $V \models \text{non(Meager)} = \kappa$. Hence $M \models \text{non(Meager)} = j(\kappa) > \kappa$ and therefore in $V[G]$, there is no non meager set of size κ: Contradiction.

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Case 2: For every new real $r \in V^{\mathbb{Q}}$, $V[r]$ is a random real extension of V: Let r be a random real in N. Then V has a non null set $X = \{r_i : i < \kappa\}$. Since $V[G]$ has a Cohen real, all sets of reals of size $< \kappa$ are null in V. Applying k to $\{r_i : i < \kappa\}$ it follows that X is null in N. Let $x \in N$ be a real coding a null Borel set covering X. But $V[x]$ is a random real extension of V and therefore it preserves old non null sets: Contradiction.

Thank You!