Local Ramsey Spaces in Matet Forcing Extensions

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CIRM, 14th International Workshop in Set Theory

October 13, 2017

The partial order $((\mathrm{FIN})^\omega, \sqsubseteq)$

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For $\bar{a}$, $\bar{b} \in (\mathsf{FIN})^\omega$, we call $\bar{b}$ a condensation of $\bar{a}$, short $\bar{b} \sqsubseteq \bar{a}$, if any member of $\bar{b}$ (strictly speaking "member of $\mathrm{range}(\bar{b})$") is a finite union of members of $\bar{a}$.

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Definition

For $\bar{b} \in (\mathsf{FIN})^\omega$ and $s \in \mathsf{FIN}$, we write $(\bar{b} \text{ past } s)$ for the part of the sequence $\bar{b}$ that starts after the maximum of s.

We write $\bar{b} \sqsubseteq^* \bar{a}$ if for some $n \in \omega$, $(\bar{b} \text{ past } \{n\}) \sqsubseteq \bar{a}$.

Definition

Let $\langle \bar{a}_n \mid n \in \omega \rangle$ be $\sqsubseteq$-descending. $\bar{b}$ is a diagonal lower bound if

$$(\forall n \in \omega)(\bar{b} \text{ past } b_{n-1}) \sqsubseteq \bar{a}_n.$$

Diagonal lower bounds and FU-sets

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Definition

Let $X \subseteq$ FIN. We let $\mathrm{FU}(X)$ be the set of unions of finitely many members of X.

Definition

A set $\mathscr{H} \subseteq (\mathrm{FIN})^\omega$ is called a Matet-adequate family if the following holds:

1. $\mathscr{H}$ is closed $\sqsubseteq^*$-upwards.
2. Every $\sqsubseteq$-descending ω-sequence of members of $\mathscr{H}$ has a diagonal lower bound in $\mathscr{H}$.
3. $\mathscr{H}$ has the Hindman property: If $A \in \mathscr{H}$ and FIN is partitioned into two pieces then there is some $\bar{b} \sqsubseteq \bar{a}$, $\bar{b} \in \mathscr{H}$ such that $\mathrm{FU}(\bar{b})$ is a subset of a single piece of the partition.

$(\mathsf{FIN})^\omega$ (Hindman)

Theorem

(Taylor) Let $\bar{a} \in (\mathsf{FIN})^\omega$, $n \in \omega$. *If* $c\colon [\mathrm{FU}(\bar{a})]^n_< \to \{0,1\}$. *Then there is a* $\bar{b} \sqsubseteq \bar{a}$ *such that* $[\mathrm{FU}(\bar{b})]^n_<$ *is monochromatic.*

Examples of Matet-adequate families

$(\mathsf{FIN})^\omega$ (Hindman)

Theorem

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same holds in any Matet-adequate family.

Centred examples

Any Milliken-Taylor ultrafilter $\mathscr{U}$.

Definition

A Milliken-Taylor ultrafilter is an ultrafilter over FIN with the following properties:

1. It has a basis of sets of the form $\mathrm{FU}(\bar{a})$ with $\bar{a} \in (\mathsf{FIN})^\omega$,
2. each $\sqsubseteq$-descending sequence of members of $\mathscr{U}$ has a $\sqsubseteq^*$-lower bound,
3. and it has the Hindman-property.

The Hindman property follows from the first two properties.

Milliken-Taylor ultrafilters are also called stable ordered-union ultrafilters.

Existence?

Under CH, MA, $\mathrm{cov}(\mathcal{M}) = \mathfrak{c}$ or in the Sacks model there is an Milliken-Taylor ultrafilter. Eisworth (2002), Yuan Yuan Zheng (2017), Fernández-Breton and Hrušák(2017).

Under NCF, so for example in the Matet model, there is none.

The issue of P-points. $\mathfrak{d} = \mathfrak{c}$. No P-points in the Silver model.

Properties

If $\mathscr{H}$ is Matet-adequate then it has solutions to colorings as in the Taylor theorem.

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We are interested in $\mathbb{M}(\mathscr{U})$, $\mathscr{U}$ and Milliken-Taylor ultrafilter.

Connections between FIN and ω

$\min[\bar{a}] = \{\min(a_n) \mid n \in \omega\}$ for $\bar{a} \in (\mathsf{FIN})^\omega$.

$\min[X] = \{\min(x) \mid x \in X\}$ for $X \subseteq \mathsf{FIN}$.

$\hat{\min}(\mathscr{F}) = \{\min[X] \mid X \in \mathscr{F}\}$ for $\mathscr{F} \subseteq \mathcal{P}(\mathsf{FIN})$.

Blass showed that for an Milliken-Taylor ultrafilter $\mathscr{U}$ the projections $\hat{\min}(\mathscr{U})$ and $\hat{\max}(\mathscr{U})$ are non-nearly coherent Ramsey ultrafilters over ω.

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$(\mathsf{FIN}, \cup)$ is a partial semigroup: We define $s \cup t$ only for $s < t$. The associative partial binary operation $\cup$ lifts to $\beta(\mathsf{FIN})$, the space of min-unbounded ultrafilters over FIN, as follows (and we write $\dot\cup$ for the lifted operation):

$$\mathscr{U}_1 \dot\cup \mathscr{U}_2 = \{X \subseteq \mathsf{FIN} \mid \text{ for } \mathscr{U}_1\text{-most } s, \text{ for } \mathscr{U}_2\text{-most } t,\ s \cup t \in X\}$$

With the topology

$$\{\{\mathscr{U} \mid X \in \mathscr{U}\} \mid X \subseteq \mathsf{FIN}\}$$

it is a compact zero-dimensional Hausdorf space. With the topology $(\beta\mathsf{FIN}, \dot\cup)$ is a semitopological semigroup.

Existence, even with a large starting point

Lemma

(Ellis) For each closed subsemigroup $\mathscr{H}$ of βFIN there is an idempotent ultrafilter.

Lemma

(Eisworth) Let $\mathscr{F}$ be an ordered-union filter. There is a min-unbounded idempotent ultrafilter $\mathscr{U} \in \beta$FIN that extends $\mathscr{F}$.

Motivating questions

Let $n \in \omega \setminus \{0, 1\}$. Is it consistent relative to ZFC that there is a model with n near coherence classes of ultrafilters?

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Let $n \in \omega \setminus \{0, 1\}$. Is it consistent relative to ZFC that there is a model with n near coherence classes of ultrafilters?

Necessary: $\mathfrak{u} < \mathfrak{d}$. No or few Cohen reals.
Try to build a model with a small P-point and an "inhomogeneous" continuum.

Definition

In the Matet forcing, $\mathbb{M}$, the conditions are pairs $(s, \bar{c})$ such that $s \in$ FIN and $\bar{c} \in (\text{FIN})^\omega$ and $s < c_0$. The forcing order is $(t, \bar{d}) \leq (s, \bar{c})$ (recall the stronger condition is the smaller one) iff $s \subseteq t$ and $t \setminus a$ is a concatenation of finitely many of the c_n and $\bar{d}$ is a condensation of $\bar{c}$.

Matet forcing

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Definition

Let $\mathscr{H}$ be a Matet-adequate family. In the subforcing $\mathbb{M}(\mathscr{H})$ the second components of the conditions are taken from $\mathscr{H}$.

Ramsey-theoretic computations in the $M(\mathscr{U})$-extension

We write and $\mathrm{set}(\bar{a}) = \bigcup\{a_n \mid n \in \omega\}$. The forcing $\mathbb{M}(\mathscr{U})$ diagonalises ("shoots a real through") $\{\mathrm{set}(\bar{a}) \mid \bar{a} \in \mathscr{C}\}$, namely the generic real

$$\mu_G := \bigcup\{s \mid \exists\bar{c} \mid (s, \bar{c}) \in G\}$$

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Definition

(1) Let $\bar{a} \in (\mathsf{FIN})^\omega$ and $\mu \in [\omega]^\omega$. $\bar{a} \restriction \mu = \langle a_n \mid n \in \omega, a_n \subseteq \mu \rangle$. Note, we do not take those a_n with $a_n \cap \mu \neq \emptyset$ that are not subsets of μ.

(2) Let $\mathscr{U} \subseteq (\mathsf{FIN})^\omega$ and $\mu \in [\omega]^\omega$. $\mathscr{U} \restriction \mu = \operatorname{fil}(\{\bar{a} \restriction \mu \mid \bar{a} \in \mathscr{U}\})$.

Preserving a P-point

Definition

Let $\mathscr{H}$ be a Matet-adequate family and let $\mathscr{E}$ be a P-point. We say $\mathscr{H}$ avoids $\mathscr{E}$ if for any $\bar{a} \in \mathscr{H}$ and finite-to-one f there is an $E \in \mathscr{E}$ and an $\bar{b} \in \mathscr{H}$ such that $\bar{b} \sqsubseteq \bar{a}$ and $f[E] \cap f[\mathrm{set}(\bar{b})] = \emptyset$.

Theorem

(Eisworth) If $\mathscr{U}$ avoids $\mathscr{E}$ then in $\mathbf{V}^{\mathbb{M}(\mathscr{U})}$ the P-point $\mathscr{E}$ is preserved, i.e. $\{Y \mid (\exists E \in \mathscr{E}) Y \supseteq X\}$ is an ultrafilter.

Extending $(\mathscr{U} \restriction \mu)$ in $\mathbf{V}^{\mathbb{M}(\mathscr{U})}$

Theorem

(M., 2017) After forcing with $\mathbb{M}(\mathscr{U})$, $(\mathscr{U} \restriction \mu)^+$ is a Matet-adequate family that avoids $\mathscr{E}$.

Corollary

Let $\mathscr{E}$ be a P-point and $\mathscr{U}$ be a Milliken-Taylor ultrafilter with $\Phi(\mathscr{U}) \not\leq_{\mathrm{RB}} \mathscr{E}$. Assume CH. Then in the forcing extension by $\mathbb{M}(\mathscr{U})$ the Milliken-Taylor ultrafilter $\mathscr{U}$ is destroyed and can be completed to an Milliken-Taylor ultrafilter $\mathscr{U}^{\mathrm{ext}} \supseteq \mathscr{U}$ with $\Phi(\mathscr{U}^{\mathrm{ext}}) \not\leq_{\mathrm{RB}} \mathscr{E}$.

Names for diagonal lower bounds

Lemma

Let $\mathscr{U}$ *be an Milliken-Taylor ultrafilter,* $\mathscr{E}$ *be a* P*-point,* $\Phi(\mathscr{U}) \not\leq_{\mathrm{RB}} \mathscr{E}$*. Let* $\mathbb{Q} = \mathbb{M}(\mathscr{U})$ *and let* μ *be the name for the generic real. Let* $\langle X_n \mid n \in \omega \rangle$ *be a sequence of* $\mathbb{Q}$*-names for elements of* $(\mathsf{FIN})^\omega$ *such that*

$$\mathbb{Q} \Vdash (\forall n \in \omega)(X_n \in (\mathscr{U} \restriction \mu)^+ \wedge X_{n+1} \sqsubseteq X_n).$$

Then

$$\begin{aligned} \underset{\sim}{D} = \{\langle \check{t}, (s, \bar{a})\rangle \mid (s, \bar{a}) \in \mathbb{Q}_\alpha \text{ is neat for } \underset{\sim}{\bar{X}} \text{ and} \\ \exists t_0 < t_1 < \dots < t_k = t \\ (s, \bar{a}) \Vdash t_0 = \min(\underset{\sim}{X}_0 \restriction \mu)) \wedge \\ \bigwedge_{i<k} t_{i+1} = \min((X_{\underset{\sim}{\max}(t_i)+1} \restriction \mu) \text{ past } t_i)\} \end{aligned}$$

fulfils

$$\mathbb{Q} \Vdash \underset{\sim}{D} \in (\mathscr{U} \restriction \mu)^+ \wedge \underset{\sim}{D} \sqsubseteq \underset{\sim}{X}_0 \wedge (\forall t \in \underset{\sim}{D})(\underset{\sim}{D} \text{ past } t \sqsubseteq X_{\underset{\sim}{\max}(t)+1}).$$

The Hindman property of the positive sets

The proof of the Hindman property includes again a proof that positive diagonal lower bounds exist.

Lemma

In $\mathbf{V}^{\mathbb{M}(\mathscr{U})}$, $(\mathscr{U} \restriction \mu)^+$ has the Hindman property.

For the proof of this lemma, we adapt a proof of a theorem of Eisworth. This says

Theorem

(Eisworth) Let $\mathscr{F}$ be an ordered-union filter generated by $< \mathrm{cov}(\mathcal{B})$ sets and let c be a partition of FIN *into finite sets. Then there is an $\bar{a} \in \mathscr{F}^+$ such that $\mathrm{FU}(\bar{a})$ is included in one piece of the partition.*

At a crucial point in the proof a Cohen real provides a name. We show that also a Matet-real can be used. For this we outline the proof. We recall the Galvin-Glazer technique.

Technique

Let c be a name for a partition of $(\bar{b}_0 \in \mathscr{U} \restriction \mu)^+$ into finitely many pieces and let $\bar{b}_n$ be a $\sqsubseteq$-descending sequence of elements $\bar{b}_n \in (\mathscr{U} \restriction \mu)^+$. Let $\mathscr{U}^i$ be such that

$$\mathbb{M}(\mathscr{U}) \Vdash \mathscr{U}^i \supseteq \left((\mathscr{U} \restriction \mu) \cup \{\bar{b}_n \mid n \in \omega\}\right) \wedge \mathscr{U}^i \dot{\cup} \mathscr{U}^i = \mathscr{U}^i.$$

For $X \subseteq \mathsf{FIN}$ and $t \in \mathsf{FIN}$ we set

$$X \ominus t = \{s \mid s \cup t \in X\}$$

If $\mathscr{U}^i$ is idempotent then for each $X \in \mathscr{U}^i$ the set $\{t \mid X \ominus t \in \mathscr{U}^i\}$ is in $\mathscr{U}^i$.

We define for $n \in \omega$ names X_n and d_n and $p_n = (s_n, \bar{a}_n)$ with the following rules:

A name for a monochromatic set in $(\mathscr{U} \upharpoonright \mu)^+$

(1) $p_0 \Vdash X_0$ is the piece of the partition c of $\mathrm{FU}(\bar{b}_0)$ that is in $\mathscr{U}^i$.

(2) $p_{n+1} = (s_{n+1}, \bar{a}_{n+1}) \Vdash d_n$ is the $\leq_{\mathrm{lex},\mathsf{FIN}}$-least element of

$$\{d \in X_n \cap \mathrm{FU}(\{a_{n,k} \mid k \in \omega\}) \cap \mathrm{FU}(\bar{b}_n) \mid X_n \ominus d \in \mathscr{U}^i \text{ and } \min(d) > \max(d_i) \text{ for } i < n\}$$

(3) $p_{n+1} \Vdash X_{n+1} = X_n \cap (X_n \ominus d_n)$.

Since $\mathscr{U}^i$ is idempotent, the set in (2) is in $\mathscr{U}^i$.

We ensure with colouring of the pure part of p_n that there is a lower bound of $\langle p_n \mid n < \omega \rangle$ that forces only the existence of the d_n, without the pinning down.

Iterating with countable support

$\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{M}(\mathscr{U}_\beta) \mid \beta < \omega_2, \alpha \leq \omega_2 \rangle$ with countable support and

$$\mathbb{P}_\beta \Vdash \mathscr{U}_\beta \supseteq \bigcup \{ (\mathscr{U}_\gamma \restriction \mu_\gamma) \mid \gamma < \beta \}$$

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Preservation theorem.

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Preservation theorem.

In $\mathbf{V}^{\mathbb{P}}$, there are at least three near coherence classes of filters.

$\hat{\min}(\mathscr{U}_{\omega_2}) = \{\min[\bar{a}] \mid \bar{a} \in \mathscr{U}_{\omega_2}\}$

$\hat{\max}(\mathscr{U}_{\omega_2}) = \{\max[\bar{a}] \mid \bar{a} \in \mathscr{U}_{\omega_2}\}$

$\mathscr{E}$.

Question

Can $\diamondsuit(S^{\aleph_2}_{\aleph_0})$ be used to arrange that there are just this three classes?