On models generated by uncountable indiscernible sequences

Hiroshi Sakai

Kobe University

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Models generated by indiscernible sequences

Let $\mathcal{M} = \langle M, \dots \rangle$ be a first order structure and $\mathcal{A} = \langle A, <_{\mathcal{A}} \rangle$ be a linearly ordered set (LO for short) with $A \subseteq M$.

- $\mathcal{A}$ is called an **indiscernible sequence** in $\mathcal{M}$ if for any formula $\varphi(v_0, \dots, v_{n-1})$ and any increasing sequences $\bar{a} = \langle a_i \mid i < n \rangle$ and $\bar{b} = \langle b_i \mid i < n \rangle$ in $\mathcal{A}$,

$$\mathcal{M} \models \varphi(\bar{a}) \quad \Leftrightarrow \quad \mathcal{M} \models \varphi(\bar{b}) \, .$$

- We say that $\mathcal{M}$ is **generated** by $\mathcal{A}$ if for any $x \in M$ there are a term $t(v_0, \dots, v_{n-1})$ and a sequence $\bar{a} = \langle a_i \mid i < n \rangle$ in A such that $x = t^{\mathcal{M}}(\bar{a})$.

Theorem (Ehrenfeucht-Mostowski)

Let T be a theory with built-in Skolem functions which has an infinite model. Then for any infinite LO $\mathcal{A}$ there is a model of T which is generated by an indiscernible sequence isomorphic to $\mathcal{A}$.

- We only discuss models of countable languages generated by uncountable indiscernible sequences. In this talk, "a structure" and "a model" mean those of countable languages.
- If $\mathcal{M}$ is generated by an indiscernible sequence $\mathcal{A}$, then $\mathcal{M}$ is somewhat "similar" as $\mathcal{A}$. We will observe this phenomenon by investigating what kinds of uncountable LO's are embeddable into relations definable in $\mathcal{M}$. In particular, we will discuss the embeddability of the following basic LO's:
  - ω_1 and ω_1^*.
  - uncountable suborders of $\mathbb{R}$.
  - Aronszajn lines, i.e. uncountable LO's into which none of ω_1, ω_1^* and uncountable suborders of $\mathbb{R}$ are embeddable.
- For an LO $\mathcal{A}$, let $\mathcal{A}^*$ denote the reversal of $\mathcal{A}$.
- For an LO $\mathcal{A} = \langle A, <_{\mathcal{A}} \rangle$ and a binary relation R on a set B, we say that $\mathcal{A}$ is **embeddable** into $\langle B, R \rangle$ if there is an injection $e : A \to B$ such that

$$a <_{\mathcal{A}} b \quad \Leftrightarrow \quad R(e(a), e(b))$$

for all $a, b \in A$.

Embeddability into LO

First, we discuss the embeddability into definable linear orderings of the universe:

Theorem 1

Let $\mathcal{M} = \langle M, \dots \rangle$ be a structure generated by an uncountable indiscernible sequence $\mathcal{A}$, and let R be an LO of M which is definable in $\mathcal{M}$. If an uncountable LO $\mathcal{X}$ is embeddable into $\langle M, R \rangle$, then there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that either $\mathcal{B}$ or $\mathcal{B}^*$ is embeddable into $\mathcal{X}$.

Corollary 2

Let $\mathcal{M}$, $\mathcal{A}$ and R be as in Theorem 1.

1. If $\mathcal{A} \cong \omega_1$, then neither uncountable suborders of $\mathbb{R}$ nor Aronszajn lines are embeddable into $\langle M, R \rangle$.
2. If $\mathcal{A} \cong \mathbb{R}$, then none of ω_1, ω_1^* and Aronszajn lines are embeddable into $\langle M, R \rangle$.
3. If $\mathcal{A}$ is an Aronszajn line, then none of ω_1, ω_1^* and uncountable suborders of $\mathbb{R}$ are embeddable in $\langle M, R \rangle$, that is, $\langle M, R \rangle$ is an Aronszajn line.

Outline of proof of Theorem 1

Let $\mathcal{M} = \langle M, \dots \rangle$ be a structure generated by an uncountable indiscernible sequence $\mathcal{A} = \langle A, <_{\mathcal{A}} \rangle$, and let R be an LO of M which is definable in $\mathcal{M}$. For simplicity, we assume R is definable without parameters.

Theorem 1 easily follows from the next theorem.
(The uncountability of $\mathcal{A}$ is not necessary for the following theorem.)

Theorem (Hodges)

Let $t(v_0, \dots, v_{n-1})$ be a term. Then there are an LO $\lhd$ of n and a function $s : n \to \{-1, 1\}$ for which the following hold:

> *Suppose $\bar{a} = \langle a_i \mid i < n \rangle$, $\bar{b} = \langle b_i \mid i < n \rangle$ are increasing sequences in $\mathcal{A}$, and $t^{\mathcal{M}}(\bar{a}) \neq t^{\mathcal{M}}(\bar{b})$. Let j be the $\lhd$-largest $i < n$ such that $a_i \neq b_i$. Then*
>
> $$R(t^{\mathcal{M}}(\bar{a}), t^{\mathcal{M}}(\bar{b})) \ \Leftrightarrow \ a_j <_{\mathcal{A}}^{s(j)} b_j \ ,$$
>
> *where $<_{\mathcal{A}}^{1} := <_{\mathcal{A}}$ and $<_{\mathcal{A}}^{-1} := <_{\mathcal{A}}^{*}$.*

Now we prove Theorem 1:

- Suppose $X \subseteq M$ is uncountable. It suffices to find an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that either $\mathcal{B}$ or $\mathcal{B}^*$ is embeddable into $\langle X, R \rangle$.
- For each $x \in X$, take a term $t_x(v_0, \ldots, v_{n_x - 1})$ and a sequence $\bar{a}_x = \langle a_i^x \mid i < n_x \rangle$ in $\mathcal{A}$ such that $x = t_x^{\mathcal{M}}(\bar{a}_x)$.
- There are an uncountable $Y \subseteq X$, a term $t(v_0, \ldots, v_{n-1})$ and $I \subseteq n$ s.t.
    - $t_x = t$ for all $x \in Y$,
    - $a_i^x = a_i^y$ for all $x, y \in Y$ and $i \in I$,
    - $a_i^x \neq a_i^y$ for all distinct $x, y \in Y$ and $i \in n \setminus I$.
- Let $\lhd$ and $s : n \to \{-1, 1\}$ be as in Hodges Theorem. Take the $\lhd$-largest $j \in n \setminus I$. Let $B := \{a_j^x \mid x \in Y\}$ and $\mathcal{B} := \langle B, <_{\mathcal{A}} \rangle$.
- Then $x \mapsto a_j^x$ is an isomorphism from $\langle Y, R \rangle$ to $\mathcal{B}$ if $s(j) = 1$ and to $\mathcal{B}^*$ if $s(j) = -1$. So either $\mathcal{B}$ or $\mathcal{B}^*$ is embeddable into $\langle X, R \rangle$.

□

Embeddability into relations

Question

Can we generalize Theorem 1 and Corollary 2 to the embeddability into a binary relation which may not be an LO?

In this general setting, we have the following:

Theorem 4

Let $\mathcal{M} = \langle M, \dots \rangle$ be a structure generated by an uncountable indiscernible sequence $\mathcal{A}$, and let R be a binary relation on M which is definable in $\mathcal{M}$.

1. If neither ω_1 nor ω_1^* is embeddable into $\mathcal{A}$, then neither ω_1 nor ω_1^* is embeddable into $\langle M, R \rangle$.
2. If no uncountable suborders of $\mathbb{R}$ are embeddable into $\mathcal{A}$, then no uncountable suborders of $\mathbb{R}$ are embeddable into $\langle M, R \rangle$.
3. If no Aronszajn lines are embeddable into $\mathcal{A}$, then no Aronszajn lines are embeddable into $\langle M, R \rangle$.

Outline of Proof of Theorem 4 (1)

Let $\mathcal{M} = \langle M, \dots \rangle$ be a structure generated by an uncountable indiscernible sequence $\mathcal{A}$, and let R be a binary relation on M which is definable in $\mathcal{M}$. For simplicity, we assume R is definable without parameters.

Suppose neither ω_1 nor ω_1^* is embeddable into $\mathcal{A}$. We prove that neither ω_1 nor ω_1^* is embeddable into $\langle M, R \rangle$.

By the assumption, $\mathcal{A}$ has the following tree representation:

Lemma

There is a tree $\mathcal{T} = \langle A, <_{\mathcal{T}} \rangle$ of height $\leq \omega_1$ such that

- $\mathcal{T}$ is countably branching,
- $\mathcal{T}$ does not have a branch of length ω_1,
- $<_{\mathcal{A}}$ coincides with a lexicographic ordering of $\mathcal{T}$.

- Suppose $e : \omega_1 \to M$ is injective. We prove that e is not monotone w.r.t. R. That is, there are $\xi_0 < \xi_1 < \xi_2 < \omega_1$ such that

$$R(e(\xi_0), e(\xi_1)) \ \Leftrightarrow \ R(e(\xi_2), e(\xi_1)) .$$

- By shrinking the domain of e if necessary, we may assume the following: There are a term $t(v_0, \ldots, v_{n-1})$ and an increasing seq. $\bar{a}^\xi = \langle a_i^\xi \mid i < n \rangle$ in $\mathcal{A}$ for each $\xi < \omega_1$ such that
  - $e(\xi) = t^{\mathcal{M}}(\bar{a}_\xi)$ for all $\xi < \omega_1$.
  - $\langle \{a_i^\xi \mid i < n\} \mid \xi < \omega_1 \rangle$ is a Δ-system.

  For simplicity, we assume that $\langle \{a_i^\xi \mid i < n\} \mid \xi < \omega_1 \rangle$ is pairwise disjoint.

- Since $\mathcal{A}$ is indiscernible, it suffices to find $\xi_0 < \xi_1 < \xi_2 < \omega_1$ such that for every $i, j < n$,

$$a_i^{\xi_0} <_{\mathcal{A}} a_j^{\xi_1} \ \Leftrightarrow \ a_i^{\xi_2} <_{\mathcal{A}} a_j^{\xi_1} .$$

- Take a sufficiently large regular cardinal θ and a countable $N \prec \mathcal{H}_\theta$ containing all relevant objects.
- Take $\xi_1 \in \omega_1 \setminus N$.

Lemma

There are an uncountable $P \subseteq \omega_1$ with $P \in N$ and $\{b_i \mid i < n\}, \{c_{ij} \mid i, j < n\} \subseteq A \cap N$ such that

- $b_i <_\mathcal{T} a_i^\xi$ for every $\xi \in P$ and $i < n$.
- $c_{ij} <_\mathcal{T} a_j^{\xi_1}$ for every $i, j < n$.
- b_i and c_{ij} are incomparable in $\mathcal{T}$ for every $i, j < n$.

- We can take $\xi_0, \xi_2 \in P$ such that $\xi_0 \in N$ (so $\xi_0 < \xi_1$) and $\xi_1 < \xi_2$, since P belongs to N and is uncountable.
- Since $<_\mathcal{A}$ is a lexicographic order of $\mathcal{T}$, we have

$$a_i^{\xi_0} <_\mathcal{A} a_j^{\xi_1} \;\Leftrightarrow\; b_i <_\mathcal{A} c_{ij} \;\Leftrightarrow\; a_i^{\xi_2} <_\mathcal{A} a_j^{\xi_1}.$$

for all $i, j < n$. □

Generalization of Corollary 2

Generalization of Corollary 2 is immediate from Theorem 4.

Corollary 5

Let $\mathcal{M}$, $\mathcal{A}$ and R be as in Theorem 4.

1. If $\mathcal{A} \cong \omega_1$, then neither uncountble suborders of $\mathbb{R}$ nor Aronszajn lines are embeddable into $\langle M, R \rangle$.
2. If $\mathcal{A} \cong \mathbb{R}$, then none of ω_1, ω_1^* and Aronszajn lines are embeddable into $\langle M, R \rangle$.
3. If $\mathcal{A}$ is an Aronszajn line, then none of ω_1, ω_1^* and uncountable suborders of $\mathbb{R}$ are embeddable in $\langle M, R \rangle$.

Generalization of Theorem 1 under PFA

Recall Moore's Five Element Basis Theorem:

Five Element Basis Theorem (Moore)

Assume PFA. Let $\mathcal{R}$ be a suborder of $\mathbb{R}$ of size $\aleph_1$ and $\mathcal{C}$ be a Countryman line. Then for any uncountable LO $\mathcal{X}$, one of the following is embeddable into $\mathcal{X}$:

$$\omega_1 \ , \ \omega_1^* \ , \ \mathcal{R} \ , \ \mathcal{C} \ , \ \mathcal{C}^*$$

Recall also that a Countryman line is an Aronszajn line.
Theorem 4 together with Five Element Basis Theorem easily implies the following generalization of Theorem 1:

Corollary 6

Assume PFA. Let $\mathcal{M}$, $\mathcal{A}$ and R be as in Theorem 4. If an uncountable LO $\mathcal{X}$ is embeddable into $\langle M, R \rangle$, then there is an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that either $\mathcal{B}$ or $\mathcal{B}^*$ is embeddable into $\mathcal{X}$.

Failure of Generalization of Theorem 1

Question

Is the generalization of Theorem 1 provable in ZFC?

Under $\diamondsuit_{\omega_1}$, the generalization of Theorem 1 fails:

Theorem 7

Assume $\diamondsuit_{\omega_1}$. Then there are a structure $\mathcal{M} = \langle M, R, \dots \rangle$ generated by some uncountable indiscernible sequence $\mathcal{A}$ and an uncountable LO $\mathcal{X}$ such that

1. R is a binary relation on M, and $\mathcal{X}$ is embeddable into $\langle M, R \rangle$,
2. for any uncountable $\mathcal{B} \subseteq \mathcal{A}$, neither $\mathcal{B}$ nor $\mathcal{B}^*$ is embeddable into $\mathcal{X}$.

Idea of Proof of Theorem 7

We use the following relation by the majority rule:

For an LO $\mathcal{A} = \langle A, <_{\mathcal{A}} \rangle$, let $R^{\mathcal{A}}_{\mathrm{maj}}$ be the following relation on A^3:

$$R^{\mathcal{A}}_{\mathrm{maj}}((a_0, a_1, a_2), (b_0, b_1, b_2)) \overset{\text{def}}{\Leftrightarrow} |\{i < 3 \mid a_i <_{\mathcal{A}} b_i\}| \geq 2 \, .$$

- $R^{\mathcal{A}}_{\mathrm{maj}}$ is not an LO. In fact, it is not transitive.
- Each of 3 coordinates equally contributes to $R^{\mathcal{A}}_{\mathrm{maj}}$. This avoids Hodges Thm.

Proposition 8

Assume $\diamondsuit_{\omega_1}$. Then there are an Aronszajn line $\mathcal{A} = \langle A, <_{\mathcal{A}} \rangle$ and an uncountable $X \subseteq A^3$ such that

1. $\langle X, R^{\mathcal{A}}_{\mathrm{maj}} \rangle$ is an LO,
2. for any uncountable $\mathcal{B} \subseteq \mathcal{A}$, neither $\mathcal{B}$ nor $\mathcal{B}^*$ is embeddable into $\langle X, R^{\mathcal{A}}_{\mathrm{maj}} \rangle$.

Proof of Theorem 7 using Proposition 8

- Let $\mathcal{L} = \langle L, <_{\mathcal{L}} \rangle$ be an infinite LO. Take a bijection $f' : L^3 \to L$, and define a binary relation R' on L by

$$R'(f'(a_0, a_1, a_2), f'(b_0, b_1, b_2)) \overset{\text{def}}{\Leftrightarrow} R^{\mathcal{L}}_{\mathrm{maj}}((a_0, a_1, a_2), (b_0, b_1, b_2)) .$$

  Let $\mathcal{M}'$ be a countable expansion of $\langle L, <_{\mathcal{L}}, f', R' \rangle$ which has built-in Skolem funcitons.

- Let $\mathcal{A}$ and X be as in Prop. 8, and let $\mathcal{X} := \langle X, R^{\mathcal{A}}_{\mathrm{maj}} \rangle$. Then $\mathcal{X}$ is an LO, and for any uncountable $\mathcal{B} \subseteq \mathcal{A}$, neither $\mathcal{B}$ nor $\mathcal{B}^*$ is embeddable into $\mathcal{X}$.

- By Ehrenfeucht-Mostowski Thm, we can take a structure $\mathcal{M} = \langle M, <_{\mathcal{M}}, f, R, \dots \rangle$ such that
  - $\mathcal{M}$ is elementary equivalent to $\mathcal{M}'$,
  - $\mathcal{A}$ is an indiscernible sequence in $\mathcal{M}$ and generates $\mathcal{M}$.

  Then $f \upharpoonright X$ is an embedding from $\mathcal{X}$ to $\langle M, R \rangle$. □

Question

Theorem 1, which is on the embeddability into definable LO's, is provable in ZFC, but Theorem 1 for definable binary relations is not provable in ZFC.

Question

For what kinds of binary relations can we prove Theorem 1 in ZFC?
How about partial orderings?