Rearrangements and Subseries

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Authors

joint work with Andreas Blass, Will Brian, Joel Hamkins, Michael Hardy, Paul Larson, and Jonathan Verner

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<u>Notes:</u> (1) $\sum a_n$ convergent $\implies a_n \to 0$

(2) If $\sum a_n$ is conditionally convergent then

$$\sum_{n \in P} a_n = +\infty \quad \text{and} \quad \sum_{n \in N} a_n = -\infty$$

where $P = \{n \in \omega : a_n > 0\}$ and $N = \{n \in \omega : a_n < 0\}$

Riemann's Rearrangement Theorem

Suppose $\sum a_n$ *is conditionally convergent and* $r \in \mathbb{R} \cup \{+\infty, -\infty\}$. *Then there is a* rearrangement $\pi \in \mathrm{Sym}(\omega)$ *such that* $\sum a_{\pi(n)} = r$.

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Also there is $\pi \in \mathrm{Sym}(\omega)$ such that $\sum a_{\pi(n)}$ diverges *by oscillation*.
($\liminf_k \sum_0^k a_{\pi(n)} < \limsup_k \sum_0^k a_{\pi(n)}$)

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Rearrangement numbers

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Π witness for $\mathfrak{rr} \implies \Pi \cup \{\sigma_\pi : \pi \in \Pi\}$ witness for $\mathfrak{rr}_o$

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Done!

$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega\ \exists f \in F\ \exists^\infty n\ (g(n) < f(n))\}$

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Then $\sum b_k = \sum a_n$ c.c. Also $\sum b_k = \sum b_{\pi(k)}$ for all $\pi \in \Pi$.
Why? Because $\forall^\infty n < m \; (\pi(i_n) < \pi(i_m))$. Done!

$\mathfrak{rr}_i, \mathfrak{rr}_f$ versus $\mathfrak{d}$

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Proof similar to Theorem 3.

$\mathfrak{rr}$ versus cov(null)

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Theorem 5

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proof based on:

Rademacher's Lemma

Let $(c_n : n \in \omega)$ *be a sequence of reals. Set*

$$A = \{f \in 2^\omega : \sum_n (-1)^{f(n)} c_n \textit{ converges}\}$$

Then

$$\mu(A) = \begin{cases} 1 & \textit{if } \sum_n c_n^2 \textit{ converges} \\ 0 & \textit{otherwise} \end{cases}$$

Consequences and a Question

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CON $(\mathfrak{d} < \mathfrak{rr})$

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Proof: In the Laver / Hechler model, $\text{cov(null)} < \mathfrak{b}$. So follows from Theorem 3.

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Question 1

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Thus the ω_1 permutations adjoined along the iteration witness $\mathfrak{rr}_i = \omega_1$.

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Proof Idea: Start with a model of $\mathfrak{c} > \omega_1$.
Make a finite support iteration of σ-linked forcing of length ω_1.
Use the Lévy-Steinitz Theorem (finite-dimensional version of Riemann's Theorem).

Upper bounds for $\mathfrak{rr}_f$?

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Conjecture 2

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More questions

If $\sum a_n$ is c.c. then it has a divergent subseries $\sum_{n \in X} a_n$, $X \in [\omega]^\omega$.

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where $P = \{n \in \omega : a_n > 0\}$ and $N = \{n \in \omega : a_n < 0\}$

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... such that $\sum_{n \in X} a_n$ diverges either to $+\infty$ or $-\infty$?

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Proofs: like for $\mathfrak{rr}$

ß versus $\mathfrak{s}$

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- either $X \cap I$ finite and $\sum_{k \in X} b_k$ finite
- or $I \subseteq^* X$ and $\sum_{k \in X} b_k$ converges because $\sum b_k$ does.

$\mathfrak{ß}$ versus $\mathfrak{b}$, 1

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$r, s \geq 0$. $(\bar{I}, \bar{B}, \bar{a})$ is (r,s)-*sequence* if, letting $b_n = \sum_{k \in B_n} a_k$ and $c_n = \sum_{k \in C_n} a_k$, we have that $\lim b_n = r$ and $\lim c_n = s$.

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$D \in [\omega]^\omega$ *almost splits* (r,s)-sequence $(\bar{I}, \bar{B}, \bar{a})$ if $\exists E \subseteq \omega$ s.t.

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$\mathfrak{s}_{\text{almost}} := \min\{|\mathcal{D}| : \mathcal{D} \subseteq [\omega]^\omega$ and $\forall (r^m, s^m)$-sequences $(\bar{I}^m, \bar{B}^m, \bar{a}^m)$ $\exists D \in \mathcal{D}$ (D almost splits all $(\bar{I}^m, \bar{B}^m, \bar{a}^m)$)$\}$

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ß versus 𝔟, 2

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Thus, $(\bar{I}, \bar{B}, \bar{a})$ is $(1,1)$-sequence.

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$D \in \mathcal{D}$ almost splits $(\bar{I}, \bar{B}, \bar{a}) \Longrightarrow \sum_{k \in D} x_k$ diverges.

ß versus $\mathfrak{b}$, 3

Theorem 13

$\mathfrak{s}_{\text{almost}} = \omega_1$ *in Laver model.*

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Corollary 14

CON$(ß < \mathfrak{b})$*; so also CON*$(ß < \mathfrak{rr})$

$\mathfrak{ß}$ versus $\mathfrak{b}$, 3

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CON$(\mathfrak{ß} < \mathfrak{b})$*; so also CON*$(\mathfrak{ß} < \mathfrak{rr})$

Question 2

CON$(\mathfrak{rr} < \mathfrak{ß})$? Even *CON*$(\mathfrak{rr} < \mathfrak{s})$?

About ß$_o$ and ß$_i$

Question 3

ß $=$ ß$_o$?

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Theorem 15

ß$_i >$ cov(*meager*)

About $\mathfrak{ß}_o$ and $\mathfrak{ß}_i$

Question 3

$\mathfrak{ß} = \mathfrak{ß}_o$?

Theorem 15

$\mathfrak{ß}_i > \mathrm{cov}(\mathit{meager})$

Question 4

$CON(\mathfrak{ß}_i < \mathfrak{c})$?