The universal triangle-free graph has finite big Ramsey degrees

Natasha Dobrinen

University of Denver

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This work commenced during the Isaac Newton Institute HIF Programme (2015) and continued at the Centre de Recerca Matemàtica (2016).

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Ramsey's Theorem

Ramsey's Theorem (finite version). Given any k, l, m, there is an n such that for each coloring of the collection of all k-element subsets of $\{0, \ldots, n-1\}$ into l colors, there is a subset $X \subseteq \{0, \ldots, n-1\}$ of size m such that each k-element subset of X has the same color.

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Ramsey's Theorem (infinite version). Given any k, l and a coloring on the collection of all k-element subsets of $\mathbb{N}$ into l colors, there is an infinite set M of natural numbers such that each k-element subset of M has the same color.

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Some Fraïssé classes of finite structures with the Ramsey property: Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

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The classes of finite graphs, hypergraphs, graphs omitting k-cliques, etc., have small Ramsey degrees.

Ramsey Theory on Infinite Structures

Def. (Kechris, Pestov, Todorcevic 2005)
Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{F} = \mathrm{Flim}(\mathcal{K})$. $\mathbf{F}$ has finite big Ramsey degrees if for each $A \in \mathcal{K}$, there is a finite number $T(A, \mathcal{K})$ such that for any coloring of $\binom{\mathbf{F}}{\mathbf{A}}$ into finitely many colors, there is a substructure $\mathbf{F}'$ of $\mathbf{F}$, with $\mathbf{F}' \cong \mathbf{F}$, in which $\binom{\mathbf{F}'}{\mathbf{A}}$ take no more than $T(A, \mathcal{K})$ colors.

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Infinite structures known to have finite big Ramsey degrees: The rationals (Devlin 1979); the Rado graph (Sauer 2006); the countable ultrametric Urysohn space (Nguyen Van Thé 2008); the $\mathbb{Q}_n$ and $\mathbf{S(2)}$, $\mathbf{S(3)}$ (Laflamme, NVT, Sauer 2010), and a few others.

Connections with Topological Dynamics

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(Nguyen Van Thé 2013) Extended above result to Fraïssé classes that have precompact expansions with the Ramsey property (small Ramsey degrees).

(Zucker 2017) Characterized universal completion flows of $\mathrm{Aut}(\mathrm{Flim}\ \mathcal{K})$ whenever $\mathrm{Flim}\ \mathcal{K}$ admits a big Ramsey structure (big Ramsey degrees).

The big Ramsey degrees in the Rado graph $\mathcal{R}$ - context

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Lower bounds for $T(\mathrm{A}, \mathcal{G})$ for any $\mathrm{A} \in \mathcal{G}$ were proved by Laflamme, Sauer, and Vuksanovic 2006 and counted by J. Larson in 2008.

Strong Trees and Milliken's Theorem

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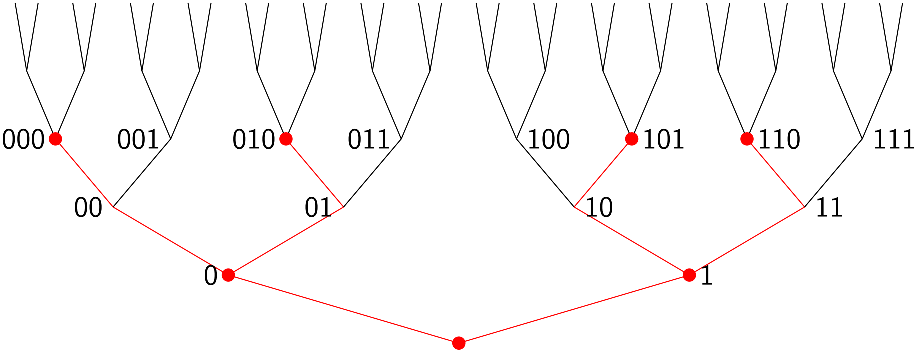
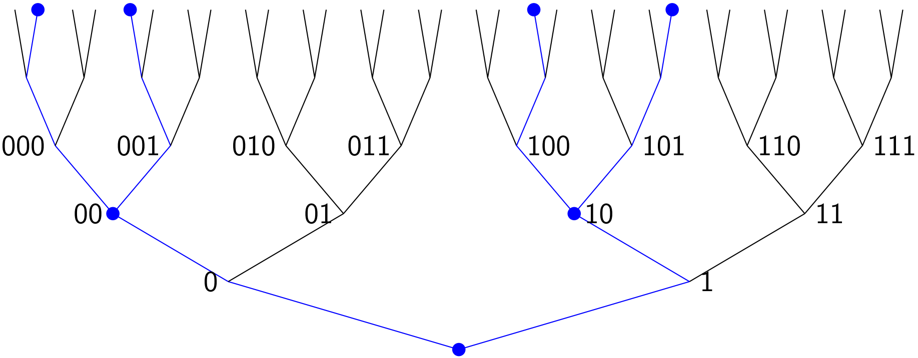
A tree $T \subseteq 2^{<\omega}$ is a strong tree iff it is either isomorphic to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

Strong Subtree $\cong 2^{\leq 2}$, Ex. 1



Strong Subtree $\cong 2^{\leq 2}$, Ex. 2



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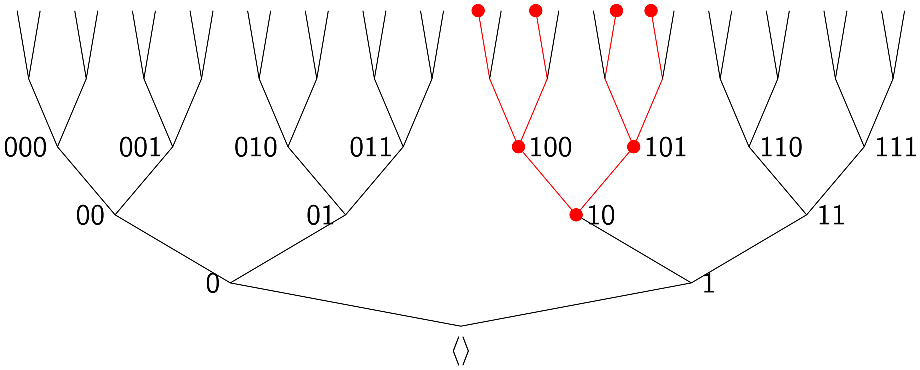

A Ramsey Theorem for Strong Trees

Thm. (Milliken 1979) Let $k \geq 0$, $l \geq 2$, and a coloring of all the subtrees of $2^{<\omega}$ which are isomorphic to $2^{\leq k}$ into l colors. Then there is an infinite strong subtree $S \subseteq 2^{<\omega}$ such that all copies of $2^{\leq k}$ in S have the same color.

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Thm. (Halpern-Läuchli 1966) Let $d \geq 1$, $l \geq 2$, and $T_i = 2^{<\omega}$ for $i < d$. Given a coloring of the product of level sets of the T_i into l colors,

$$f : \bigcup_{n<\omega} \prod_{i<d} T_i(n) \to l,$$

there are infinite strong trees $S_i \leq T_i$ and an infinite sets of levels $M \subseteq \omega$ where the splitting in S_i occurs, such that f is constant on $\bigcup_{m \in M} \prod_{i<d} S_i(m)$.

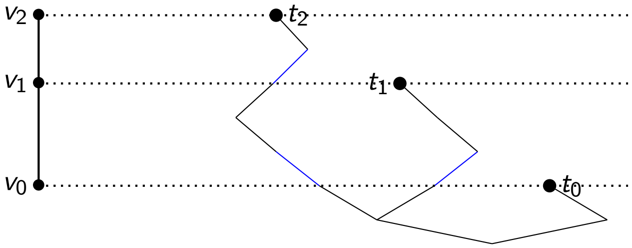
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

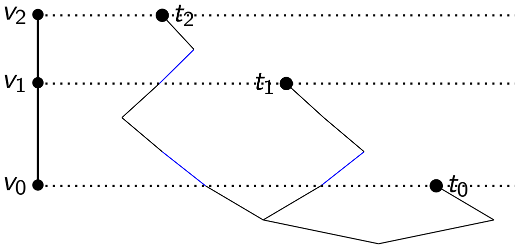
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Diagonal Trees Code Graphs

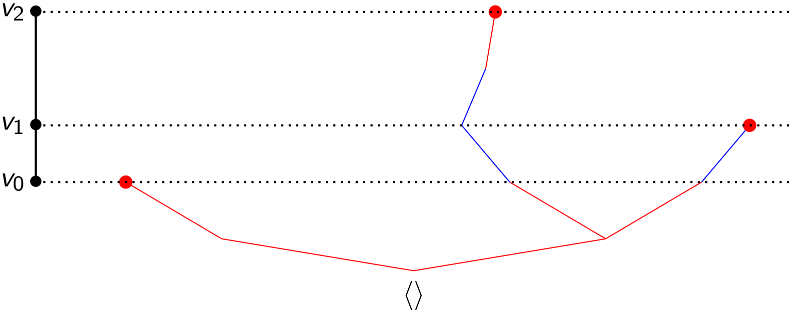
A tree T is diagonal if there is at most one meet or terminal node per level.

T is strongly diagonal if passing numbers at splitting levels are all 0 (except for the right extension of the splitting node).

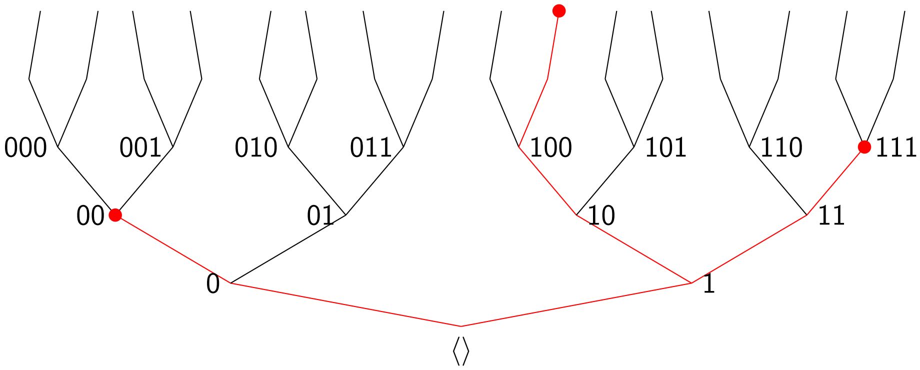


Every graph can be coded by the terminal nodes of a diagonal tree. Moreover, there is a strongly diagonal tree which codes $\mathcal{R}$.

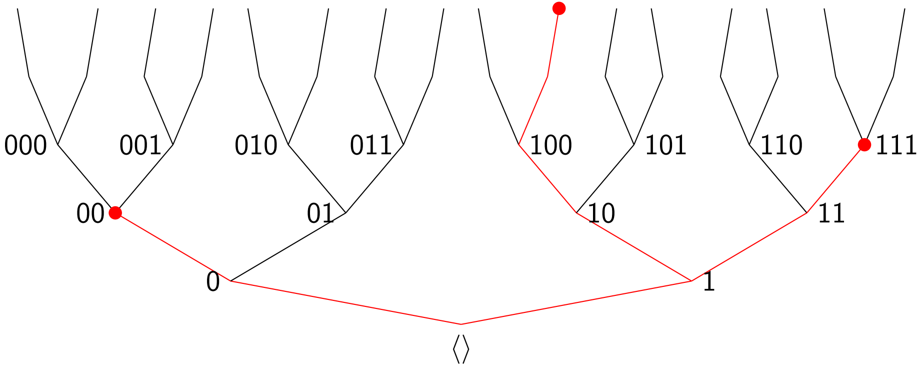
A Different Strongly Diagonal Tree Coding a Path



Strongly diagonal trees can be enveloped into strong trees



Another strong tree envelope



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3. Each strongly diagonal tree can be enveloped into a finite strong tree.
4. Apply Milliken's Theorem finitely many times to obtain one color for each type.
5. Choose a strongly diagonal subtree coding the Rado graph.

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Then I read Sauer's 1998 paper on the universal triangle-free graph, $\mathcal{H}_3$, where he got edge colorings to have big Ramsey degree of two.

Why did he stop with edges? Why wouldn't Sauer's methods for the Rado graph generalize to give finite Ramsey degrees for $\mathcal{H}_3$?

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$\mathcal{H}_3$ was constructed by Henson in 1971. Henson also constructed universal k-clique-free graphs for each $k \geq 3$.

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What about big Ramsey degrees in $\mathcal{H}_3$ for other finite triangle-free graphs?

Main Obstacles

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"So far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties." (Nguyen Van Thé, 2013 Habilitation)

Main Theorem: $\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem. (D.) For each finite triangle-free graph A, there is a positive integer $T(\mathrm{A}, \mathcal{K}_3)$ such that for any coloring of all copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H} \le \mathcal{H}_3$, again universal triangle-free, such that all copies of A in $\mathcal{H}$ take no more than $T(\mathrm{A}, \mathcal{K}_3)$ colors.

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This is the first result on big Ramsey degrees of a homogeneous structure omitting a non-trivial substructure.

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III Construct a strongly diagonal subset of coding nodes coding $\mathcal{H}_3$ and apply the Ramsey Theorem for strictly similar antichains.

Part I: Strong Coding Trees

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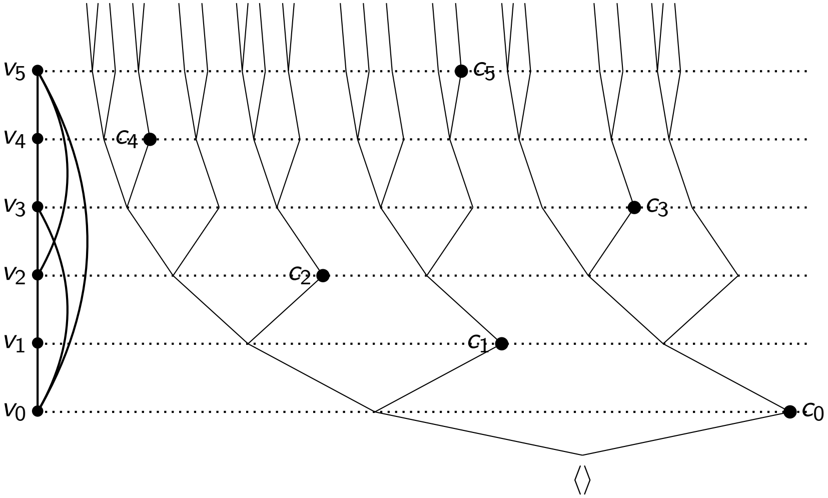
Non-splitting nodes extend left.

Strong triangle-free tree $\mathbb{S}$



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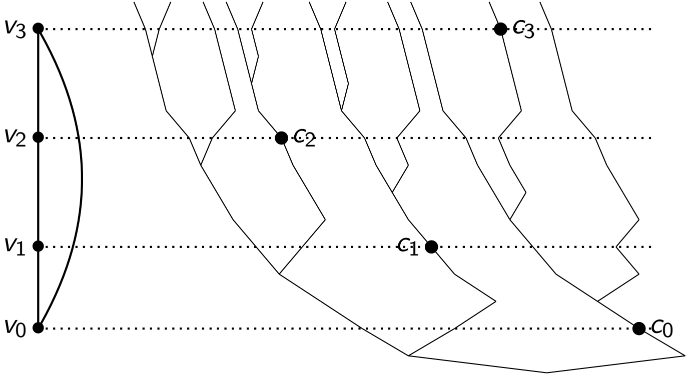
To get around this, we stretch and skew the trees so that at most one coding or one splitting node occurs at each level.

Strong coding tree $\mathbb{T}$



Write $T \leq \mathbb{T}$ if T is a subtree of $\mathbb{T}$ strongly similar to T.
Every tree $T \leq \mathbb{T}$ is a strong coding tree: Its coding nodes are dense and code $\mathcal{H}_3$, and the "zip up" forms a strong triangle-free tree.

A subset A of a strong coding tree is a tree if A is meet closed, $A = \bigcup \{t \restriction |s| : s, t \in A \text{ and } |t| \geq |s|\}$, and the lengths of members of A are exactly the lengths of its coding nodes and splitting nodes.

Guarantees for extending subtrees to copies of $\mathbb{T}$

Parallel 1's Criterion: "New sets of parallel 1's are witnessed by a coding node." A tree satisfying the Parallel 1's Criterion is a preserving tree: "all types are preserved".

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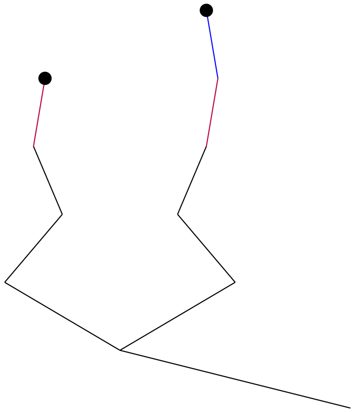
Facts. (1) Any preserving subtree of $\mathbb{T}$ in which the splitting is maximal and the coding nodes are dense and non-terminal is a strong coding tree, where the coding nodes code $\mathcal{H}_3$.

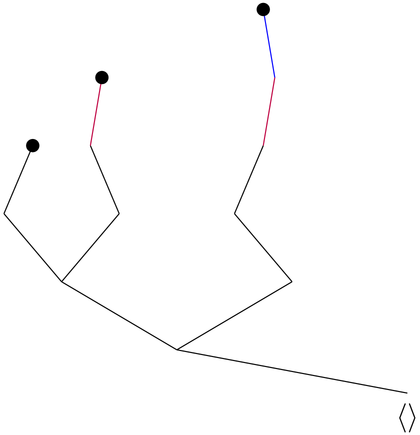
(2) Any finite valid preserving subtree of a strong coding tree T can be extended to a strong coding subtree of T.

A tree which is not preserving

It has parallel 1's not witnessed by a coding node.



A preserving tree



Part II: A Ramsey Theorem for Strictly Similar Finite Antichains.

Some Key Concepts

Two finite subtrees A, B of a strong coding tree T are strictly similar if there is a tree isomorphism $\varphi : A \to B$ which sends coding (splitting) nodes to coding (splitting) nodes, preserves relative lengths, passing numbers at levels of coding nodes, and first instances of parallel 1's.

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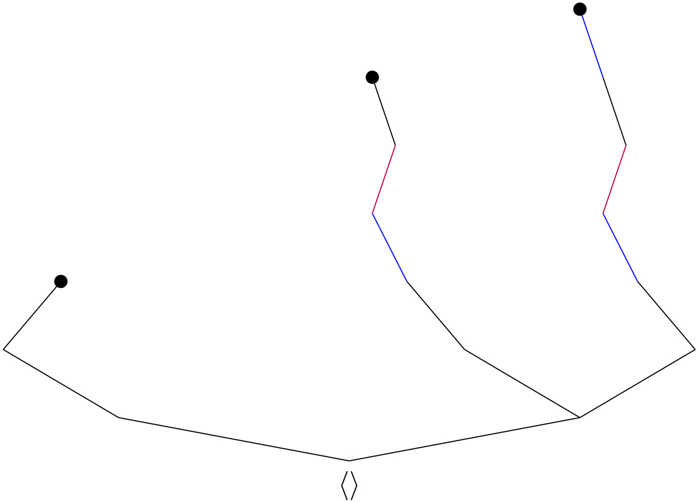
A subtree A of a strong coding tree T is incremental if whenever a new set of parallel 1's occurs in A, all of its proper subsets occur as new parallel 1's at a lower level.

G a graph with three vertices and no edges

An incremental tree A coding G



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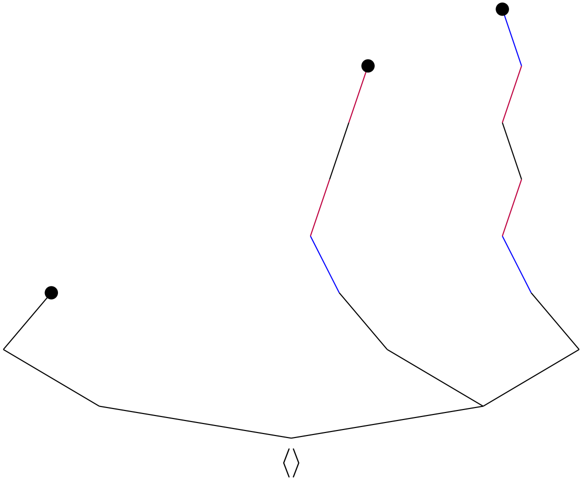
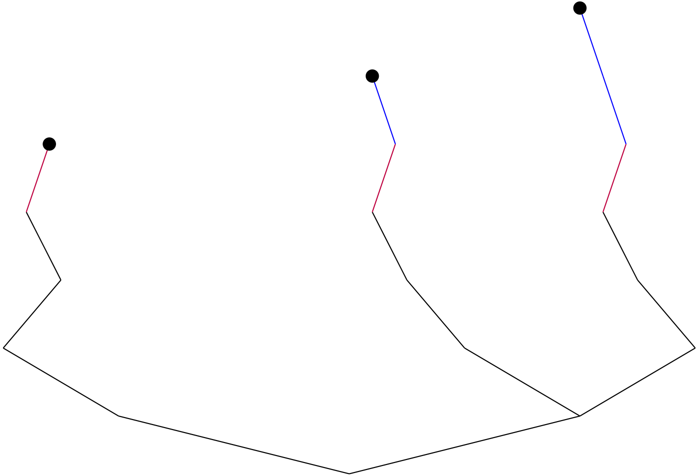
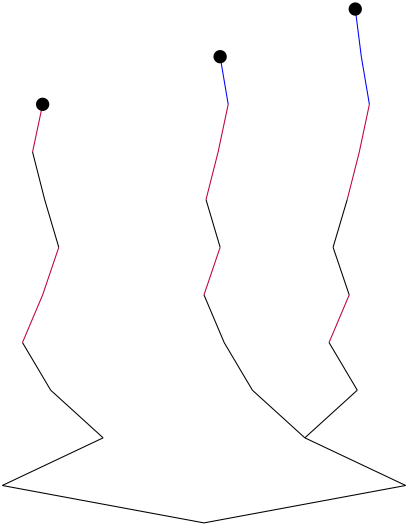
An incremental tree B coding G. B is strictly similar to A.



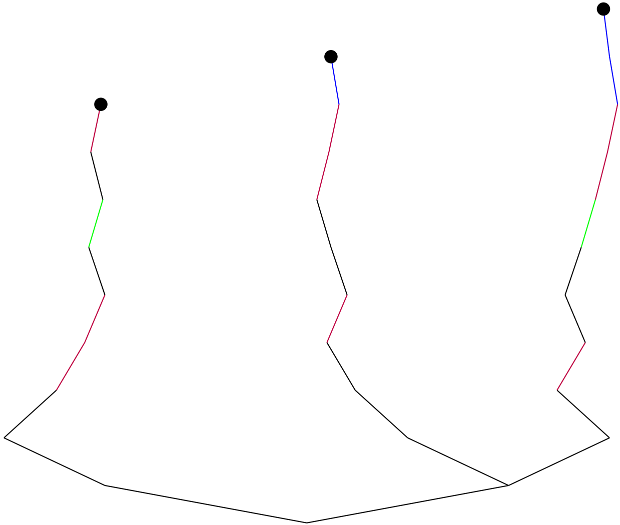
A non-incremental tree coding G



An incremental tree C coding G



An incremental tree D coding G strictly similar to C



Ramsey Theorem for Strictly Similar Antichains

Theorem. (D.) Let A be a finite antichain of coding nodes. Associate A with the tree it induces, and let c color all strictly similar copies of A in T into finitely many colors.

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(The theorem works for more than antichains, but only antichains are used in the proof of the Main Theorem. The proof takes four sections of the paper.)

Part III: Apply the Ramsey Theorem for Strictly Similar Antichains and construct a diagonal subtree coding $\mathcal{H}_3$ to obtain the Main Theorem.

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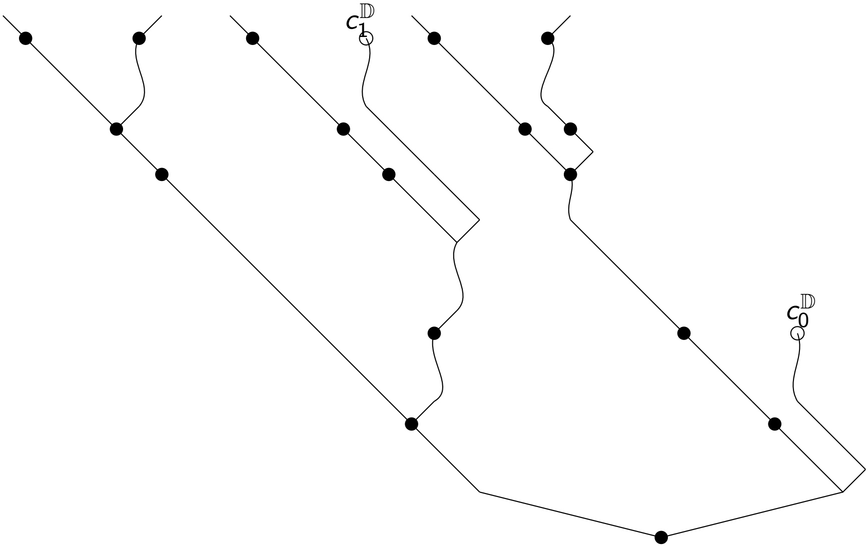
5. Take a strongly diagonal subtree $\mathbb{D}$ in S which codes $\mathcal{H}_3$, and let $\mathcal{H}'$ be the subgraph of $\mathcal{H}_3$ coded by $\mathbb{D}$.
6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of strict similarity types of antichains coding G.

Constructing a diagonal set $\mathbb{D}$ of coding nodes coding $\mathcal{H}_3$



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(c) New notion of envelope.
  – Involves new notions of incremental strong coding tree and sets of witnessing coding nodes.

(a) Halpern-Läuchli-style Theorem

Thm. (D.) Given: T a strong coding tree, B a finite valid strong coding subtree of T, A a finite subtree of B with $\max(A) \subseteq \max(B)$, and X a level set extending A into T with $A \cup X$ a valid preserving tree. Color all end-extensions Y of A in T for which $A \cup Y$ is strictly similar to $A \cup X$ into finitely many colors.
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Remark. The proof uses three different forcings. The forcings are best thought of as conducting unbounded searches for finite objects in ZFC.

The forcing ideas - Case (i): X contains a splitting node

Let T be a strong coding tree.

List the maximal nodes of A^+ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \le d$.

Fix κ large enough so that $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

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$q \leq p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \geq l_p$, and

(i) $q(d) \supset p(d)$, and $q(i,\delta) \supset p(i,\delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) The set $\{q(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{q(d)\}$ has no new sets of parallel 1's above $\{p(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{p(d)\}$.

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(3) The assumption that $A \cup X$ satisfies the Parallel 1's Criterion is necessary.

Case (ii): X contains a coding node

We use a different forcing.

We obtain end-homogneity.

To homogenize over these, we need a third forcing. This is where the strict similarity comes into play.

(b) Ramsey Theorem for Strict Preserving Trees

Thm. (D.) Let T be a strong coding tree. Let A be a finite strict preserving subtree of T. Suppose all the strictly similar copies of A in T are colored in finitely many colors.

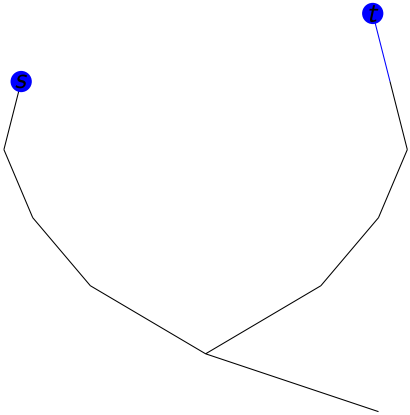
Then there is a subtree $S \leq T$ which is isomorphic to T (hence codes $\mathcal{H}_3$) such that all strictly similar copies of A in S have the same color.

A tree is a strict preserving tree if each new set of parallel 1's is witnessed by a coding node before anything else happens (other occurrences of new parallel 1's, splits, or coding nodes).

Strict similarity takes into account isomorphism as trees with coding nodes, passing numbers, and placements of new sets of parallel 1's.

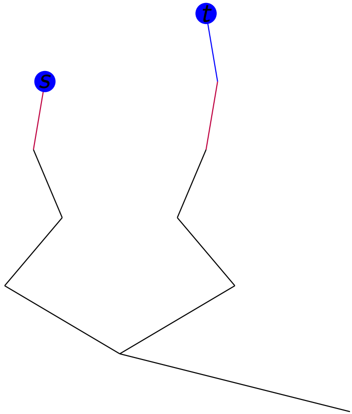
(c) Envelopes, Incremental Trees, and Witnessing Coding Nodes

A codes a non-edge



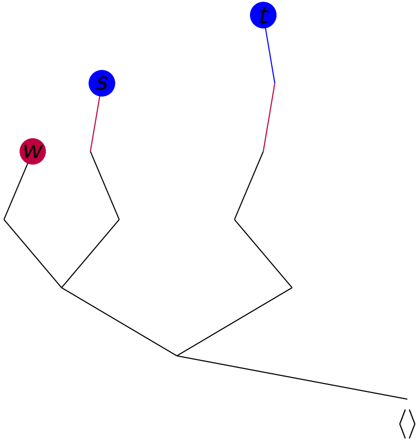
This satisfies the Parallel 1's Criterion, so A is its own envelope.

B codes a non-edge



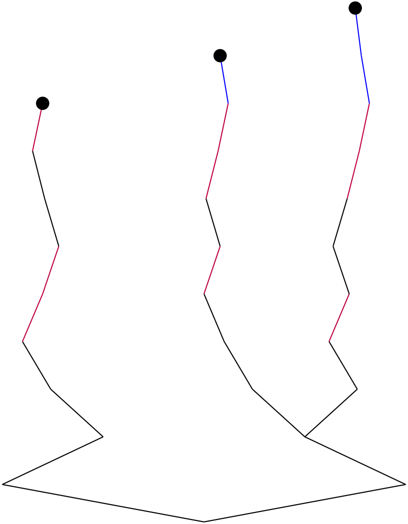
B does not satisfy the Parallel 1's Criterion.

An Envelope E(B)

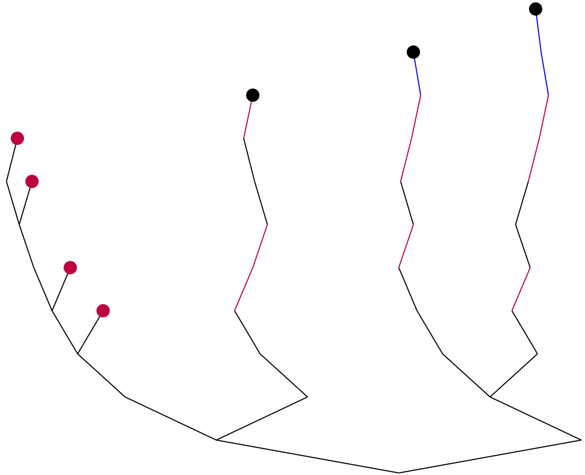


The envelope $E(B)$ satisfies the Parallel 1's Criterion.

An incremental tree D coding three vertices with no edges



An envelope of the incremental tree D



Towards Ramsey Theorem for Strictly Similar Antichains

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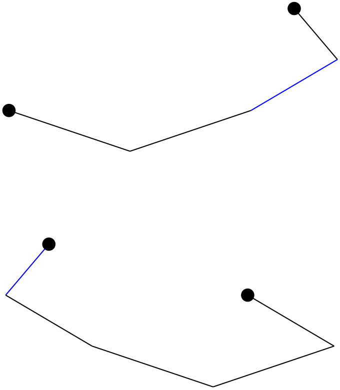
Then each copy of A in S has an envelop in T', by adding in some nodes from W.

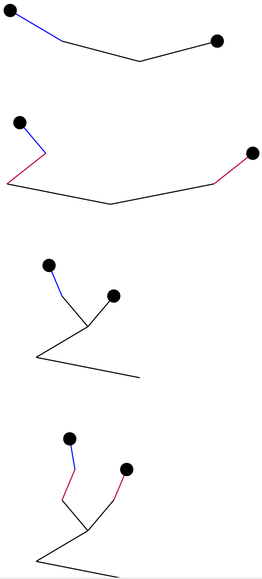
Thus, each copy of A in S has the same color.

To finish: Some Examples

The two strict similarity types of Edge Codings



Non-edges have eight strict similarity types



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Thank you for your attention!