Bounding, splitting and almost disjointness can be quite different

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Creature forcing: S. Shelah

Assume CH. There is a cardinal preserving extension such that

- $\mathfrak{b} = \aleph_1 < \mathfrak{a} = \mathfrak{s} = \aleph_2$.

Rank Arguments: S. Hechler

- Consistently $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{a} = \kappa$.

Larger values

Obtaining larger values, i.e. the consistency of $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \kappa^+$ (Brendle; Steprans and V.F.) led to:

- development of ccc suborders of proper creature posets, which behave analogously;
- construction of ultrafilters (filters) for which $\mathbb{M}_{\mathscr{U}}$ preserves the unboundedness of certain unbounded families (weak Canjar filters);

Larger spread

Models in which $\mathfrak{b}$, $\mathfrak{s}$ and $\mathfrak{a}$ are distinct, but do not have consecutive values, have been obtained via matrix, or 2D-iterations. Let $\kappa < \lambda$ be arbitrary regular cardinals.

- (Blass, Shelah) Con($\mathfrak{u} = \kappa < \mathfrak{d} = \lambda$);
- (Brendle, V.F.) Con($\mathfrak{b} = \mathfrak{a} = \kappa < \mathfrak{s} = \lambda$).

In particular these constructions brought:

- the construction of strong Canjar Ultrafilters;
- a new method of preserving mad families along an iteration, strong diagonalization;

Large cardinals assumptions: Ultrapowers of Posets

Let μ be a measurable cardinal. Then

- Con($\mathfrak{d} < \mathfrak{a}$);
- If μ is a measurable cardinal such that $\mu < \kappa$, then there is a generic extension in which cofinalities have not been changed and $\mathfrak{b} = \kappa < \mathfrak{s} = \mathfrak{a} = \lambda$.

Template iterations

- Eliminating the measurable in the consistency proof of $\mathfrak{d} < \mathfrak{a}$ led to the development of a powerful forcing technique, known as template iterations.
- In the original template model: $\mathfrak{s} = \aleph_1 < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$. The fact that $\mathfrak{s} = \aleph_1$ in the model follows from the existence of preservation theorems for template iterations, which appear natural generalizations of preservation properties of standard finite support iterations.

Con($\aleph_1 < \theta = \mathfrak{s} < \mathfrak{b} = \mathfrak{d} < \mathfrak{a}$)

- The result is a consequence of a very careful analysis of the local homogeniety properties of posets, which have been obtained via the iteration of non-definable ccc posets along templates.
- In particular, isomorphism-of-names arguments typical for forcing notions obtained via the iteration of definable posets along a template, have been substituted with the notion of a width of a template.

For a linear order L and $x \in L$, let $L_x = \{z \in L : z < x\}$.

Definition

An indexed template is a pair $\langle L, \mathscr{I} \rangle := \langle \mathscr{I}_x \rangle_{x \in L}$ such that L is a linear order, $\mathscr{I}_x \subseteq \mathscr{P}(L_x)$ for all $x \in L$ and

- $\emptyset \in \mathscr{I}_x$,
- $\mathscr{I}_x$ is closed under finite unions and intersections,
- $\mathscr{I}_x \subseteq \mathscr{I}_y$ if $x < y$ and
- $\mathscr{I}(L) := \bigcup_{x \in L} \mathscr{I}_x \cup \{L\}$ is well-founded by the subset relation.

For $x \in L$, let $\hat{\mathscr{I}}_x$ denote the ideal (on L_x) generated by $\mathscr{I}_x$.

Let $A \subseteq L$. Then for each $x \in A$ denote $\mathscr{I}_x \restriction A = \{A \cap H : H \in \mathscr{I}_x\}$. Furthermore, let

- $\bar{\mathscr{I}} \restriction A := \langle \mathscr{I}_x \restriction A \rangle_{x \in A}$ and
- $\mathscr{I}(A) = \bigcup_{x \in A} \mathscr{I}_x \restriction A \cup \{A\}$.

Fact

$\langle A, \bar{\mathscr{I}} \restriction A \rangle$ is an indexed template.

Definition (Rank Function)

Let $\mathrm{Dp} = \mathrm{Dp}^{\bar{\mathscr{I}}} : \mathscr{P}(L) \to \mathbf{ON}$ by $\mathrm{Dp}(X) = \mathrm{rank}_{\mathscr{I}(X)}(X)$.

Lemma

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Lemma

For $X \subseteq Y \subseteq L$,

- $\mathrm{Dp}(X) \leq \mathrm{Dp}(Y)$ *and*
- $\mathrm{Dp}(X) < \mathrm{Dp}(Y)$ *whenever $x \in Y$ and $X \in \mathscr{I}_x \restriction Y$. Furthermore, if $X \subsetneq Y \cap L_x$, then* $\mathrm{Dp}(X \cup \{x\}) < \mathrm{Dp}(Y)$.

Definition

Let θ be an uncountable regular cardinal, $\langle L, \bar{\mathscr{I}} \rangle$ be an indexed template, H, M disjoint sets such that $L = H \cup M$ and for each $x \in M$ let $C_x \in \hat{\mathscr{I}}_x$ of size $< \theta$. Now, for $A \subseteq L$, by recursion on $\mathrm{Dp}(A)$, define

- a poset $\mathbb{P} \restriction A$, and
- for each $x \in A$, and each $B \in \hat{\mathscr{I}}_x \restriction A$, define a $\mathbb{P} \restriction B$-name $\dot{\mathbb{Q}}_x^B$ as follows:

Definition (Continued:)

(a) If $x \in H$ then $\dot{\mathbb{Q}}_x^B$ is a $\mathbb{P}\restriction B$-name for $\mathbb{D}^{V^{\mathbb{P}\restriction B}}$.

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(b) If $x \in M$, for a fixed $\mathbb{P} \restriction C_x$-name $\dot{F}_x$ for a filter base of size $< \theta$, $\dot{\mathbb{Q}}_x^B = \mathbb{M}_{\dot{F}_x}$ if $C_x \subseteq B$ or $\mathbb{1}$ otherwise.

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(c) $p \in \mathbb{P}\restriction A$ iff p is a finite sequence of names such that $\operatorname{dom} p \subseteq A$ and either $p = \varnothing$ or, if $x = \max(\operatorname{dom} p)$, then there exists a $B \in \mathscr{I}_x\restriction A$ such that $p\restriction L_x \in \mathbb{P}\restriction B$ and $p(x)$ is a $\mathbb{P}\restriction B$-name for a condition in $\dot{\mathbb{Q}}_x^B$.

Definition (The extension relation)

For $p, q \in \mathbb{P}\restriction A$, $q \leq_A p$ if $\operatorname{dom} p \subseteq \operatorname{dom} q$ and either

(i) $p = \emptyset$, or

(ii) $x = y$ and there exists a $B \in \mathscr{I}_y\restriction A$ such that $p\restriction L_y, q\restriction L_y \in \mathbb{P}\restriction B$, $q\restriction L_y \leq_B p\restriction L_y$ and $p(y), q(y)$ are $\mathbb{P}\restriction B$-names for conditions in $\dot{\mathbb{Q}}^B_y$ such that $q\restriction L_y$ forces that $q(y) \leq p(y)$, or

(iii) $x := \max(\operatorname{dom} p) < y := \max(\operatorname{dom} q)$ and there exists a $B \in \mathscr{I}_y\restriction A$ such that $p, q\restriction L_y \in \mathbb{P}\restriction B$, $q\restriction L_y \leq_B p$ and $q(y)$ is a $\mathbb{P}\restriction B$-name for a condition in $\dot{\mathbb{Q}}^B_y$.

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(b) *if $p \in \mathbb{P}\restriction A$ and $\dot{x}$ is a $\mathbb{P}\restriction A$-name for a real, then there is $C \subseteq A$ of size $< \theta$ such that $p \in \mathbb{P}\restriction C$ and $\dot{x}$ is a $\mathbb{P}\restriction C$-name.*

Definition (Shelah's template)

Fix uncountable regular cardinals $\theta < \mu < \lambda$. For $\delta \leq \lambda$ define

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linearly ordered by $x < y$ iff one of the following holds:

(i) there is some $k < \min\{|x|, |y|\}$ such that $x \restriction k = y \restriction k$ and $x(k) < y(k)$;

(ii) $x \subseteq y$ and $y(|x|)$ is positive.

(iii) $y \subseteq x$ and $x(|y|)$ is negative.

Definition

The family $\mathscr{I}^\delta$ if formed by finite unions of sets from

$$\{L^\delta_\alpha : \alpha \in \lambda\mu\} \cup \{[x\restriction(|x|-1), x) : x \in L^\delta \text{ is } \theta\text{-relevant}\} \cup \{\{z\} \ : z \in L^\delta\}.$$

Lemma

$\langle L^\delta, \bar{\mathscr{I}}^\delta\rangle$ is an indexed template, where $\mathscr{I}^\delta_x := \{A \in \mathscr{I}^\delta : A \subseteq L^\delta_x\}$.

Assume $\theta^{<\theta} = \theta$ and $\lambda^{<\lambda} = \lambda$.

Main Lemma

Let $\theta^+ < \delta < \lambda$, $\mathbb{P}^\delta$ be an iteration of $\mathbb{D}$ and Mathias-Prikry forcings of size $< \theta$ along L^δ and $\dot{A}$ a $\mathbb{P}\restriction L^\delta$-name of an a.d. family of size $\kappa \in (\theta, \lambda)$.

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(a) $\mathbb{P}^\delta\restriction X = \mathbb{P}^{\delta'}\restriction X$ for all $X \subseteq L^\delta$,

(b) for any $\mathbb{P}^{\delta'}\restriction L^{\delta'}$-name $\dot{F}$ for a filter base of size $< \theta$, there is an $x \in M^{\delta'}$ such that $\Vdash_{\delta'} \dot{F} = \dot{F}_x$ and

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(c) $\mathbb{P}^{\delta'}\restriction L^{\delta'}$ forces that $\dot{A}$ is not mad.

Theorem (F. and Mejia)

There is an iteration $\mathbb{P}^\lambda$ along L^λ that forces $\mathfrak{s} = \theta < \mathfrak{b} = \mathfrak{d} = \mu < \mathfrak{a} = \mathfrak{c} = \lambda$.

- To show $\mathfrak{s} \leq \theta$ we rely on preservation theorems, which are natural (though quite technical) generalizations of preservation theorems known for finite support iterations of ccc posets.
- To show $\mathfrak{s} \geq \theta$ we diagonalize all small (i.e. of size $< \theta$) filter bases. In particular we provide $\mathfrak{p} = \theta$ in the final extension.

Boolean Ultrapowers: Raghavan, Shelah

Under the assumption of the existence of large cardinals, it is consistent that $\mathfrak{b} < \mathfrak{s} < \mathfrak{a}$.

3D-Iterations (?)

The consistency of $\mathfrak{b} < \mathfrak{a} < \mathfrak{s}$ is still open!

Thank you for your attention!