Non-Archimedean Abelian Polish Groups and Their Actions

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- Closed subgroups of S_∞

Non-Archimedean Polish Groups

Theorem (Becker–Kechris)
A Polish group is non-archimedean iff it is isomorphic to a closed subgroup of S_∞.

Non-Archimedean Abelian Polish Groups

Theorem (folklore)

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(iv) G is pro-countable abelian, i.e., there is an inverse system of countable discrete abelian groups

$$\Gamma_0 \leftarrow \Gamma_1 \leftarrow \cdots \leftarrow \Gamma_n \leftarrow \cdots$$

with $\pi_{i,j} : \Gamma_i \to \Gamma_j$, $i > j$, such that G is the inverse limit

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E, F: equivalence relations on Polish spaces X, Y, respectively
$E \leq_B F$, or E is Borel reducible to F:
there is a Borel function $\varphi : X \to Y$ such that

$$xEx' \iff \varphi(x) F \varphi(x')$$

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Theorem (Feldman–Moore)
Any countable Borel equivalence relation is the orbit equivalence relation of a Borel action of a countable discrete group.

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$$xE_{0,n}y \iff \forall m \geq n\ x(m) = y(m)$$

Each $E_{0,n}$ is finite, and E_0 is the increasing union of $E_{0,n}$.

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This can be viewed as the **countable case** of the actions of non-achimedean abelian Polish groups.

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Conjecture If G is any abelian Polish group and E_G^X is essentially countable, then E_G^X is essentially hyperfinite.

Actions of Non-Archimedean Abelian Polish Groups

Theorem (Ding–G.)
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If G is a non-archimedian abelian Polish group and $E \leq_B E_G^X$ is essentially countable, then E is essentially hyperfinite.

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If G is a locally compact non-archimedean abelian Polish group, then E_G^X is essentially hyperfinite.

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This is the **locally compact case** of the actions of non-archimedean abelian Polish groups.

Another deviation: Lower Bounds

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Theorem (Hjorth–Kechris)
If G is a non-archimedean Polish group, then either E_G^X is essentially countable or $E_0^\omega \leq_B E_G^X$.

Tame Groups

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Other work on tame groups were done by Hjorth and recently by Malicki.

Tame Groups

Theorem (Solecki)
A group $\prod H_n$, where each H_n countable discrete abelian, is tame iff both

(1) for all but finitely many n, H_n is torsion, and
(2) for any prime p, for all but finitely many n, the p-component of H_n is of the form $F \oplus \mathbb{Z}(p^\infty)^k$, where F is a finite p-group and $k \in \mathbb{N}$.

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$\mathbb{Z}(p^\infty)$ is the quasicyclic or Prüfer group: the additive mod 1 group of $\left\{ \dfrac{m}{p^l} \mid m \in \mathbb{Z}, l \in \mathbb{N} \right\}$

Wild Groups

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If G involves H and H is wild, then so is G.

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Theorem (Ding–G.)
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Corollary
Let G be a non-archimedean abelian Polish group. If G involves $\mathbb{Z}^\omega$ then $\mathbb{Z}^\omega$ is isomorphic to a closed subgroup of G.

Actions of Tame Groups

For any equivalence relation E on X, the jump of E, E^+, is defined on $X^{\mathbb{N}}$:

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Theorem (Ding–G.)
Let G be a non-archimedean abelian Polish group. If G is tame then $E_G^X \leq_B (E_0^\omega)^{+++}$. In particular, every E_G^X is potentially $\mathbf{\Pi}_6^0$.

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The bound is not sharp.

Actions of Tame Groups

The previous theorem is in contrast with

Theorem (Hjorth)
For every $\alpha < \omega_1$ there is a tame group of the form $\prod H_n$, where each H_n is countable discrete, such that some E_G^X is not pontentially $\mathbf{\Pi}^0_\alpha$.

Structure of Tame Groups

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Definition (Solecki)
A countable group H is p-compact if for any decreasing sequence of subgroups $G_k < \mathbb{Z}(p) \times H$ such that $\pi_1[G_k] = \mathbb{Z}(p)$, where $\pi_1 : \mathbb{Z}(p) \times H \to \mathbb{Z}(p)$ is the projection, we have $\pi_1[\bigcap_k G_k] = \mathbb{Z}(p)$.

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If $\prod H_n$, each H_n countable discrete, is wild, then there is some prime p such that for infinitely many n, H_n is not p-compact.

He showed that the converse is true in the abelian case.

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(v) H is torsion and H does not involve $\mathbb{Z}(p)^{<\omega}$.

Structure of Tame Groups

Lemma (Ding–G.)

The following are also equivalent to p-compactness:

(vi) H is torsion and $H[p] = \{g \in H \mid pg = 0\}$ is finite.

(vii) H does not contain either $\mathbb{Z}$ or $\mathbb{Z}(p)^{<\omega}$ as a subgroup.

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Let H be a countable abelian group and $L \leq H$. Then H is p-compact iff both L and H/L are p-compact.

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Hjorth showed that this is false in the non-abelian case.

Structure of Tame Groups

Theorem (Ding–G.)
Let G be a non-archimedean abelian Polish group. Then G is tame iff there is a nbhd base $\{G_n\}$ of the identity of G consisting of open subgroups such that

(i) for all but finitely many n, G_n/G_{n+1} is torsion, and
(ii) for any prime p, for all but finitely many n, G_n/G_{n+1} contains only finitely many elements of order p.

Structure of Tame Groups

Malicki has recently obtained similar characterizations of tameness. He also showed that tameness coincides with relative tameness.

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Definition A Polish group G is relatively tame if whenever $G \curvearrowright X$ and $G \curvearrowright Y$ are such that E_G^X and E_G^Y are both Borel, we have that the diagonal action $G \curvearrowright X \times Y$ gives Borel $E_G^{X\times Y}$.

Structure of Tame Groups

Theorem (Ding–G.)
Let G be a closed subgroup of $\prod H_n$, where each H_n countable discrete abelian. If G is tame then any group tree $T \subseteq T_G$ has rank $< \omega \cdot 4$.

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Structure of Tame Groups

Consider the class $\mathcal{P}$ of all tame groups of the form $\prod_n H_n$, where each H_n is countable discrete abelian.

Theorem (Ding–G.)
$\mathcal{P}$ has a universal element $H_\infty = \prod_n H_n$, where

$$H_0 = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{<\omega} \oplus \mathbb{Q}^{<\omega},$$

and

$$H_{n+1} = \bigoplus_{0\leq i\leq n} \mathbb{Z}(p_i^\infty) \oplus \bigoplus_{i>n} \mathbb{Z}(p_i^\infty)^{<\omega}.$$

Open Problems

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Conjecture Let G be any tame non-archimedean abelian Polish group. Then every G-orbit equivalence relation is Borel reducible to E_0^ω, and therefore is potentially $\mathbf{\Pi}_3^0$.

Thank you for your attention!