Embedding theorem and regularity properties under AD^+

Itay Neeman

Department of Mathematics
University of California Los Angeles
Los Angeles, CA 90095

www.math.ucla.edu/~ineeman

Luminy, October 2017

There is a long line of absoluteness results for concrete (within the realm of determinacy) statements on reals and ordinals, assuming large cardinals.

We present a specific absoluteness result, similar to the embedding theorem of Neeman-Zapletal.

We prove the absoluteness result under large cardinals, and under AD^+. We present several applications, for example proving that AD^+ implies that there are no (infinite) MAD families.

MAD

Recall (infinte) $A \subseteq [\omega]^\omega$ is almost disjoint if $(\forall x, y \in A)$ $x \cap y$ is finite. A is maximal almost disjoint (MAD) if A is maximal with this property.

Under the axiom of choice, MAD families exist using Zorn's lemma. But:

Theorem (Mathias '70s)

1. *There are no analytic MAD families.*
2. *If κ is Mahlo, and G is generic for $\mathrm{Col}(\omega, < \kappa)$, then in $L(\mathbb{R})^{V[G]}$ (the Solovay model at κ) there are no MAD families.*

MAD, cont.

Theorem (Mathias ’70s)

1. *There are no analytic MAD families.*
2. *If κ is Mahlo, and G is generic for $\mathrm{Col}(\omega, <\kappa)$, then in $L(\mathbb{R})^{V[G]}$ (the Solovay model at κ) there are no MAD families.*

Raises several questions:

Is the Mahlo needed?

What about projective sets beyond analytic assuming large cardinals? all sets in $L(\mathbb{R})$?

Does AD imply no MAD families?

MAD, cont.

Theorem (Mathias '70s)

1. *There are no analytic MAD families.*
2. *If κ is Mahlo, and G is generic for $\mathrm{Col}(\omega, < \kappa)$, then in $L(\mathbb{R})^{V[G]}$ (the Solovay model at κ) there are no MAD families.*

Raises several questions:

Is the Mahlo needed? No, Tornquist (2015) in Solovay model assuming inaccessible, Horowitz-Shelah (2016) in other models assuming ZFC.

What about projective sets beyond analytic assuming large cardinals? all sets in $L(\mathbb{R})$?

Does AD imply no MAD families?

MAD, cont.

Theorem (Mathias '70s)

1. *There are no analytic MAD families.*
2. *If κ is Mahlo, and G is generic for $\mathrm{Col}(\omega, <\kappa)$, then in $L(\mathbb{R})^{V[G]}$ (the Solovay model at κ) there are no MAD families.*

Raises several questions:

Is the Mahlo needed? No, Tornquist (2015) in Solovay model assuming inaccessible, Horowitz-Shelah (2016) in other models assuming ZFC.

What about projective sets beyond analytic assuming large cardinals? all sets in $L(\mathbb{R})$? Holds, Todorcevic (<1998).

Does AD imply no MAD families?

MAD, cont.

Theorem (Mathias '70s)

1. *There are no analytic MAD families.*
2. *If κ is Mahlo, and G is generic for $\mathrm{Col}(\omega, < \kappa)$, then in $L(\mathbb{R})^{V[G]}$ (the Solovay model at κ) there are no MAD families.*

Raises several questions:

Is the Mahlo needed? No, Tornquist (2015) in Solovay model assuming inaccessible, Horowitz-Shelah (2016) in other models assuming ZFC.

What about projective sets beyond analytic assuming large cardinals? all sets in $L(\mathbb{R})$? Holds, Todorcevic (<1998).

Does AD imply no MAD families? Open.

Happy

Mathias's proof had combinatorial content beyond the inexistence of MAD families.

Definition (Mathias '70s)

$\emptyset \neq H \subseteq [\omega]^\omega$ *is happy (aka selective co-ideal) if:*

1. *(Upward closure)* $y \in H \wedge z \supseteq y \rightarrow z \in H$.
2. *(Pigeonhole)* $y_0 \cup \cdots \cup y_n \in H \rightarrow (\exists i) y_i \in H$.
3. *(Selectivity) If* $y_0 \supseteq y_1 \supseteq y_2 \ldots$ *all in* H*, then* $(\exists y_\infty \in H)$ *so that* $(\forall m \in y_\infty) y_\infty - (m+1) \subseteq y_m$. *Such* y_∞ *diagonalizes* $\langle y_m \mid m < \omega \rangle$.

If A is almost disjoint, then $H = \{y \mid y \not\subseteq^* x_1 \cup \cdots \cup x_k$ for any $x_1, \ldots, x_k \in A\}$ satisfies upward closure, pigeonhole.

If A is MAD, then H is also selective, hence happy. To see this, note (for MAD A) that $y \in H$ iff y has infinite intersection with infinitely many $x \in A$.

Happy (cont.)

Definition

$X \subseteq [\omega]^\omega$ is H-Ramsey if there is $y \in H$ so that either $[y]^\omega \subseteq X$ or $[y]^\omega \subseteq [\omega]^\omega - X$.

For H as above, assuming A is MAD, H is *not* H-Ramsey.

Theorem (Mathias '70s)

1. *If H is happy, then every analytic X is H-Ramsey.*
2. *Let κ be Mahlo and let G be generic for $\mathrm{Col}(\omega, <\kappa)$. If $H \in V[G]$ is happy, then every $X \in L(\mathbb{R})^{V[G]}$ is H-Ramsey.*

Proved using *Mathias forcing*.

Gives the results on inexistence of MAD families.

Mahlo needed here (Eisworth 1999).

Happy (cont.)

Definition

$X \subseteq [\omega]^\omega$ is H-Ramsey if there is $y \in H$ so that either $[y]^\omega \subseteq X$ or $[y]^\omega \subseteq [\omega]^\omega - X$.

For H as above, assuming A is MAD, H is *not* H-Ramsey.

Theorem (Mathias ’70s)

1. *If H is happy, then every analytic X is H-Ramsey.*
2. *Let κ be Mahlo and let G be generic for $\mathrm{Col}(\omega, < \kappa)$. If $H \in V[G]$ is happy, then every $X \in L(\mathbb{R})^{V[G]}$ is H-Ramsey.*

Proved using *Mathias forcing*.

Gives the results on inexistence of MAD families.

Mahlo needed here (Eisworth 1999).

For $H \in L(\mathbb{R})$, an inaccessible is enough (N-Norwood), gives Tornquist’s result through Mathias’s methods.

Equivalence relations with simple classes

Definition (Zapletal)

Given a σ-ideal I on ω^ω, let $\mathbb{P}_I$ be the forcing notion consisting of Borel sets in I^+ ordered by inclusion mod I, that is $B \leq A$ iff $B - A \in I$.

Zapletal (2000s) initiated a program of studying ideals for which $\mathbb{P}_I$ is proper, under determinacy or large cardinal assumptions.

Theorem (Chan, Chan-Magidor 2016)

1. *(Assuming sharps.) Let E be an analytic (or co-analytic) equivalence relation with Borel classes. Let I be a σ-ideal on ω^ω so that $\mathbb{P}_I$ is proper. Then there is $C \in I^+$ so that $E \restriction C$ is Borel.*
2. *(Assuming Woodin cardinals.) The same is true for $E \in L(\mathbb{R})$. Also true replacing Borel with analytic, co-analytic.*

Absoluteness

Theorem (Woodin '80s)

Assuming large cardinals, the theory of $L(\mathbb{R})$ with real parameters cannot be changed by forcing.

Theorem (Foreman-Magidor 1995)

Assuming large cardinals, proper (or reasonable) forcing does not change the length of projective prewellorderings on reals (or prewellorderings in $L(\mathbb{R})$).

Theorem (Neeman-Zapletal embedding theorem 1998)

Assuming large cardinals, if $\mathbb{P}$ is proper (reasonable) and G is generic for $\mathbb{P}$, then there is an elementary embedding $j\colon L(\mathbb{R}) \to L(\mathbb{R}^{V[G]})$ which fixes reals and ordinals.

Absoluteness, cont.

Absoluteness, cont.

The embedding theorem is proved using Woodin's genericity iterations.

Absoluteness, cont.

The embedding theorem is proved using Woodin's genericity iterations.

Let Q be a fully iterable class model, suppose Q has ω Woodin cardinals, with supremum δ_Q, and $\mathcal{P}(\delta_Q) \cap Q$ countable in V.

Absoluteness, cont.

The embedding theorem is proved using Woodin's genericity iterations.

Let Q be a fully iterable class model, suppose Q has ω Woodin cardinals, with supremum δ_Q, and $\mathcal{P}(\delta_Q) \cap Q$ countable in V.

Using Woodin's methods can iterate Q to some Q^*, and find g generic for $\mathrm{Col}(\omega, < \delta_{Q^*})$, so that $L(\mathbb{R}^V)$ is the Solovay model for Q^* at δ_{Q^*} using g.

Absoluteness, cont.

The embedding theorem is proved using Woodin's genericity iterations.

Let Q be a fully iterable class model, suppose Q has ω Woodin cardinals, with supremum δ_Q, and $\mathcal{P}(\delta_Q) \cap Q$ countable in V.

Using Woodin's methods can iterate Q to some Q^*, and find g generic for $\mathrm{Col}(\omega, < \delta_{Q^*})$, so that $L(\mathbb{R}^V)$ is the Solovay model for Q^* at δ_{Q^*} using g.

Can do the same in $V[G]$ to get $L(\mathbb{R}^{V[G]})$.

Absoluteness, cont.

The embedding theorem is proved using Woodin's genericity iterations.

Let Q be a fully iterable class model, suppose Q has ω Woodin cardinals, with supremum δ_Q, and $\mathcal{P}(\delta_Q) \cap Q$ countable in V.

Using Woodin's methods can iterate Q to some Q^*, and find g generic for $\mathrm{Col}(\omega, < \delta_{Q^*})$, so that $L(\mathbb{R}^V)$ is the Solovay model for Q^* at δ_{Q^*} using g.

Can do the same in $V[G]$ to get $L(\mathbb{R}^{V[G]})$.

Key to the embedding theorem is finding an iteration Q^* which works simultaneously for $\mathbb{R}^V$ and $\mathbb{R}^{V[G]}$.

A triangular embedding theorem

A triangular embedding theorem

A similar proof, but done over a countable M embedded in V_θ, gives the following:

Theorem (N-Norwood)

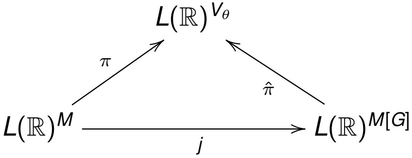
(Assuming large cardinals.) Let $\pi : M \to V_\theta$ be elementary, M countable. Let $\mathbb{P}$ be proper in M, G generic for $\mathbb{P}$ over M.

A triangular embedding theorem

A similar proof, but done over a countable M embedded in V_θ, gives the following:

Theorem (N-Norwood)

(Assuming large cardinals.) Let $\pi\colon M \to V_\theta$ *be elementary, M countable. Let* $\mathbb{P}$ *be proper in M, G generic for* $\mathbb{P}$ *over M.*
Then there is $j\colon L(\mathbb{R}^M) \to L(\mathbb{R}^{M[G]})$ *which fixes reals and ordinals, and* $\hat\pi\colon L(\mathbb{R}^{M[G]}) \to L(\mathbb{R})^{V_\theta}$*, both elementary, with* $\pi = \hat\pi \circ j$*.*



Ramsey property for happy families

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Theorem

(Assuming large cardinals.) Every $X \subseteq [\omega]^\omega$ in $L(\mathbb{R})$ is H-Ramsey for every happy family H.

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Theorem

(Assuming large cardinals.) Every $X \subseteq [\omega]^\omega$ in $L(\mathbb{R})$ is H-Ramsey for every happy family H.

To prove (following Mathias), force over a countable substructure M with Mathias forcing relative to $H \cap M$.

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Theorem

(Assuming large cardinals.) Every $X \subseteq [\omega]^\omega$ in $L(\mathbb{R})$ is H-Ramsey for every happy family H.

To prove (following Mathias), force over a countable substructure M with Mathias forcing relative to $H \cap M$.

Using happiness of H, can find M-generic $g \in H$.

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Theorem

(Assuming large cardinals.) Every $X \subseteq [\omega]^\omega$ in $L(\mathbb{R})$ is H-Ramsey for every happy family H.

To prove (following Mathias), force over a countable substructure M with Mathias forcing relative to $H \cap M$.

Using happiness of H, can find M-generic $g \in H$.

Wlog g belongs to reinterpretation of X over $L(\mathbb{R})^{M[g]}$.

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Theorem

(Assuming large cardinals.) Every $X \subseteq [\omega]^\omega$ in $L(\mathbb{R})$ is H-Ramsey for every happy family H.

To prove (following Mathias), force over a countable substructure M with Mathias forcing relative to $H \cap M$.

Using happiness of H, can find M-generic $g \in H$.

Wlog g belongs to reinterpretation of X over $L(\mathbb{R})^{M[g]}$.

By Mathias property, every $\bar{g} \subseteq g$ also generic.

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Theorem

(Assuming large cardinals.) Every $X \subseteq [\omega]^\omega$ in $L(\mathbb{R})$ is H-Ramsey for every happy family H.

To prove (following Mathias), force over a countable substructure M with Mathias forcing relative to $H \cap M$.

Using happiness of H, can find M-generic $g \in H$.

Wlog g belongs to reinterpretation of X over $L(\mathbb{R})^{M[g]}$.

By Mathias property, every $\bar{g} \subseteq g$ also generic. By Prikry property for Mathias forcing still forced into reinterpretation of X over $L(\mathbb{R})^{M[\bar{g}]}$.

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Theorem

(Assuming large cardinals.) Every $X \subseteq [\omega]^\omega$ in $L(\mathbb{R})$ is H-Ramsey for every happy family H.

To prove (following Mathias), force over a countable substructure M with Mathias forcing relative to $H \cap M$.

Using happiness of H, can find M-generic $g \in H$.

Wlog g belongs to reinterpretation of X over $L(\mathbb{R})^{M[g]}$.

By Mathias property, every $\bar{g} \subseteq g$ also generic. By Prikry property for Mathias forcing still forced into reinterpretation of X over $L(\mathbb{R})^{M[\bar{g}]}$.

By embedding theorem, reinterpretation is $X \cap M[\bar{g}]$.

Ramsey property for happy families

As first application we obtain the following theorem of Todorcevic 1998 (reducing large cardinal assumptions).

Theorem

(Assuming large cardinals.) Every $X \subseteq [\omega]^\omega$ in $L(\mathbb{R})$ is H-Ramsey for every happy family H.

To prove (following Mathias), force over a countable substructure M with Mathias forcing relative to $H \cap M$.

Using happiness of H, can find M-generic $g \in H$.

Wlog g belongs to reinterpretation of X over $L(\mathbb{R})^{M[g]}$.

By Mathias property, every $\bar{g} \subseteq g$ also generic. By Prikry property for Mathias forcing still forced into reinterpretation of X over $L(\mathbb{R})^{M[\bar{g}]}$.

By embedding theorem, reinterpretation is $X \cap M[\bar{g}]$.

So $\bar{g} \in X$.

Embedding theorem under AD^+

AD^+ is a strengthening of AD due to Woodin.

It adds the following:

1. $\mathrm{DC}_{\mathbb{R}}$;
2. all sets of reals are ∞-Borel;
3. for every $\lambda < \Theta$, continuous $f\colon \lambda^\omega \to \omega^\omega$, and $A \subseteq \omega^\omega$, $f^{-1}{}''A$ is determined.

Every known model of AD in fact satisfies AD^+.

It is open whether the two are equivalent.

AD^+ allows finding nice witnesses for $\boldsymbol{\Sigma}^2_1$ statements.

Embedding theorem under AD^+

AD^+ is a strengthening of AD due to Woodin.

It adds the following:

1. $\mathrm{DC}_{\mathbb{R}}$;
2. all sets of reals are ∞-Borel;
3. for every $\lambda < \Theta$, continuous $f\colon \lambda^\omega \to \omega^\omega$, and $A \subseteq \omega^\omega$, $f^{-1}{}''A$ is determined.

Every known model of AD in fact satisfies AD^+.

It is open whether the two are equivalent.

AD^+ allows finding nice witnesses for $\boldsymbol{\Sigma}^2_1$ statements.

Embedding theorem under AD^+, cont.

Embedding theorem under AD^+, cont.

Inner model theory has progressed enough that (assuming AD^+) if a $\mathbf{\Sigma}^2_1$ statement is true, then one can find a witness F for the $\mathbf{\Sigma}^2_1$ statement, a countable model Q with ω Woodin cardinals, with supremum δ_Q say, a $\mathrm{Col}(\omega, < \delta_Q)$-name $\dot{F} \in Q$, and an iteration strategy for Q which move $\dot{F}$ to names with interpretations that agree with F.

Embedding theorem under AD$^+$, cont.

Inner model theory has progressed enough that (assuming AD$^+$) if a $\mathbf{\Sigma}^2_1$ statement is true, then one can find a witness F for the $\mathbf{\Sigma}^2_1$ statement, a countable model Q with ω Woodin cardinals, with supremum δ_Q say, a $\mathrm{Col}(\omega, < \delta_Q)$-name $\dot{F} \in Q$, and an iteration strategy for Q which move $\dot{F}$ to names with interpretations that agree with F.

Can run the proof of the triangular embedding theorem, replacing the Solovay model $L(\mathbb{R}^*)$, where $\mathbb{R}^*$ are the reals added by g over $Q|\delta_Q$, with $L(\mathbb{R}^*, \dot{F}[g])$.

Embedding theorem under AD^+, cont.

Inner model theory has progressed enough that (assuming AD^+) if a $\mathbf{\Sigma}^2_1$ statement is true, then one can find a witness F for the $\mathbf{\Sigma}^2_1$ statement, a countable model Q with ω Woodin cardinals, with supremum δ_Q say, a $\mathrm{Col}(\omega, < \delta_Q)$-name $\dot{F} \in Q$, and an iteration strategy for Q which move $\dot{F}$ to names with interpretations that agree with F.

Can run the proof of the triangular embedding theorem, replacing the Solovay model $L(\mathbb{R}^*)$, where $\mathbb{R}^*$ are the reals added by g over $Q|\delta_Q$, with $L(\mathbb{R}^*, \dot{F}[g])$.

Get the embedding theorem for $L(\mathbb{R}, F)$.

Embedding theorem under AD^+, cont.

Inner model theory has progressed enough that (assuming AD^+) if a $\mathbf{\Sigma}^2_1$ statement is true, then one can find a witness F for the $\mathbf{\Sigma}^2_1$ statement, a countable model Q with ω Woodin cardinals, with supremum δ_Q say, a $\mathrm{Col}(\omega, < \delta_Q)$-name $\dot{F} \in Q$, and an iteration strategy for Q which move $\dot{F}$ to names with interpretations that agree with F.

Can run the proof of the triangular embedding theorem, replacing the Solovay model $L(\mathbb{R}^*)$, where $\mathbb{R}^*$ are the reals added by g over $Q|\delta_Q$, with $L(\mathbb{R}^*, \dot{F}[g])$.

Get the embedding theorem for $L(\mathbb{R}, F)$.

Phrase the theorem so that its failure is a $\mathbf{\Sigma}^2_1$ statement; then get that it holds.

Embedding theorem under AD^+, cont.

Embedding theorem under AD^+, cont.

Some care with meaning of properness. Need properness in models of choice generated by the iteration strategies.

Embedding theorem under AD^+, cont.

Some care with meaning of properness. Need properness in models of choice generated by the iteration strategies.

Definition (N-Norwood)

(In ZF.) A poset $\mathbb{P} \subseteq \mathbb{R}$ is absolutely proper if there is a club $C \subseteq \mathcal{P}_{<\omega_1}(\mathbb{R})$ and $A \subseteq \mathbb{R}$ so that for all $U \in C$ and all transitive $N \models \mathsf{ZFC}$ with $\mathbb{R}^N = U$ and $\mathbb{P} \cap U, A \cap U \in N$, $\mathbb{P} \cap U$ is proper in N.

Embedding theorem under AD^+, cont.

Some care with meaning of properness. Need properness in models of choice generated by the iteration strategies.

Definition (N-Norwood)

(In ZF.) A poset $\mathbb{P} \subseteq \mathbb{R}$ is absolutely proper if there is a club $C \subseteq \mathcal{P}_{<\omega_1}(\mathbb{R})$ and $A \subseteq \mathbb{R}$ so that for all $U \in C$ and all transitive $N \models \mathsf{ZFC}$ with $\mathbb{R}^N = U$ and $\mathbb{P} \cap U, A \cap U \in N$, $\mathbb{P} \cap U$ is proper in N.

If $\mathbb{P}$ is proper by a sufficiently absolute proof, can run the proof in any N as above, and get absolute properness.

Embedding theorem under AD^+, cont.

Some care with meaning of properness. Need properness in models of choice generated by the iteration strategies.

Definition (N-Norwood)

(In ZF.) A poset $\mathbb{P} \subseteq \mathbb{R}$ is absolutely proper if there is a club $C \subseteq \mathcal{P}_{<\omega_1}(\mathbb{R})$ and $A \subseteq \mathbb{R}$ so that for all $U \in C$ and all transitive $N \models \mathsf{ZFC}$ with $\mathbb{R}^N = U$ and $\mathbb{P} \cap U, A \cap U \in N$, $\mathbb{P} \cap U$ is proper in N.

If $\mathbb{P}$ is proper by a sufficiently absolute proof, can run the proof in any N as above, and get absolute properness.

In particular Mathias forcing is absolutely proper.

Embedding theorem under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+*.) For every* $\alpha < \Theta$*, every* $A \subseteq \mathbb{R}$*, stationarily many* $Z \preceq L_\alpha(\mathbb{R}, A)$*, every absolutely proper* $\mathbb{P}$ *in the transitive collapse* M *of* Z*,*

Embedding theorem under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+*.) For every* $\alpha < \Theta$*, every* $A \subseteq \mathbb{R}$*, stationarily many* $Z \preceq L_\alpha(\mathbb{R}, A)$*, every absolutely proper* $\mathbb{P}$ *in the transitive collapse* M *of* Z*,*

there is a transitive $N \models \mathsf{ZFC}$ *with* $\mathbb{R}^N = \mathbb{R}^M$*,* $A \cap M \in N$*,* $\bar{\alpha} = M \cap \mathrm{Ord} \in N$*, and a* $\mathbb{P}$*-name* $\dot{A}^* \in N$*,*

Embedding theorem under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+.) For every $\alpha < \Theta$, every $A \subseteq \mathbb{R}$, stationarily many $Z \preceq L_\alpha(\mathbb{R}, A)$, every absolutely proper $\mathbb{P}$ in the transitive collapse M of Z,

there is a transitive $N \models \mathsf{ZFC}$ with $\mathbb{R}^N = \mathbb{R}^M$, $A \cap M \in N$, $\bar{\alpha} = M \cap \mathrm{Ord} \in N$, and a $\mathbb{P}$-name $\dot{A}^* \in N$,

so that for every G generic for $\mathbb{P}$ over N

Embedding theorem under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+.) For every $\alpha < \Theta$, every $A \subseteq \mathbb{R}$, stationarily many $Z \preceq L_\alpha(\mathbb{R}, A)$, every absolutely proper $\mathbb{P}$ in the transitive collapse M of Z,

there is a transitive $N \models \mathsf{ZFC}$ with $\mathbb{R}^N = \mathbb{R}^M$, $A \cap M \in N$, $\bar{\alpha} = M \cap \mathrm{Ord} \in N$, and a $\mathbb{P}$-name $\dot{A}^* \in N$,

so that for every G generic for $\mathbb{P}$ over N

there are embeddings $j\colon M \to L_{\bar{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G])$ fixing reals and ordinals, and $\hat{\pi}\colon L_{\bar{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G]) \to L_\alpha(\mathbb{R}, A)$ commuting with the anticollapse $\pi\colon M \to L_\alpha(\mathbb{R}, A)$.

Embedding theorem under AD^+, cont.

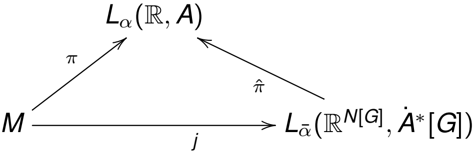
Theorem (N-Norwood)

(Assuming AD^+.) For every $\alpha < \Theta$, every $A \subseteq \mathbb{R}$, stationarily many $Z \preceq L_\alpha(\mathbb{R}, A)$, every absolutely proper $\mathbb{P}$ in the transitive collapse M of Z,

there is a transitive $N \models \mathsf{ZFC}$ with $\mathbb{R}^N = \mathbb{R}^M$, $A \cap M \in N$, $\bar{\alpha} = M \cap \mathrm{Ord} \in N$, and a $\mathbb{P}$-name $\dot{A}^* \in N$,

so that for every G generic for $\mathbb{P}$ over N

there are embeddings $j \colon M \to L_{\bar{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G])$ fixing reals and ordinals, and $\hat{\pi} \colon L_{\bar{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G]) \to L_\alpha(\mathbb{R}, A)$ commuting with the anticollapse $\pi \colon M \to L_\alpha(\mathbb{R}, A)$.



Embedding theorem under AD^+, cont.

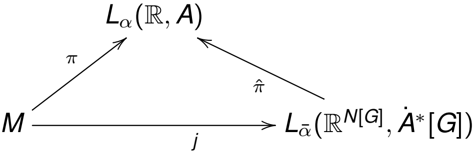
Theorem (N-Norwood)

(Assuming AD^+.) For every $\alpha < \Theta$, every $A \subseteq \mathbb{R}$, stationarily many $Z \preceq L_\alpha(\mathbb{R}, A)$, every absolutely proper $\mathbb{P}$ in the transitive collapse M of Z,

there is a transitive $N \models \mathrm{ZFC}$ with $\mathbb{R}^N = \mathbb{R}^M$, $A \cap M \in N$, $\bar{\alpha} = M \cap \mathrm{Ord} \in N$, and a $\mathbb{P}$-name $\dot{A}^* \in N$,

so that for every G generic for $\mathbb{P}$ over N

there are embeddings $j\colon M \to L_{\bar{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G])$ fixing reals and ordinals, and $\hat{\pi}\colon L_{\bar{\alpha}}(\mathbb{R}^{N[G]}, \dot{A}^*[G]) \to L_\alpha(\mathbb{R}, A)$ commuting with the anticollapse $\pi\colon M \to L_\alpha(\mathbb{R}, A)$.



Applications under AD^+

Applications under AD^+

Theorem (N-Norwood)

(Assuming AD^+*.) Let* I *be a* σ*-ideal on* ω^ω *so that* $\mathbb{P}_I$ *is absolutely proper. Let* Γ *be closed under Borel substitutions, with a universal set. Let* E *be an equivalence relation with* Γ *classes (respectively* $\Gamma \cap \check{\Gamma}$*). Then there is* $C \in I^+$ *so that* $E \restriction C$ *is in* Γ *(respectively* $\Gamma \cap \check{\Gamma}$*).*

Applications under AD^+

Theorem (N-Norwood)

(Assuming AD^+.) Let I be a σ-ideal on ω^ω so that $\mathbb{P}_I$ is absolutely proper. Let Γ be closed under Borel substitutions, with a universal set. Let E be an equivalence relation with Γ classes (respectively $\Gamma \cap \check{\Gamma}$). Then there is $C \in I^+$ so that $E \restriction C$ is in Γ (respectively $\Gamma \cap \check{\Gamma}$).

An AD^+ strengthening of the Chan-Magidor result.

Applications under AD$^+$

Theorem (N-Norwood)

(Assuming AD$^+$.) Let I be a σ-ideal on ω^ω so that $\mathbb{P}_I$ is absolutely proper. Let Γ be closed under Borel substitutions, with a universal set. Let E be an equivalence relation with Γ classes (respectively $\Gamma \cap \check{\Gamma}$). Then there is $C \in I^+$ so that $E \restriction C$ is in Γ (respectively $\Gamma \cap \check{\Gamma}$).

An AD$^+$ strengthening of the Chan-Magidor result.

To prove (the Γ case), take a universal U, force over a countable substructure M.

Applications under AD^+

Theorem (N-Norwood)

(Assuming AD^+.) Let I be a σ-ideal on ω^ω so that $\mathbb{P}_I$ is absolutely proper. Let Γ be closed under Borel substitutions, with a universal set. Let E be an equivalence relation with Γ classes (respectively $\Gamma \cap \check{\Gamma}$). Then there is $C \in I^+$ so that $E \restriction C$ is in Γ (respectively $\Gamma \cap \check{\Gamma}$).

An AD^+ strengthening of the Chan-Magidor result.

To prove (the Γ case), take a universal U, force over a countable substructure M. For generic x, embedding theorem allows recovering y so that $[x]_E = U_y$ in a Borel manner from x

Applications under AD^+

Theorem (N-Norwood)

(Assuming AD^+.) Let I be a σ-ideal on ω^ω so that $\mathbb{P}_I$ is absolutely proper. Let Γ be closed under Borel substitutions, with a universal set. Let E be an equivalence relation with Γ classes (respectively $\Gamma \cap \check{\Gamma}$). Then there is $C \in I^+$ so that $E \restriction C$ is in Γ (respectively $\Gamma \cap \check{\Gamma}$).

An AD^+ strengthening of the Chan-Magidor result.

To prove (the Γ case), take a universal U, force over a countable substructure M. For generic x, embedding theorem allows recovering y so that $[x]_E = U_y$ in a Borel manner from x (because there is a name for y in M).

Applications under AD^+

Theorem (N-Norwood)

(Assuming AD^+*.) Let* I *be a* σ*-ideal on* ω^ω *so that* $\mathbb{P}_I$ *is absolutely proper. Let* Γ *be closed under Borel substitutions, with a universal set. Let* E *be an equivalence relation with* Γ *classes (respectively* $\Gamma \cap \check{\Gamma}$*). Then there is* $C \in I^+$ *so that* $E \restriction C$ *is in* Γ *(respectively* $\Gamma \cap \check{\Gamma}$*).*

An AD^+ strengthening of the Chan-Magidor result.

To prove (the Γ case), take a universal U, force over a countable substructure M. For generic x, embedding theorem allows recovering y so that $[x]_E = U_y$ in a Borel manner from x.

Applications under AD^+

Theorem (N-Norwood)

(Assuming AD^+.) Let I be a σ-ideal on ω^ω so that $\mathbb{P}_I$ is absolutely proper. Let Γ be closed under Borel substitutions, with a universal set. Let E be an equivalence relation with Γ classes (respectively $\Gamma \cap \check{\Gamma}$). Then there is $C \in I^+$ so that $E \restriction C$ is in Γ (respectively $\Gamma \cap \check{\Gamma}$).

An AD^+ strengthening of the Chan-Magidor result.

To prove (the Γ case), take a universal U, force over a countable substructure M. For generic x, embedding theorem allows recovering y so that $[x]_E = U_y$ in a Borel manner from x. So restriction of E to generic reals is in Γ.

Applications under AD^+, cont.

Applications under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+.) Every $X \subseteq [\omega]^\omega$ is H-Ramsey for every happy family H. Consequently there are no MAD families.

Applications under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+.) Every $X \subseteq [\omega]^\omega$ is H-Ramsey for every happy family H. Consequently there are no MAD families.

Proof is similar to the one under large cardinals, but using the AD^+ embedding theorem.

Applications under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+.) Every $X \subseteq [\omega]^\omega$ is H-Ramsey for every happy family H. Consequently there are no MAD families.

Proof is similar to the one under large cardinals, but using the AD^+ embedding theorem.

Since AD implies AD^+ in $L(\mathbb{R})$, gives for example that $\mathsf{AD}^{L(\mathbb{R})}$ implies no MAD families in $L(\mathbb{R})$.

Applications under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+.) Every $X \subseteq [\omega]^\omega$ is H-Ramsey for every happy family H. Consequently there are no MAD families.

Proof is similar to the one under large cardinals, but using the AD^+ embedding theorem.

Since AD implies AD^+ in $L(\mathbb{R})$, gives for example that $\mathsf{AD}^{L(\mathbb{R})}$ implies no MAD families in $L(\mathbb{R})$.

Same in all *known* models of AD....

Applications under AD^+, cont.

Theorem (N-Norwood)

(Assuming AD^+.) Every $X \subseteq [\omega]^\omega$ is H-Ramsey for every happy family H. Consequently there are no MAD families.

Proof is similar to the one under large cardinals, but using the AD^+ embedding theorem.

Since AD implies AD^+ in $L(\mathbb{R})$, gives for example that $\mathsf{AD}^{L(\mathbb{R})}$ implies no MAD families in $L(\mathbb{R})$.

Same in all *known* models of AD....

But still open under AD.

Thank you!