The Hurewicz dichotomy for definable subsets of generalized Baire spaces

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Its closed subsets are the sets

$$[T] = \{x \in {}^{\kappa}\kappa \mid \forall \alpha < \kappa \ (x{\restriction}\alpha) \in T\}$$

of branches through subtrees T of ${}^{<\kappa}\kappa$. A subtree is a downwards closed subset.

The classical Hurewicz dichotomy

A K_σ set is a countable union of compact sets.

Theorem (Hurewicz)

Any Polish space is either K_σ or it contains a closed subset that is homeomorphic to the Baire space ${}^\omega\omega$.

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A subset of a Polish space X satisfies the *Hurewicz dichotomy* if it is either contained in a K_σ subset of X or it contains a closed subset that is homeomorphic to the Baire space.

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Theorem

In Solovay's model all subsets of the Baire space satisfy the Hurewicz dichotomy.

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Definition

A subset A of ${}^\kappa\kappa$ satisfies the *topological Hurewicz dichotomy* if either A is contained in a K_κ-subset of ${}^\kappa\kappa$ or A contains a closed subset of ${}^\kappa\kappa$ homeomorphic to ${}^\kappa\kappa$.

Facts about κ-compact sets

Lemma

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A subtree T of ${}^{<\kappa}\kappa$ is *pruned* if through every node in T, there is a cofinal branch.

Lemma (Halko)

For any pruned subtree T of ${}^{<\kappa}\kappa$, the following statements are equivalent.

- *$[T]$ is κ-compact.*
- *T is a κ-tree without κ-Aronszajn subtrees.*

A previous result

Theorem (Lücke–Motto Ros–S. 2016)

There is a $<\kappa$*-closed* κ^+*-c.c. partial order which forces that every* $\boldsymbol{\Sigma}^1_1$*-subset of* ${}^\kappa\kappa$ *satisfies the topological Hurewicz Dichotomy.*

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- Variants that respect not only the topology, but the structure of ${}^{<\kappa}\kappa$

If κ is not weakly compact, then the topological Hurewicz dichotomy follows from the perfect set property.

Theorem (S. 201^∞)

If $\lambda > \kappa$ is inaccessible, then in any $\mathrm{Col}(\kappa, <\lambda)$-generic extension $V[G]$, every subset of ${}^\kappa\kappa$ that is definable from an element of ${}^\kappa\kappa$ satisfies the perfect set property.

Superperfect sets

We write $\mathrm{l}(s) = \mathrm{dom}(s)$ for $s \in {}^{<\kappa}\kappa$.

Definition

Suppose that T is a subtree of ${}^{<\kappa}\kappa$ and A is a subset of ${}^{\kappa}\kappa$.

- T is $<\kappa$-splitting if every node in T has strictly less than κ many direct successors.

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- T is *weakly superperfect* if for every s in T, there is a level on which s has κ many successors.

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- A is *weakly superperfect* if $A = [S]$ for some weakly superperfect subtree S of ${}^{<\kappa}\kappa$.

The superperfect set game

Definition

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In the β-th round, player I loses immediately unless $s_\alpha \subsetneq s_\beta$ and $s_{\alpha+1}(\mathrm{l}(s_\alpha)) > \gamma_\alpha$ for all $\alpha < \beta$.

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If player I has not lost before stage κ, then $x = \bigcup_{\alpha<\kappa} s_\alpha$ is an element of ${}^{\kappa}\kappa$ and player I wins if $x \in A$.

A characterization of the superperfect set property

Lemma

Assume that A is a subset of ${}^{\kappa}\kappa$.

- *Player I has a winning strategy in $G(A)$ if and only if A contains a superperfect subset.*

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- *Player I has a winning strategy in $G(A)$ if and only if A contains a superperfect subset.*
- *Player II has a winning strategy in $G(A)$ if and only if there is a sequence $\vec{T} = \langle T_\alpha \mid \alpha < \kappa \rangle$ of $<\kappa$-splitting subtrees of ${}^{<\kappa}\kappa$ with $A \subseteq \bigcup_{\alpha<\kappa}[T_\alpha]$.*

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- *Player I has a winning strategy in $G^*(A)$ if and only if A contains a weak superperfect subset.*
- *Player II has a winning strategy in $G^*(A)$ if and only if A is eventually bounded.*

Statement of the result

Definition

A subset of ${}^\kappa\kappa$ satisfies the *Hurewicz dichotomy* if it is either contained in a union of κ many sets of the form $[T]$ where T is a $<\kappa$-splitting subtree of ${}^\kappa\kappa$, or it contains a set of the form $[S]$ where S is superperfect.

Theorem

If $\lambda > \kappa$ is inaccessible, then in any $\mathrm{Col}(\kappa, <\lambda)$-generic extension $V[G]$, every subset of ${}^\kappa\kappa$ that is definable from an element of ${}^\kappa\kappa$ satisfies the Hurewicz dichotomy.

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- The proof builds on the proofs of the perfect set property and the Hurewicz dichotomy for $\mathbf{\Sigma}^1_1$ subsets of ${}^{\kappa}\kappa$.

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- The proof builds on the proofs of the perfect set property and the Hurewicz dichotomy for $\mathbf{\Sigma}^1_1$ subsets of ${}^{\kappa}\kappa$.
- The usual forcing arguments for the Hurewicz dichotomy don't work in our situation because of bad quotients. More precisely, if $V[G]$ is an $\mathrm{Add}(\kappa, 1)$-generic extension then there is an $\mathrm{Add}(\kappa, 1)$-generic extension $V[h] \subseteq V[G]$ such that no quotient forcing for $V[h]$ is equivalent to $\mathrm{Add}(\kappa, 1)$.

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To prove the dichotomy for A, we distinguish two cases and will need the following forcing $\mathbb{P}$.

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Definition

A condition in $\mathbb{P}$ is a pair (S, α), where S is a ${<}\kappa$-splitting subtree of ${}^{<\kappa}\kappa$ of height κ and $\alpha < \kappa$. Let $(S, \alpha) \leq (T, \beta)$ if $S \supseteq T$, $\alpha \geq \beta$ and $\mathrm{Lev}_{\leq\beta}(S) = \mathrm{Lev}_{\leq\beta}(T)$.

The case of covering by small sets

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If T is a subtree of ${}^{<\kappa}\kappa$, $s, t \in \kappa^\alpha$ for some $\alpha < \kappa$ with $s \in T$, then we call the tree $T_{s \frown t} = \{t^\frown u \mid s^\frown u \in T\}$ a *local translate* of T.

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- for all α with $2^\kappa < \alpha < \lambda$ the following holds in $V[G{\restriction}\alpha]$: $A^{G\restriction\alpha}$ is covered by the sets $[S]$ for all local translates S of $\dot{T}^{G\restriction\alpha}$

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Then the first case of the dichotomy holds.

The case of a superperfect subset

Assuming that this fails

- we can choose $\alpha = \nu + 1$, where $\nu^{<\kappa} = \nu$ and
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For any $p \in \mathrm{Col}(\kappa, {<}\alpha)$ let

$$T^{\dot{x},p} = \{t \in {}^{<\kappa}\kappa \mid \exists q \leq p \ q \Vdash^{V}_{\mathrm{Col}(\kappa,<\alpha)} t \subseteq \dot{x}\}$$

denote the *tree of possible values* for $\dot{x}$ below p.

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The tree $T^{\dot{x},p}$ has a κ-splitting node for all $p \in \mathrm{Col}(\kappa, {<}\alpha)$ and moreover, $1_{\mathrm{Col}(\kappa,<\alpha)}$ forces that $\Vdash_{\mathrm{Col}(\kappa,<\lambda)} \dot{x} \in \dot{A}$

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By a factoring argument, we can show that there is an $\mathrm{Add}(\kappa, 1)$-name with the same property.

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- For any $p \in v$ and any direct successor q in u of p, the tree $T^{\dot{x},p}$ has a κ-splitting node $r = r_q \in \kappa^\alpha$ for some $\alpha = \alpha_q < \kappa$ such that $q \Vdash \dot{x} \restriction \alpha = r$ and q decides $\dot{x}(\alpha)$ as some $\gamma_q < \kappa$.

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- For any $p \in v$ and any distinct direct successors q, q' in u of p, we have $r_q = r_{q'}$, $\alpha_q = \alpha_{q'}$ but $\gamma_q \neq \gamma_{q'}$.

Definition

Let $\mathbb{P}_{\dot{x}}$ denote the set of triples (t, u, v) with the following properties.

- t is a subtree of ${}^{<\kappa}\kappa$ of size strictly less then κ.
- u is a disjoint union of segments ξ (linearly ordered convex subsets) of t with a maximum $m(\xi)$ that is the only element of ξ that can split in t. Moreover, v is the collection of all these nodes $m(\xi)$.
- For any $p \in v$ and any direct successor q in u of p, the tree $T^{\dot{x},p}$ has a κ-splitting node $r = r_q \in \kappa^\alpha$ for some $\alpha = \alpha_q < \kappa$ such that $q \Vdash \dot{x}{\restriction}\alpha = r$ and q decides $\dot{x}(\alpha)$ as some $\gamma_q < \kappa$.
- For any $p \in v$ and any distinct direct successors q, q' in u of p, we have $r_q = r_{q'}$, $\alpha_q = \alpha_{q'}$ but $\gamma_q \neq \gamma_{q'}$.
- Any maximal node in t is in u (and hence in v).

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Let $(t', u', v') \leq (t, u, v)$ if $t' \supseteq t$, $u' \cap t = u \cap t$ and $v' \cap t = v \cap t$.

Let G be a $\mathbb{P}_{\dot{x}}$-generic filter over V. The forcing adds a tree

$$T = \bigcup_{(t,u,v)\in G} t$$

and subsets of T

$$U = \bigcup_{(t,u,v)\in G} u$$

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The branches in (T, U) induce a superperfect tree via $\dot{x}$.

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The forcing $\mathbb{P}_{\dot{x}}$ is equivalent to $\mathrm{Add}(\kappa, 1)$, since it is $<\kappa$-closed, nonatomic and has size κ.

We replace g with a $\mathbb{P}_{\dot{x}}$-generic filter g^* with $V[G] = V[g^* \times h]$. This will induce a superperfect subset of A.

Open questions

A different version of the dichotomy:

Theorem (Hurewicz)

Suppose that A is an analytic subset of a Polish space X. Then A is an F_σ set or there is a subset C of X such that

- *C is homeomorphic to the Cantor space and*

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Question

Is a variant of this result consistent for ${}^\kappa\kappa$?

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The schema for the proofs of the consistency of the perfect set property and the superperfect set property is similar.

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Is the inaccessible cardinal necessary?