Tall, Strong, and Strongly Compact Cardinals

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We begin with the following definition due to Hamkins.

Definition: Suppose κ is a cardinal and $\lambda \geq \kappa$ is an arbitrary ordinal. κ is λ *tall* if there is an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\kappa \subseteq M$. κ is *tall* if κ is λ tall for every ordinal λ.

In his 2009 *MLQ* paper "Tall Cardinals", Hamkins made a systematic study of tall cardinals and established many of their basic properties. He also made the interesting observation that

"strongness is to tallness as supercompactness is to strong compactness"

and established many results that either support this thesis directly or are analogues of conjectures believed true about strongly compact and supercompact cardinals. For instance, Hamkins mentions that Gitik's work yields the equiconsistency of the theories "ZFC + There is a strong cardinal" and "ZFC + There is a tall cardinal", a positive answer to the analogue of the question asking whether the theories "ZFC + There is a supercompact cardinal" and "ZFC + There is a strongly compact cardinal" are equiconsistent. Hamkins also shows that any measurable limit of tall cardinals is tall, an analogue of Menas' result that any measurable limit of strongly compact cardinals is strongly compact, and that any strong or strongly compact cardinal is in addition tall. By definition, any tall cardinal is also measurable.

Turning now to the main topic of my lecture, the following is a joint theorem with James Cummings.

Theorem 1 *(***A.-Cummings 2001***)*

Con(ZFC + There is a proper class of supercompact cardinals) $\Longrightarrow$ Con(ZFC + There is a proper class of strongly compact cardinals + No strongly compact cardinal κ is $2^\kappa = \kappa^+$ supercompact + $\forall\kappa[\kappa$ is strongly compact iff κ is a strong cardinal$]$).

Since this theorem was proven prior to Hamkins' work on tall cardinals, the question of whether there could be non-strong tall cardinals in the model witnessing the conclusions of Theorem 1 was not addressed. This will be the focus of my talk. In fact, this is indeed possible, as witnessed by the following theorem.

Theorem 2 *Con(ZFC + There is a proper class of supercompact cardinals) $\Longrightarrow$ Con(ZFC + There is a proper class of strongly compact cardinals + No strongly compact cardinal κ is $2^\kappa = \kappa^+$ supercompact + $\forall\kappa[\kappa$ is strongly compact iff κ is a strong cardinal] + Every strongly compact cardinal is a limit of (non-strong) tall cardinals).*

If we are willing to restrict the large cardinal structure of our universe, then it is possible to obtain an even better result. In particular, we also have the following theorem.

Theorem 3 *Suppose $V \vDash$ "ZFC + GCH + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable". Then there is a partial ordering $\mathbb{P} \in V$ such that $V^{\mathbb{P}} \vDash$ "ZFC + κ is both the only strong and only strongly compact cardinal + κ is not $2^\kappa = \kappa^+$ supercompact + Every measurable cardinal is tall + No cardinal $\lambda > \kappa$ is measurable".*

The proof of Theorem 3 will use joint work with Gitik. It will be obtained by forcing over a model containing a supercompact cardinal κ in which there are no measurable cardinals above κ, and in which every measurable cardinal is also tall.

I will now discuss the main ideas involved in the proofs of these theorems, beginning with the proof of Theorem 2. In the interest of time, I will only speak about the proof of this theorem for one cardinal. (The proof in the general case proceeds by forcing over a model of ZFC containing a proper class of indestructible supercompact cardinals with an Easton support product of the appropriate versions of the partial ordering used in the case of one cardinal.) In particular, starting with a model $V \vDash$ "ZFC + GCH + κ is supercompact", I will sketch how to construct a model $V^{\mathbb{P}} \vDash$ "ZFC + κ is both the least strongly compact and least strong cardinal + κ is not $2^\kappa = \kappa^+$ supercompact + κ is a limit of non-strong tall cardinals".

The definition of the forcing conditions $\mathbb{P}$ is motivated by the definition of the partial ordering used in the proof of Theorem 1. For that theorem, one uses the Easton support iteration of length κ which begins by adding a Cohen subset of ω (to introduce a gap so that Hamkins' Gap Forcing Theorem can be applied) and then adds a non-reflecting stationary set of ordinals of cofinality ω to each V-strong cardinal $\delta < \kappa$. This poset, however, will not ensure that there are any non-strong tall cardinals below κ. We therefore modify the definition so that $\mathbb{P}$ is taken as the Easton support iteration of length κ which begins by adding a Cohen subset of ω and then adds a non-reflecting stationary set of ordinals of cofinality ω to each member of the set $A = \{\delta < \kappa \mid \delta$ is a V-strong cardinal which is a limit of V-strong cardinals$\}$. Note that since κ is supercompact, A is unbounded in κ.

The same arguments used in the proof of Theorem 1 show that $V^{\mathbb{P}} \models$ "ZFC + κ is the least strongly compact cardinal + κ is strong + κ is not $2^\kappa = \kappa^+$ supercompact". We must now show that $V^{\mathbb{P}} \models$ "No cardinal $\delta < \kappa$ is strong + κ is a limit of non-strong tall cardinals".

Lemma 1 $V^{\mathbb{P}} \models$ *"No cardinal $\delta < \kappa$ is strong".*

Proof: By the Gap Forcing Theorem, we may infer that if $V^{\mathbb{P}} \models$ "δ is a strong cardinal", then $V \models$ "δ is a strong cardinal" as well. Therefore, to prove Lemma 1, it suffices to show that if $V \models$ "$\delta < \kappa$ is strong", then $V^{\mathbb{P}} \models$ "δ is not a strong cardinal". This is clearly true if $V \models$ "δ is a strong cardinal which is a limit of strong cardinals". This is since under these circumstances, by the definition of $\mathbb{P}$, $V^{\mathbb{P}} \models$ "There is $S \subseteq \delta$ which is a non-reflecting stationary set of ordinals of cofinality ω and thus δ is not weakly compact". Hence, to complete the proof of

Lemma 1, we must show that if $V \vDash$ "δ is a strong cardinal which is not a limit of strong cardinals", then $V^{\mathbb{P}} \vDash$ "δ is not a strong cardinal".

To do this, suppose to the contrary that $V^{\mathbb{P}} \vDash$ "δ is a strong cardinal". Because $V \vDash$ "δ is not a limit of strong cardinals", we may write $\mathbb{P} = \mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$, where $|\mathbb{R}^*| < \delta$, $\mathbb{R}^*$ adds a Cohen subset of ω and also adds non-reflecting stationary sets of ordinals of cofinality ω to each cardinal below δ which is a V-strong limit of V-strong cardinals, $\dot{\mathbb{R}}^{**}$ is a term for the partial ordering adding a non-reflecting stationary set of ordinals of cofinality ω to the least V-strong cardinal $\delta' > \delta$ which is a limit of V-strong cardinals, and $\dot{\mathbb{R}}$ is a term for the rest of $\mathbb{P}$. Since $\Vdash_{\mathbb{R}^* * \dot{\mathbb{R}}^{**}}$ "$\dot{\mathbb{R}}$ is σ-strategically closed for σ the least inaccessible cardinal above δ'", it is the case that $V^{\mathbb{P}} = V^{\mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}} \vDash$ "δ is $\delta' + 2$ strong" iff $V^{\mathbb{R}^* * \dot{\mathbb{R}}^{**}} \vDash$ "δ is $\delta' + 2$ strong". The proof of

Lemma 1 will therefore be complete if we can show that $V^{\mathbb{R}^* * \dot{\mathbb{R}}^{**}} \models$ "δ is not $\delta' + 2$ strong".

Towards this end, let G^* be V-generic over $\mathbb{R}^*$ and G^{**} be $V[G^*]$-generic over $\mathbb{R}^{**}$. Since $V[G^*][G^{**}] \models$ "δ is $\delta' + 2$ strong", we may let $j^* : V[G^*][G^{**}] \to M[j^*(G^*)][j^*(G^{**})]$ be an elementary embedding having critical point δ which witnesses the $\delta' + 2$ strongness of δ. By the Gap Forcing Theorem, j^* must lift some elementary embedding $j : V \to M$ witnessing the $\delta' + 2$ strongness of δ in V, where $M \subseteq V$, $V_{\delta'+2} \in M$, and $j(\delta) > \delta' + 2$. Further, as $|\mathbb{R}^*| < \delta$ and δ is the critical point of both j and j^*, $j(\mathbb{R}^*) = \mathbb{R}^*$ and $j^*(G^*) = G^*$, i.e., $j^* : V[G^*][G^{**}] \to M[G^*][j^*(G^{**})]$.

Because $V_{\delta'+2} \subseteq M$, $(V_{\delta'+1})^{V[G^*]} = (V_{\delta'+1})^{M[G^*]}$. Thus, as $\mathbb{R}^{**} \in (V_{\delta'+1})^{V[G^*]}$, $\mathbb{R}^{**} \in (V_{\delta'+1})^{M[G^*]}$. Therefore, since $M[G^*] \subseteq V[G^*]$, G^{**} is also $M[G^*]$-generic over $\mathbb{R}^{**}$, so that in particular,

G^{**} is not a member of either $V[G^*]$ or $M[G^*]$. However, because

$$(V_{\delta'+2})^{V[G^*][G^{**}]} \in M[G^*][[j^*(G^{**})]$$

and

$$G^{**} \in (V_{\delta'+2})^{V[G^*][G^{**}]},$$

$G^{**} \in M[G^*][[j^*(G^{**})]$. Since $j^*(G^{**})$ is $M[G^*]$-generic over a suitably strategically closed partial ordering, $G^{**} \in M[G^*]$. This contradiction completes the proof of Lemma 1.

$\square$

Lemma 2 $V^{\mathbb{P}} \models$ "*κ is a limit of non-strong tall cardinals*".

Proof: Since $V \models$ "κ is supercompact", the set $B = \{\delta < \kappa \mid \delta$ is a V-strong cardinal which is not a limit of V-strong cardinals$\}$ is unbounded in κ. We show that $V^{\mathbb{P}} \models$ "Every $\delta \in B$ is a tall cardinal". This will suffice, since by Lemma 1, $V^{\mathbb{P}} \vDash$ "No $\delta \in B$ is a strong cardinal".

Towards this end, fix $\delta \in B$. With the same meaning as in the proof of Lemma 1, write $\mathbb{P} = \mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$. Since $|\mathbb{R}^*| < \delta$, $V^{\mathbb{R}^*} \vDash$ "δ is a strong cardinal". As we have already noted, it then immediately follows that $V^{\mathbb{R}^*} \vDash$ "δ is a tall cardinal". By a result of Hamkins, δ's tallness is indestructible under (δ, ∞)-distributive forcing. Because by its definition, $\Vdash_{\mathbb{R}^*}$ "$\dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$ is (δ, ∞)-distributive", $V^{\mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}} = V^{\mathbb{P}} \vDash$ "δ is a tall cardinal". This completes the proof of Lemma 2.

$\square$

Lemmas 1 and 2 complete the proof sketch of Theorem 2 for one cardinal. $\square$

To prove Theorem 3, by joint work with Gitik, we assume our ground model $V \vDash$ "ZFC + κ is supercompact + No cardinal $\lambda > \kappa$ is measurable + Every measurable cardinal is tall". We then force over V with the partial ordering $\mathbb{P}$ used in the proof of Theorem 2. $V^{\mathbb{P}}$ satisfies the same conclusions as in Theorem 2. We must now verify that in $V^{\mathbb{P}}$, every measurable cardinal is also tall. To do this, suppose $V^{\mathbb{P}} \vDash$ "$\delta < \kappa$ is measurable". By the Gap Forcing Theorem, $V \vDash$ "δ is measurable", so by our assumptions on V, $V \vDash$ "δ is tall" as well.

We consider now the following two cases.

Case 1: δ is not a limit of V-strong cardinals which are limits of V-strong cardinals. We also know, by the definition of $\mathbb{P}$, that $V \vDash$ "δ is not a strong cardinal which is a limit of strong cardinals". We may therefore use the factorization $\mathbb{P} = \mathbb{R}^* * \dot{\mathbb{R}}^{**} * \dot{\mathbb{R}}$ given in the proofs of

Lemmas 1 and 2 and employ the same argument found in the proof of Lemma 2 to infer that $V^{\mathbb{P}} \models$ "δ is a tall cardinal".

Case 2: δ is a limit of V-strong cardinals which are limits of V-strong cardinals. It then immediately follows that δ must be a limit of V-strong cardinals which are themselves not limits of V-strong cardinals. For any such γ, by Lemma 2, $V^{\mathbb{P}} \models$ "γ is a tall cardinal". Since $V^{\mathbb{P}} \models$ "δ is a measurable limit of tall cardinals", $V^{\mathbb{P}} \models$ "δ is a tall cardinal".

Cases 1 and 2 complete the proof sketch of Theorem 3. $\square$

We conclude by asking if it is possible to prove a version of Theorem 2 in which the only tall cardinals in the witnessing model are both strongly compact and strong. This seems to be an extremely challenging question to answer, both since every tall cardinal κ is automatically indestructible under (κ, ∞)-distributive forcing, and since as we have already mentioned, it is consistent to assume in a model containing a supercompact cardinal that the statement "κ is measurable iff κ is tall" is true. One cannot therefore kill a tall cardinal κ by adding a non-reflecting stationary set of ordinals of small cofinality above κ (as can be done with a strongly compact cardinal), because the forcing is (κ, ∞)-distributive. Also, forcing to eliminate the "bad" tall cardinals below a supercompact cardinal λ by adding non-reflecting stationary sets of ordinals to them might preserve λ's strong compactness but kill λ's strongness, since all measurable cardinals below λ may be destroyed.

Thank you all very much for your attention!