Local Ramsey Spaces in Matet Forcing Extensions

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The partial order \(((\text{FIN})^\omega, \sqsubseteq)\)

**Definition**

The set of non-empty finite subsets of \(\omega\) is denoted by \(\text{FIN}\). \((\text{FIN})^\omega\) is the set of block sequences.
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For \(\bar{a}, \bar{b} \in (\text{FIN})^\omega\), we call \(\bar{b}\) a **condensation of** \(\bar{a}\), short \(\bar{b} \sqsubseteq \bar{a}\), if any member of \(\bar{b}\) (strictly speaking “member of \(\text{range}(\bar{b})\)”) is a finite union of members of \(\bar{a}\).
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For \(\bar{b} \in (\text{FIN})^\omega\) and \(s \in \text{FIN}\), we write \((\bar{b} \text{ past } s)\) for the part of the sequence \(\bar{b}\) that starts after the maximum of \(s\).

We write \(\bar{b} \sqsubseteq^* \bar{a}\) if for some \(n \in \omega\), \((\bar{b} \text{ past } \{n\}) \sqsubseteq \bar{a}\).
Definition
Let $\langle \bar{a}_n \mid n \in \omega \rangle$ be $\sqsubseteq$-descending. $\bar{b}$ is a diagonal lower bound if

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Definition
A set $\mathcal{H} \subseteq (\text{FIN})^\omega$ is called a Matet-adequate family if the following holds:

1. $\mathcal{H}$ is closed $\sqsubseteq^*$-upwards.
2. Every $\sqsubseteq$-descending $\omega$-sequence of members of $\mathcal{H}$ has a diagonal lower bound in $\mathcal{H}$.
3. $\mathcal{H}$ has the Hindman property: If $A \in \mathcal{H}$ and $\text{FIN}$ is partitioned into two pieces then there is some $\bar{b} \sqsubseteq \bar{a}$, $\bar{b} \in \mathcal{H}$ such that $FU(\bar{b})$ is a subset of a single piece of the partition.
Examples of Matet-adequate families

(FIN)ω (Hindman)

Theorem

(Taylor) Let $\bar{a} \in (\text{FIN})^\omega$, $n \in \omega$. If $c : [\text{FU}(\bar{a})]^n_\ WL \to \{0, 1\}$. Then there is a $\bar{b} \subseteq \bar{a}$ such that $[\text{FU}(\bar{b})]^n_\ WL$ is monochromatic.
Examples of Matet-adequate families

(\text{FIN})^\omega \ (\text{Hindman})

\textbf{Theorem}
\textit{(Taylor)} Let $\bar{a} \in (\text{FIN})^\omega$, $n \in \omega$. If $c : [\text{FU}(\bar{a})]^n_\prec \rightarrow \{0, 1\}$. Then there is a $\bar{b} \subseteq \bar{a}$ such that $[\text{FU}(\bar{b})]^n_\prec$ is monochromatic.

same holds in any Matet-adequate family.
Any Milliken-Taylor ultrafilter $\mathcal{U}$.

**Definition**

A **Milliken-Taylor ultrafilter** is an ultrafilter over $\text{FIN}$ with the following properties:

1. It has a basis of sets of the form $\text{FU}(\bar{a})$ with $\bar{a} \in (\text{FIN})^\omega$,
2. each $\sqsubseteq$-descending sequence of members of $\mathcal{U}$ has a $\sqsubseteq^*$-lower bound,
3. and it has the Hindman-property.

The Hindman property follows from the first two properties.

Milliken-Taylor ultrafilters are also called stable ordered-union ultrafilters.
Existence?


Under NCF, so for example in the Matet model, there is none.

The issue of \( P \)-points. \( \diamond = c \). No \( P \)-points in the Silver model.
If $H$ is Matet-adequate then it has solutions to colorings as in the Taylor theorem.
If $\mathcal{H}$ is Matet-adequate then it has solutions to colorings as in the Taylor theorem.

We are interested in $\mathbb{M}(\mathcal{U})$, $\mathcal{U}$ and Milliken-Taylor ultrafilter.
Connections between FIN and $\omega$

\[ \min[\bar{a}] = \{\min(a_n) \mid n \in \omega\} \text{ for } \bar{a} \in (\text{FIN})^\omega. \]

\[ \min[X] = \{\min(x) \mid x \in X\} \text{ for } X \subseteq \text{FIN}. \]

\[ \hat{\min}(\mathcal{F}) = \{\min[X] \mid X \in \mathcal{F}\} \text{ for } \mathcal{F} \subseteq \mathcal{P}(\text{FIN}). \]

Blass showed that for an Milliken-Taylor ultrafilter $\mathcal{U}$ the projections $\hat{\min}(\mathcal{U})$ and $\hat{\max}(\mathcal{U})$ are non-nearly coherent Ramsey ultrafilters over $\omega$. 
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Blass showed that for an Milliken-Taylor ultrafilter $\mathcal{U}$ the projections $\hat{\min}(\mathcal{U})$ and $\hat{\max}(\mathcal{U})$ are non-nearly coherent Ramsey ultrafilters over $\omega$. 
(FIN, ∪) is a partial semigroup: We define \( s \cup t \) only for \( s < t \). The associative partial binary operation \( \cup \) lifts to \( \beta(FIN) \), the space of min-unbounded ultrafilters over FIN, as follows (and we write \( \dot{\cup} \) for the lifted operation):

\[
U_1 \dot{\cup} U_2 = \{ X \subseteq FIN \mid \text{for } U_1\text{-most } s, \text{ for } U_2\text{-most } t, s \cup t \in X \}
\]

With the topology

\[
\{ \{ U \mid X \in U \} \mid X \subseteq FIN \}
\]

it is a compact zero-dimensional Hausdorff space. With the topology \((\beta FIN, \dot{\cup})\) is a semitopological semigroup.
Existence, even with a large starting point

Lemma
(Ellis) For each closed subsemigroup $\mathcal{H}$ of $\beta\text{FIN}$ there is an idempotent ultrafilter.

Lemma
(Eisworth) Let $\mathcal{F}$ be an ordered-union filter. There is a min-unbounded idempotent ultrafilter $\mathcal{U} \in \beta\text{FIN}$ that extends $\mathcal{F}$. 
Motivating questions

Let $n \in \omega \setminus \{0, 1\}$. Is it consistent relative to ZFC that there is a model with $n$ near coherence classes of ultrafilters?
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Let $n \in \omega \setminus \{0, 1\}$. Is it consistent relative to ZFC that there is a model with $n$ near coherence classes of ultrafilters?

Necessary: $u < \frak{u}$. No or few Cohen reals.
Try to build a model with a small $P$-point and an “inhomogeneous” continuum.
Definition

In the Matet forcing, $M$, the conditions are pairs $(s, \overline{c})$ such that $s \in \text{FIN}$ and $\overline{c} \in (\text{FIN})^\omega$ and $s < c_0$. The forcing order is $(t, \overline{d}) \leq (s, \overline{c})$ (recall the stronger condition is the smaller one) iff $s \subseteq t$ and $t \setminus a$ is a concatenation of finitely many of the $c_n$ and $\overline{d}$ is a condensation of $\overline{c}$. 
Matet forcing

Definition
In the Matet forcing, $\mathbb{M}$, the conditions are pairs $(s, \bar{c})$ such that $s \in \text{FIN}$ and $\bar{c} \in (\text{FIN})^\omega$ and $s < c_0$. The forcing order is $(t, \bar{d}) \leq (s, \bar{c})$ (recall the stronger condition is the smaller one) iff $s \subseteq t$ and $t \setminus a$ is a concatenation of finitely many of the $c_n$ and $\bar{d}$ is a condensation of $\bar{c}$.

Definition
Let $\mathcal{H}$ be a Matet-adequate family. In the subforcing $\mathbb{M}(\mathcal{H})$ the second components of the conditions are taken from $\mathcal{H}$. 

We write and \( \text{set}(\bar{a}) = \bigcup \{ a_n \mid n \in \omega \} \). The forcing \( \mathbb{M}(\mathcal{U}) \) diagonalises ("shoots a real through") \( \{ \text{set}(\bar{a}) \mid \bar{a} \in \mathcal{C} \} \), namely the generic real

\[
\mu_G := \bigcup \{ s \mid \exists c \mid (s, c) \in G \}
\]

is a pseudo-intersection of this set.
Ramsey-theoretic computations in the $M(U)$-extension

We write and $\text{set}(\bar{a}) = \bigcup\{a_n \mid n \in \omega\}$. The forcing $M(U)$ diagonalises (“shoots a real through”) $\{\text{set}(\bar{a}) \mid \bar{a} \in C\}$, namely the generic real

$$\mu_G := \bigcup\{s \mid \exists \bar{c} \mid (s, \bar{c}) \in G\}$$

is a pseudo-intersection of this set.

**Definition**

1. Let $\bar{a} \in (FIN)^\omega$ and $\mu \in [\omega]^\omega$. $\bar{a} \upharpoonright \mu = \langle a_n \mid n \in \omega, a_n \subseteq \mu\rangle$. Note, we do not take those $a_n$ with $a_n \cap \mu \neq \emptyset$ that are not subsets of $\mu$.

2. Let $U \subseteq (FIN)^\omega$ and $\mu \in [\omega]^\omega$. $U \upharpoonright \mu = \text{fil}(\{\bar{a} \upharpoonright \mu \mid \bar{a} \in U\})$. 

Preserving a $P$-point

Definition
Let $\mathcal{H}$ be a Matet-adequate family and let $\mathcal{E}$ be a $P$-point. We say $\mathcal{H}$ avoids $\mathcal{E}$ if for any $\bar{a} \in \mathcal{H}$ and finite-to-one $f$ there is an $E \in \mathcal{E}$ and an $\bar{b} \in \mathcal{H}$ such that $\bar{b} \subseteq \bar{a}$ and $f[E] \cap f[\text{set}(\bar{b})] = \emptyset$.

Theorem
(Eisworth) If $\mathcal{U}$ avoids $\mathcal{E}$ then in $\mathcal{V}^{\mathcal{M}(\mathcal{U})}$ the $P$-point $\mathcal{E}$ is preserved, i.e. $\{Y \mid (\exists E \in \mathcal{E})Y \supseteq X\}$ is an ultrafilter.
Extending $(\mathcal{U} \upharpoonright \mu)$ in $V^\mathbb{M}(\mathcal{U})$

**Theorem**

(M., 2017) After forcing with $\mathbb{M}(\mathcal{U})$, $(\mathcal{U} \upharpoonright \mu)^+$ is a Matet-adequate family that avoids $\mathcal{E}$.

**Corollary**

Let $\mathcal{E}$ be a $P$-point and $\mathcal{U}$ be a Milliken-Taylor ultrafilter with $\Phi(\mathcal{U}) \not\leq_{ RB } \mathcal{E}$. Assume CH. Then in the forcing extension by $\mathbb{M}(\mathcal{U})$ the Milliken-Taylor ultrafilter $\mathcal{U}$ is destroyed and can be completed to an Milliken-Taylor ultrafilter $\mathcal{U}^{ext} \supseteq \mathcal{U}$ with $\Phi(\mathcal{U}^{ext}) \not\leq_{ RB } \mathcal{E}$. 
Lemma

Let $\mathcal{U}$ be an Milliken-Taylor ultrafilter, $\mathcal{E}$ be a $P$-point, $\Phi(\mathcal{U}) \not\subseteq_{RB} \mathcal{E}$. Let $\mathcal{Q} = \mathcal{M}(\mathcal{U})$ and let $\mu$ be the name for the generic real. Let $\langle X_n \mid n \in \omega \rangle$ be a sequence of $\mathcal{Q}$-names for elements of $(\text{FIN})^\omega$ such that

$$\mathcal{Q} \models (\forall n \in \omega)(X_n \in (\mathcal{U} \upharpoonright \mu)^+ \land X_{n+1} \sqsubseteq X_n).$$

Then

$$\tilde{D} = \{ \langle \tilde{t}, (s, \bar{a}) \rangle \mid (s, \bar{a}) \in \mathcal{Q}_\alpha \text{ is neat for } \tilde{X} \text{ and}$$

$$\exists t_0 < t_1 < \cdots < t_k = t$$

$$(s, \bar{a}) \models t_0 = \min(\tilde{X}_0 \upharpoonright \mu)) \land$$

$$\bigwedge_{i<k} t_{i+1} = \min((\tilde{X}_{\max(t_i)} \upharpoonright \mu) \text{ past } t_i) \}$$

fulfils

$$\mathcal{Q} \models \tilde{D} \in (\mathcal{U} \upharpoonright \mu)^+ \land \tilde{D} \sqsubseteq \tilde{X}_0 \land (\forall t \in \tilde{D})(\tilde{D} \text{ past } t \sqsubseteq \tilde{X}_{\max(t)+1}).$$
The proof of the Hindman property includes again a proof that positive diagonal lower bounds exist.

**Lemma**

In $\mathcal{V}^\mathbb{M}(\mathcal{U})$, $(\mathcal{U} \upharpoonright \mu)^+$ has the Hindman property.

For the proof of this lemma, we adapt a proof of a theorem of Eisworth. This says

**Theorem**

(Eisworth) Let $\mathcal{F}$ be an ordered-union filter generated by $< \text{cov}(\mathcal{B})$ sets and let $c$ be a partition of $\text{FIN}$ into finite sets. Then there is an $\bar{a} \in \mathcal{F}^+$ such that $\text{FU}(\bar{a})$ is included in one piece of the partition.

At a crucial point in the proof a Cohen real provides a name. We show that also a Matet-real can be used. For this we outline the proof. We recall the Galvin-Glazer technique.
Let $c$ be a name for a partition of $(\bar{b}_0 \in \mathcal{U} \upharpoonright \mu)^+$ into finitely many pieces and let $\bar{b}_n$ be a $\sqsubseteq$-descending sequence of elements $\bar{b}_n \in (\mathcal{U} \upharpoonright \mu)^+$. Let $\mathcal{U}^i$ be such that

$$\mathcal{M}(\mathcal{U}) \models \mathcal{U}^i \supseteq ((\mathcal{U} \upharpoonright \mu) \cup \{\bar{b}_n \mid n \in \omega\}) \wedge \mathcal{U}^i \cup \mathcal{U}^i = \mathcal{U}^i.$$

For $X \subseteq \text{FIN}$ and $t \in \text{FIN}$ we set

$$X \ominus t = \{s \mid s \cup t \in X\}$$

If $\mathcal{U}^i$ is idempotent then for each $X \in \mathcal{U}^i$ the set

$$\{t \mid X \ominus t \in \mathcal{U}^i\}$$

is in $\mathcal{U}^i$.

We define for $n \in \omega$ names $X_n$ and $d_n$ and $p_n = (s_n, \bar{a}_n)$ with the following rules:
A name for a monochromatic set in $(\mathcal{U} \upharpoonright \mu)^+$

(1) $p_0 \models X_0$ is the piece of the partition $c$ of $FU(\bar{b}_0)$ that is in $\mathcal{U}^i$.

(2) $p_{n+1} = (s_{n+1}, \bar{a}_{n+1}) \models d_n$ is the $\leq_{\text{lex},\mathbb{N}}$-least element of

$$\{d \in X_n \cap FU(\{a_{n,k} \mid k \in \omega\}) \cap FU(\bar{b}_n) \mid X_n \ominus d \in \mathcal{U}^i \text{ and } \min(d) > \max(d_i) \text{ for } i < n\}$$

(3) $p_{n+1} \models X_{n+1} = X_n \cap (X_n \ominus d_n)$.

Since $\mathcal{U}^i$ is idempotent, the set in (2) is in $\mathcal{U}^i$.

We ensure with colouring of the pure part of $p_n$ that there is a lower bound of $\langle p_n \mid n < \omega \rangle$ that forces only the existence of the $d_n$, without the pinning down.
Iterating with countable support

\[ \mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{M}(\mathcal{U}_\beta) \mid \beta < \omega_2, \alpha \leq \omega_2 \rangle \text{ with countable support and } \]

\[ \mathbb{P}_\beta \models \mathcal{U}_\beta \supseteq \bigcup \{(\mathcal{U}_\gamma \upharpoonright \mu_\gamma) \mid \gamma < \beta \} \]
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Preservation theorem.
Iterating with countable support

\[ P = \langle P_\alpha, M(\mathcal{U}_\beta) \mid \beta < \omega_2, \alpha \leq \omega_2 \rangle \text{ with countable support and} \]

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Preservation theorem.

In \( V^P \), there are at least three near coherence classes of filters.

\[ \hat{\min}(\mathcal{U}_{\omega_2}) = \{ \min[\bar{a}] \mid \bar{a} \in \mathcal{U}_{\omega_2} \} \]

\[ \hat{\max}(\mathcal{U}_{\omega_2}) = \{ \max[\bar{a}] \mid \bar{a} \in \mathcal{U}_{\omega_2} \} \]

Question

Can \( \Diamond (S^{\omega_2}_{\omega_0}) \) be used to arrange that there are just this three classes?