Borel complexity of equivalence relations

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Origins

Theorem (Lusin-Suslin)

Let X be a Polish space, and B be a Borel subset of X. Then we can find a closed set $C \subseteq \omega^\omega$ and $f : C \to B$ bijective continuous.

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- It provides a good subsequence of any $\alpha \in 2^\omega$, viewed as the sequence $(\alpha|l)_{l\in\omega}$ of its initial segments. It can help to prove the

Theorem (Hurewicz)

Let $\mathbb{C} := \{\alpha \in 2^\omega \mid \exists^\infty n \in \omega \ \ \alpha(n) = 1\}$, X be a Polish space, and B be a Borel subset of X. Exactly one of the following holds:

1. *B is in $\mathbf{\Sigma}^0_2$,*
2. *we can find $f : 2^\omega \rightarrow X$ injective continuous such that $\mathbb{C} = f^{-1}(B)$.*

Definition (Debs-Saint Raymond)

• *A partial order relation* R *on* $2^{<\omega}$ *is a* **tree relation** *if, for* $s\in 2^{<\omega}$,

1. $\emptyset\ R\ s$,
2. *the set* $P_R(s):=\{t\in 2^{<\omega} \mid t\ R\ s\}$ *is finite and linearly ordered by* R.

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• $[R]$ *is the set of all infinite* R*-branches.*

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- *Let R be a tree relation.* $[R]\subseteq(2^{<\omega})^\omega$: *induced topology*, *Polish.*
- *Basic clopen sets:* $N^R_s:=\{\gamma\in[R]\mid\gamma\big(h_R(s)\big)=s\}$, $s\in 2^{<\omega}$.

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- *Let R, S be tree relations with $R \subseteq S$. The* **canonical map**

$\Pi : [R] \to [S]$ *is defined by*

$$\Pi(\gamma) := \text{ the unique } S\text{-branch containing } \gamma.$$

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- *Let S be a tree relation. We say that $R\subseteq S$ is* **distinguished** *in S if*

$$\forall s,t,u\in 2^{<\omega}\quad \left.\begin{array}{c} s\ S\ t\ S\ u \\ \\ s\ R\ u \end{array}\right\}\Rightarrow\ s\ R\ t.$$

The representation theorem (continued)

Definition (Debs-Saint Raymond)

• *Let* $\eta < \omega_1$. *A family* $(R^\rho)_{\rho \leq \eta}$ *of tree relations is a* **resolution family** *if*

1. $R^{\rho+1}$ *is a distinguished subtree of* R^ρ, *for each* $\rho < \eta$.
2. $R^\lambda = \bigcap_{\rho < \lambda} R^\rho$, *for each limit ordinal* $\lambda \leq \eta$.

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Theorem (Debs-Saint Raymond)

Let $\eta<\omega_1$, and $P\in\mathbf{\Pi}^0_{\eta+1}([\subseteq])$. Then there is a resolution family $(R^\rho)_{\rho\leq\eta}$ such that

1. $R^0=\subseteq$,
2. *the canonical map* $\Pi:[R^\eta]\rightarrow[R^0]$ *is a continuous bijection with* $\mathbf{\Sigma}^0_{\eta+1}$*-measurable inverse,*
3. *the set* $\Pi^{-1}(P)$ *is a closed subset of* $[R^\eta]$.

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- $\mathbb{C} := \begin{cases} \{\alpha \in 2^\omega \mid \exists^\infty n \in \omega \;\; \alpha(n) = 1\} \text{ if } \mathbf{\Gamma} = \mathbf{\Sigma}^0_2, \\ \{\alpha \in 2^\omega \mid \forall^\infty n \in \omega \;\; \alpha(n) = 0\} \text{ if } \mathbf{\Gamma} = \mathbf{\Pi}^0_2, \\ \{0\} \text{ if } \mathbf{\Gamma} = \mathbf{\Sigma}^0_1, \\ \mathbb{K} \setminus \{0\} \text{ if } \mathbf{\Gamma} = \mathbf{\Pi}^0_1. \end{cases}$

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Theorem (Louveau-Saint Raymond)

Let $\boldsymbol{\Gamma} \neq \check{\boldsymbol{\Gamma}}$ be a Borel class, $\mathbb{K}, \mathbb{C}$ as above, X be a Polish space, and A, B be disjoint analytic subsets of X. Exactly one of the following holds:

1. *A is separable from B by a $\boldsymbol{\Gamma}$ set,*
2. *we can find $f : \mathbb{K} \to X$ injective continuous such that $\mathbb{C} \subseteq f^{-1}(A)$ and $\neg\mathbb{C} \subseteq f^{-1}(B)$.*

Extensions of the Louveau-Saint Raymond theorem

Theorem 1

Let $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$ be a Borel class, $\mathbb{K}, \mathbb{C}$ as above, X be an analytic space, and A, B be disjoint analytic relations on X, A having sections in $\mathbf{\Gamma}$. Exactly one of the following holds:

1. *A is separable from B by a $\mathbf{\Gamma}$ set,*
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• Let $2\leq\eta<\omega_1$, and $\mathbb{C}\in\mathbf{\Pi}^0_{\eta+1}([\subseteq])$. The representation theorem gives $(R^\rho)_{\rho\leq\eta}$ such that $\Pi^{-1}(\mathbb{C})$ is a closed subset of $[R^\eta]$. We can find $\mathbb{I}\subseteq\omega$ and $(s_n)_{n\in\mathbb{I}}$ such that $\neg\Pi^{-1}(\mathbb{C})$ is the disjoint union of the $N^{R^\eta}_{s_n}$'s. We set $\mathbb{C}_n:=\Pi[N^{R^\eta}_{s_n}]$, so that $(\mathbb{C}_n)_{n\in\mathbb{I}}$ is a partition of $\neg\mathbb{C}$ into $\mathbf{\Delta}^0_{\eta+1}$ sets.

Theorem 2

Let $2 \leq \eta < \omega_1$, $\mathbb{C} \in \mathbf{\Pi}^0_{\eta+1}([\subseteq])$, X be an analytic space, A be an analytic subset of X, and $(D_n)_{n\in\omega}$ be a sequence of pairwise disjoint analytic subsets of X such that A is both disjoint from $\bigcup_{n\in\omega} D_n$ and separable from any of the D_n's by a $\mathbf{\Sigma}^0_{\eta+1}$ set. One of the following holds:

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If moreover $\mathbb{C} \notin \mathbf{\Sigma}^0_{\eta+1}$, then this is a dichotomy.

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- There are versions of this for $\eta \leq 1$ and limit ordinals.

Another application of the representation theorem

Theorem 3

Let $\boldsymbol{\Gamma} \neq \check{\boldsymbol{\Gamma}}$ be a Borel class of rank $3 \leq \xi < \omega_1^{CK}$, $\mathbb{C} \in \Delta^1_1 \cap \check{\boldsymbol{\Gamma}}(2^\omega)$, and R be a Δ^1_1 relation on 2^ω with F_σ vertical sections. We assume that there is a Σ^1_1 subset V of 2^ω disjoint from $\Delta^1_1 \cap 2^\omega$ such that $R \cap V^2$ is GH^2-meager in V^2, and $V \cap \mathbb{C}$ is not separable from $V \setminus \mathbb{C}$ by a set in $\boldsymbol{\Gamma}$. Then there is $f : 2^\omega \to 2^\omega$ injective continuous such that $\mathbb{C} = f^{-1}(\mathbb{C})$ and $\big(f(\alpha), f(\beta)\big) \notin R$ if $\alpha \neq \beta$.

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Corollary

Let $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$ be a Borel class of rank at least three, $\mathbb{C}$ in $\check{\mathbf{\Gamma}}(2^\omega) \setminus \mathbf{\Gamma}$, and R be a Borel relation on 2^ω with countable vertical sections. Then we can find $f : 2^\omega \to 2^\omega$ injective continuous such that $\mathbb{C} = f^{-1}(\mathbb{C})$ and $(f(\alpha), f(\beta)) \notin R$ if $\alpha \neq \beta$.

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- This cannot be extended to lower levels.

Borel equivalence relations

• If $E \subseteq X^2$, $F \subseteq Y^2$, then (X, E) $\sqsubseteq_c$ (Y, F) means that there is $f : X \to Y$ injective continuous with $\big(f(x), f(x')\big) \in F$ iff $(x, x') \in E$.

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Questions

1. *When is a Borel equivalence relation* $\mathbf{\Sigma}^0_\xi$ *(or* $\mathbf{\Pi}^0_\xi$*)?*
2. *When are the classes of a Borel equivalence relation* $\mathbf{\Sigma}^0_\xi$ *(or* $\mathbf{\Pi}^0_\xi$*)?*

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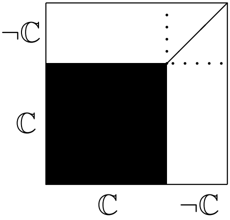
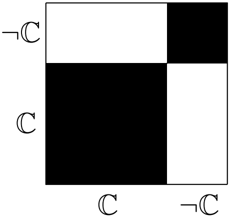
- We define equivalence relations on $\mathbb{K}$ by

$$\begin{cases} x\mathbb{E}_0^{\boldsymbol{\Gamma}}y \Leftrightarrow (x,y\in\mathbb{C})\vee(x=y), \\ x\mathbb{E}_1^{\boldsymbol{\Gamma}}y \Leftrightarrow (x,y\in\mathbb{C})\vee(x,y\notin\mathbb{C}), \\ x\mathbb{E}_2^{\boldsymbol{\Sigma}^0_\xi}y \Leftrightarrow (x,y\in\mathbb{C})\vee(\exists n\in\omega\ \ x,y\in\mathbb{C}_n). \end{cases}$$

Some examples



$\mathbb{E}_0^{\Gamma}$



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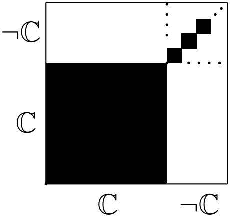

$\mathbb{E}_2^{\mathbf{\Sigma}^0_\xi}$

- We set

$$\mathcal{A}^{\boldsymbol{\Gamma}} := \begin{cases} \{(\mathbb{K}, \mathbb{E}_0^{\boldsymbol{\Gamma}})\} \text{ if } \boldsymbol{\Gamma} = \boldsymbol{\Pi}_1^0, \\ \{(\mathbb{K}, \mathbb{E}_n^{\boldsymbol{\Gamma}}) \mid n \leq 1\} \text{ if } \boldsymbol{\Gamma} \in \{\boldsymbol{\Sigma}_\xi^0 \mid \xi \leq 2\} \cup \{\boldsymbol{\Pi}_\xi^0 \mid \xi \geq 2\}, \\ \{(\mathbb{K}, \mathbb{E}_n^{\boldsymbol{\Gamma}}) \mid n \leq 2\} \text{ if } \boldsymbol{\Gamma} \in \{\boldsymbol{\Sigma}_\xi^0 \mid \xi \geq 3\}. \end{cases}$$

The first two ranks

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Theorem

Let $\boldsymbol{\Gamma} \neq \check{\boldsymbol{\Gamma}}$ be a Borel class of rank at most two, $\mathbb{K}, \mathbb{C}$ as above, X be an analytic space, and E be a Borel equivalence relation on X. Exactly one of the following holds:

1. *the equivalence classes of E are in $\boldsymbol{\Gamma}$,*
2. *there is $(\mathbb{X}, \mathbb{E}) \in \mathcal{A}^{\boldsymbol{\Gamma}}$ such that $(\mathbb{X}, \mathbb{E}) \sqsubseteq_c (X, E)$.*

Moreover, $\mathcal{A}^{\boldsymbol{\Gamma}}$ is a $\leq_c$-antichain (and thus a $\sqsubseteq_c$ and a $\leq_c$-antichain basis).

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Conjecture

This holds for any Borel class $\mathbf{\Gamma}$.

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Theorem

Let $1 \leq \xi < \omega_1$, $\mathbb{K}, \mathbb{C} \in \mathbf{\Sigma}^0_\xi$ *as above,* X *be an analytic space, and* E *be a Borel equivalence relation on* X *with countably many classes. Exactly one of the following holds:*

1. *the equivalence classes of* E *are* $\mathbf{\Pi}^0_\xi$,
2. $(\mathbb{K}, \mathbb{E}_1^{\mathbf{\Pi}^0_\xi}) \sqsubseteq_c (X, E)$.

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Let $1 \leq \xi < \omega_1$, $\mathbb{K}, \mathbb{C} \in \mathbf{\Pi}^0_\xi$ as above, X be an analytic space, and E be a Borel equivalence relation on X with countably many classes. Exactly one of the following holds:

1. *the equivalence classes of E are $\mathbf{\Sigma}^0_\xi$,*
2. *there is $n \in \{1, 2\}$ such that $(\mathbb{K}, \mathbb{E}_n^{\mathbf{\Sigma}^0_\xi}) \sqsubseteq_c (X, E)$.*

Moreover, $\{(\mathbb{K}, \mathbb{E}_n^{\mathbf{\Sigma}^0_\xi}) \mid 1 \leq n \leq 2\}$ is a $\leq_c$-antichain.

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- In the next result, we assume that $\mathbb{C}\cap N_s\in\check{\mathbf{\Gamma}}(N_s)\setminus\mathbf{\Gamma}$ for each $s\in 2^{<\omega}$ if the rank of $\mathbf{\Gamma}$ is at least two (assumption (*)).

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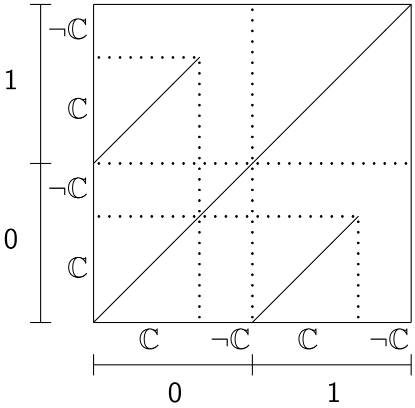
Theorem

Let $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$ be a Borel class, $\mathbb{K}, \mathbb{C}$ as above satisfying (*), X be an analytic space, and E be a Borel equivalence relation on X whose classes are in $\mathbf{\Gamma}$. Exactly one of the following holds:

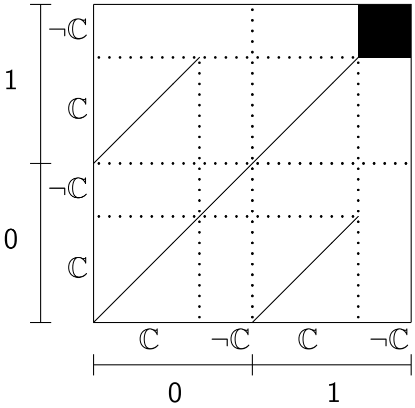
1. *E is in $\mathbf{\Gamma}$,*
2. *there is a Borel equivalence relation $\mathbb{E}$ on $\mathbb{H} := 2 \times \mathbb{K}$ such that $\{((0,\alpha),(1,\alpha)) \mid \alpha \in \mathbb{C}\} \subseteq \mathbb{E}$, $\{((0,\alpha),(1,\alpha)) \mid \alpha \notin \mathbb{C}\} \subseteq \neg\mathbb{E}$ and $(\mathbb{H}, \mathbb{E}) \sqsubseteq_c (X, E)$.*

Some other examples



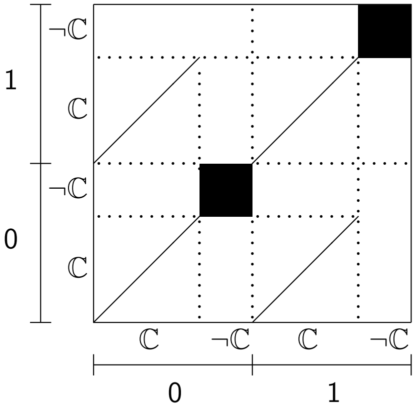
$\mathbb{E}_3^{\Gamma}$

Some other examples



$\mathbb{E}_4^{\Gamma}$

Some other examples



$\mathbb{E}_5^{\Gamma}$

Equivalence relations of rank at most two

- We set

$$\mathcal{B}^{\boldsymbol{\Gamma}} := \mathcal{A}^{\boldsymbol{\Gamma}} \cup \begin{cases} \emptyset \text{ if } \boldsymbol{\Gamma} = \boldsymbol{\Sigma}^0_1, \\ \{(\mathbb{H}, \mathbb{E}^{\boldsymbol{\Gamma}}_3)\} \text{ if } \boldsymbol{\Gamma} = \boldsymbol{\Pi}^0_1, \\ \{(\mathbb{H}, \mathbb{E}^{\boldsymbol{\Gamma}}_n) \mid 3 \leq n \leq 5\} \text{ if the rank of } \boldsymbol{\Gamma} \text{ is two.} \end{cases}$$

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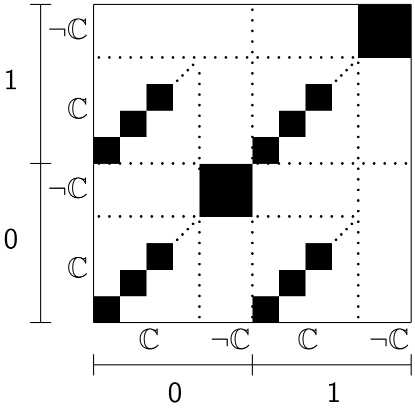
Theorem

Let $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$ be a Borel class of rank at most two, $\mathbb{K}, \mathbb{C}$ as above, X be an analytic space, and E be a Borel equivalence relation on X. Exactly one of the following holds:

1. *E is in $\mathbf{\Gamma}$,*
2. *there is $(\mathbb{X}, \mathbb{E}) \in \mathcal{B}^{\mathbf{\Gamma}}$ such that $(\mathbb{X}, \mathbb{E}) \sqsubseteq_c (X, E)$.*

Moreover, $\mathcal{B}^{\mathbf{\Gamma}}$ is a $\leq_c$-antichain.

Some other example



$\mathbb{E}_8^{\mathbf{\Pi}^0_\xi}$

Equivalence relations with countably many classes

- The following is an application of Theorems 1 and 2.

Equivalence relations with countably many classes

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Theorem

Let $3 \leq \xi < \omega_1$, $\mathbb{K}, \mathbb{C} \in \mathbf{\Sigma}^0_\xi$ as above satisfying (*), X be an analytic space, and E be a Borel equivalence relation on X with countably many classes. Exactly one of the following holds:

1. *E is in $\mathbf{\Pi}^0_\xi$,*
2. *there is $(\mathbb{X}, \mathbb{E}) \in \{(\mathbb{K}, \mathbb{E}_1^{\mathbf{\Pi}^0_\xi}), (\mathbb{H}, \mathbb{E}_8^{\mathbf{\Pi}^0_\xi})\}$ such that $(\mathbb{X}, \mathbb{E}) \sqsubseteq_c (X, E)$.*

Moreover, $\{(\mathbb{K}, \mathbb{E}_1^{\mathbf{\Pi}^0_\xi}), (\mathbb{H}, \mathbb{E}_8^{\mathbf{\Pi}^0_\xi})\}$ is a $\leq_c$-antichain.

Countable equivalence relations

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Countable equivalence relations

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Theorem

Let $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$ be a Borel class of rank at least three, $\mathbb{C}$ as above satisfying (*), X be an analytic space, and E be a Borel equivalence relation on X with F_σ classes. Exactly one of the following holds:

1. *E is in $\mathbf{\Gamma}$,*
2. $(\mathbb{H}, \mathbb{E}_3^{\mathbf{\Gamma}}) \sqsubseteq_c (X, E)$.

Countable equivalence relations

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Theorem

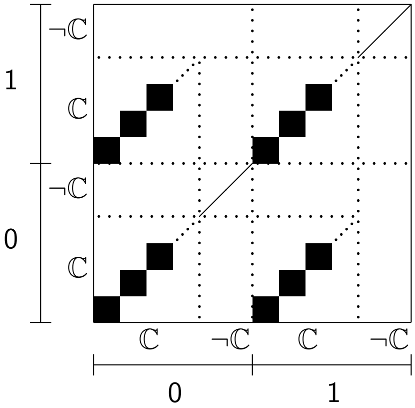
Let $\mathbf{\Gamma} \neq \check{\mathbf{\Gamma}}$ be a Borel class of rank at least three, $\mathbb{C}$ as above satisfying (*), X be an analytic space, and E be a Borel equivalence relation on X with F_σ classes. Exactly one of the following holds:

1. *E is in $\mathbf{\Gamma}$,*
2. *$(\mathbb{H}, \mathbb{E}_3^{\mathbf{\Gamma}}) \sqsubseteq_c (X, E)$.*

- First levels: replace $\{(\mathbb{H}, \mathbb{E}_3^{\mathbf{\Gamma}})\}$ with

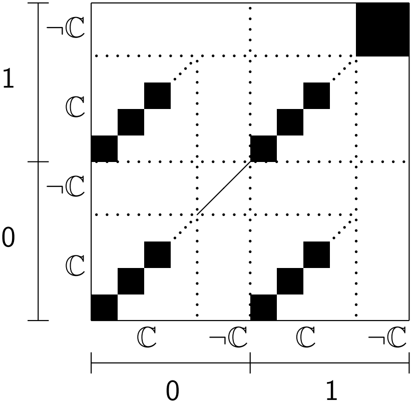
$$\begin{cases} \{(\mathbb{K}, \mathbb{E}_0^{\mathbf{\Gamma}}), (\mathbb{K}, \mathbb{E}_1^{\mathbf{\Gamma}})\} \text{ if } \mathbf{\Gamma} = \mathbf{\Sigma}_1^0, \\ \{(\mathbb{K}, \mathbb{E}_0^{\mathbf{\Gamma}}), (\mathbb{H}, \mathbb{E}_3^{\mathbf{\Gamma}})\} \text{ if } \mathbf{\Gamma} \in \{\mathbf{\Pi}_1^0, \mathbf{\Pi}_2^0\}, \\ \{(\mathbb{H}, \mathbb{E}_n^{\mathbf{\Gamma}}) \mid 3 \leq n \leq 5\} \text{ if } \mathbf{\Gamma} = \mathbf{\Sigma}_2^0. \end{cases}$$

Some other examples



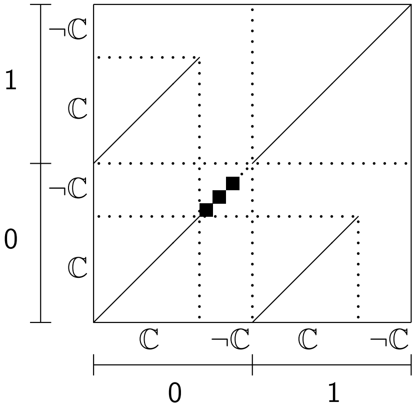
$\mathbb{E}_6^{\mathbf{\Pi}^0_\xi}$

Some other examples



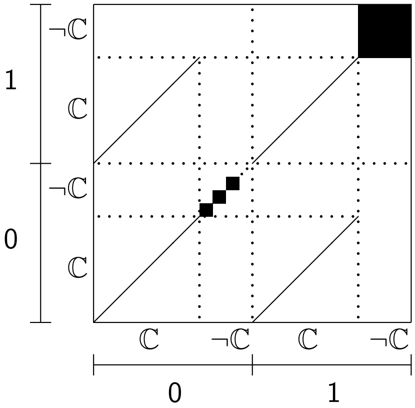
$\mathbb{E}_7^{\mathbf{\Pi}^0_\xi}$

Some other examples



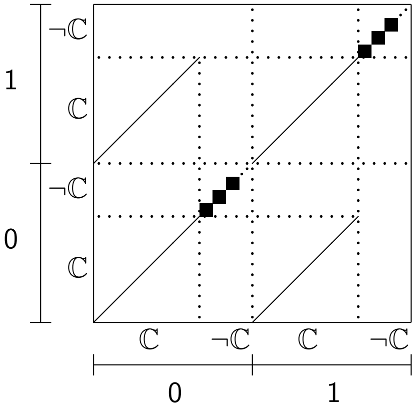
$\mathbb{E}_6^{\boldsymbol{\Sigma}^0_\xi}$

Some other examples



$\mathbb{E}_7^{\mathbf{\Sigma}^0_\xi}$

Some other examples



$\mathbb{E}_8^{\mathbf{\Sigma}^0_\xi}$

A general conjecture

• We set $\mathcal{B}^{\mathbf{\Gamma}} := \mathcal{A}^{\mathbf{\Gamma}} \cup \{(\mathbb{H}, \mathbb{E}_n^{\mathbf{\Gamma}}) \mid 3 \leq n \leq 8\}$ if the rank of $\mathbf{\Gamma}$ is at least three.

A general conjecture

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Theorem

Let $\boldsymbol{\Gamma} \neq \check{\boldsymbol{\Gamma}}$ be a Borel class, and $\mathbb{K}, \mathbb{C}$ as above satisfying (*). Then $\mathcal{B}^{\boldsymbol{\Gamma}}$ is a $\leq_c$-antichain made of non-$\boldsymbol{\Gamma}$ Borel equivalence relations.

A general conjecture

• We set $\mathcal{B}^{\Gamma} := \mathcal{A}^{\Gamma} \cup \{(\mathbb{H}, \mathbb{E}_n^{\Gamma}) \mid 3 \leq n \leq 8\}$ if the rank of Γ is at least three.

Theorem

Let $\Gamma \neq \check{\Gamma}$ be a Borel class, and $\mathbb{K}, \mathbb{C}$ as above satisfying (*). Then $\mathcal{B}^{\Gamma}$ is a $\leq_c$-antichain made of non-Γ Borel equivalence relations.

Conjecture

Let $\Gamma \neq \check{\Gamma}$ be a Borel class of rank at least three, $\mathbb{K}, \mathbb{C}$ as above satisfying (*), X be an analytic space, and E be a Borel equivalence relation on X. One of the following holds:

1. *E is in Γ,*
2. *there is $(\mathbb{X}, \mathbb{E}) \in \mathcal{B}^{\Gamma}$ such that $(\mathbb{X}, \mathbb{E}) \sqsubseteq_c (X, E)$.*