The isomorphism and bi-embeddability relations for countable torsion abelian groups

Simon Thomas
(Joint work with Filippo Calderoni)

Rutgers University

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Countable torsion abelian groups

Countable torsion abelian groups

Fact

If A is a countable torsion abelian group, then $A = \bigoplus_{p \in P} A_p$ is the direct sum of its (possibly finite) *p-primary components*

$$A_p = \{\, a \in A \mid (\, \exists n \geq 0 \,)\ p^n a = 0 \,\}.$$

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Furthermore, if $B = \bigoplus_{p \in P} B_p$ is a second countable torsion abelian group, then:

- $A \cong B$ iff $A_p \cong B_p$ for every prime p;

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Fact

Furthermore, if $B = \bigoplus_{p \in P} B_p$ is a second countable torsion abelian group, then:

- $A \cong B$ iff $A_p \cong B_p$ for every prime p;
- A, B are bi-embeddable iff A_p, B_p are bi-embeddable for every prime p.

Isomorphism vs bi-embeddability of abelian p-groups

Notation

For each prime p,

- $\mathcal{A}_p$ is the space of countable abelian p-groups;
- $\cong_p$ is the isomorphism relation on $\mathcal{A}_p$;
- $\equiv_p$ is the bi-embeddability relation on $\mathcal{A}_p$.

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Theorem

$\cong_p$ and $\equiv_p$ are incomparable with respect to Borel reducibility.

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Hypothesis (*RC*)

There exists a Ramsey cardinal.

Theorem (*RC*)

$\cong_p$ is strictly more complex than $\equiv_p$ with respect to $\mathbf{\Delta}^1_2$ reducibility.

Comparing different primes

Comparing different primes

Theorem

If $p \neq q$ are distinct primes, then the isomorphism relations $\cong_p$ and $\cong_q$ are $\mathbf{\Delta}^1_2$ bireducible.

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If $p \neq q$ are distinct primes, then the bi-embeddability relations $\equiv_p$ and $\equiv_q$ are $\mathbf{\Delta}^1_2$ bireducible.

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Conjecture

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If $p \neq q$ are distinct primes, then the bi-embeddability relations $\equiv_p$ and $\equiv_q$ are incomparable with respect to Borel reducibility.

The Ulm analysis of countable abelian p-groups

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Definition

If A is a countable abelian p-group, then the α-th Ulm subgroup A^α is defined inductively by:

- $A^0 = A$;
- $A^{\alpha+1} = \bigcap_{n<\omega} p^n A^\alpha$;
- $A^\delta = \bigcap_{\alpha<\delta} A^\alpha$, *if δ is a limit ordinal.*

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Definition

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Remarks/Definition

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Remarks/Definition

- $\tau(A)$ is a countable ordinal.
- $A^{\tau(A)}$ is the maximal divisible subgroup of A.
- A is reduced if $A^{\tau(A)} = 0$.

Definition

For each $\alpha < \tau(A)$, the αth Ulm factor is $A_\alpha = A^\alpha / A^{\alpha+1}$.

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Fact

Each Ulm factor A_α is a Σ-cyclic p-group; i.e.

$$A_\alpha \cong \bigoplus_{n \geq 1} C_{p^n}^{(s_n)} = \bigoplus_{n \geq 1} \underbrace{C_{p^n} \oplus \cdots \oplus C_{p^n}}_{s_n \text{ times}}$$

where C_{p^n} is cyclic of order p^n and $s_n \in \omega \cup \{ \omega \}$.

The Ulm analysis of countable abelian p-groups

Theorem (Ulm)

If A and B are countable abelian p-groups, then $A \cong B$ iff the following conditions are satisfied:

(i) $\tau(A) = \tau(B)$;

(ii) $A_\alpha \cong B_\alpha$ *for each* $\alpha < \tau(A) = \tau(B)$;

(ii) *The (possibly trivial) divisible subgroups* $A^{\tau(A)}$, $B^{\tau(B)}$ *are isomorphic.*

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Remark

- $A^{\tau(A)}$ is isomorphic to a direct sum of d copies of the quasi-cyclic group $\mathbb{Z}(p^\infty)$ for some $d \in \omega \cup \{ \omega \}$.

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Remark

- $A^{\tau(A)}$ is isomorphic to a direct sum of d copies of the quasi-cyclic group $\mathbb{Z}(p^\infty)$ for some $d \in \omega \cup \{\omega\}$.
- We write $\text{rk}(A^{\tau(A)}) = d$.

The Zippin realization theorem

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Theorem (Zippin)

Suppose that $0 < \tau < \omega_1$ *and that* $(C_\alpha \mid \alpha < \tau)$ *is a sequence of nontrivial countable (possibly finite)* Σ*-cyclic p-groups. Then the following statements are equivalent:*

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(i) *There exists a countable reduced abelian p-group A with* $\tau(A) = \tau$ *such that* $A_\alpha \cong C_\alpha$ *for all* $\alpha < \tau$.

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Suppose that $0 < \tau < \omega_1$ and that $(C_\alpha \mid \alpha < \tau)$ is a sequence of nontrivial countable (possibly finite) Σ-cyclic p-groups. Then the following statements are equivalent:

(i) *There exists a countable reduced abelian p-group A with $\tau(A) = \tau$ such that $A_\alpha \cong C_\alpha$ for all $\alpha < \tau$.*

(ii) *C_α is unbounded for each α such that $\alpha + 1 < \tau$.*

Definition

A Σ-cyclic p-group $G = \bigoplus_{n\geq 1} C_{p^n}^{(s_n)}$ is bounded if there exists an integer $m \geq 0$ such that $s_n = 0$ for all $n \geq m$.

Comparing different primes

Notation

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Theorem

If $p \neq q$ are distinct primes, then the isomorphism relations $\cong_p$ and $\cong_q$ are $\mathbf{\Delta}^1_2$ bireducible.

Proof.

There exist $\mathbf{\Delta}^1_2$ maps

$$A \in \mathcal{A}_p \mapsto \mathbf{u}(A) \mapsto A' \in \mathcal{A}_q$$

such that $\mathbf{u}(A') = \mathbf{u}(A)$. □

Bi-embeddability: the Barwise-Eklof analysis

Theorem

The countable abelian p-groups A, B are bi-embeddable iff either:

Bi-embeddability: the Barwise-Eklof analysis

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The countable abelian p-groups A, B are bi-embeddable iff either:

(a) $\operatorname{rk}(A^{\tau(A)}) = \operatorname{rk}(B^{\tau(B)}) = \omega$;

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The countable abelian p-groups A, B are bi-embeddable iff either:

(a) $\text{rk}(A^{\tau(A)}) = \text{rk}(B^{\tau(B)}) = \omega$; *or*

(b) $\text{rk}(A^{\tau(A)}) = \text{rk}(B^{\tau(B)}) < \omega$ *and the following conditions hold:*

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(b) $\mathrm{rk}(A^{\tau(A)}) = \mathrm{rk}(B^{\tau(B)}) < \omega$ *and the following conditions hold:*

  (i) $\tau(A) = \tau(B)$;

  (ii) *if* $\tau(A) = \tau(B) = \beta + 1$, *then the final Ulm factors* A_β, B_β *are bi-embeddable.*

Bi-embeddability of countable Σ-cyclic p-groups

Observation

If $G = \bigoplus_{n \geq 1} C_{p^n}^{(s_n)}$ and $H = \bigoplus_{n \geq 1} C_{p^n}^{(t_n)}$, where each s_n, $t_n \in \omega \cup \{ \omega \}$, then G and H are bi-embeddable iff one of the following holds:

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(i) G and H are both unbounded.

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(i) G and H are both unbounded.

(ii) G and H are both infinite bounded Σ-cyclic p-groups and

- $m_G = \max\{ n \mid s_n = \omega \} = \max\{ n \mid t_n = \omega \} = m_H$;
- $s_n = t_n$ for all $n \geq m_G = m_H$.

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(iii) G and H are isomorphic finite p-groups.

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Remark

In particular, there are only countably many countable Σ-cyclic p-groups up to bi-embeddability.

Bi-embeddability of countable Σ-cyclic p-groups

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- $s_n = t_n$ for all $n \geq m_G = m_H$.

(iii) G and H are isomorphic finite p-groups.

Remark

Each bi-embeddability class countable Σ-cyclic p-groups contains a "maximal" isomorphism class.

Bi-embeddability of countable Σ-cyclic p-groups

Observation

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(iii) G and H are isomorphic finite p-groups.

Theorem

If $p \neq q$ are distinct primes, then the bi-embeddability relations $\equiv_p$ and $\equiv_q$ are $\boldsymbol{\Delta}^1_2$ bireducible.

Bi-embeddability vs isomorphism

Lemma

The bi-embeddability relation $\equiv_p$ is $\mathbf{\Delta}^1_2$ reducible to the isomorphism relation $\cong_p$.

Bi-embeddability vs isomorphism

Lemma

The bi-embeddability relation $\equiv_p$ *is* $\mathbf{\Delta}^1_2$ *reducible to the isomorphism relation* $\cong_p$*.*

Proof.

In fact, there exists a $\mathbf{\Delta}^1_2$ map which selects the maximal isomorphism class within each bi-embeddability class. □

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Proof.

- $D_\infty = \{ A \in \mathcal{A}_p \mid \mathrm{rk}(A^{\tau(A)}) = \omega \}$ is a single $\equiv_p$-class.

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The bi-embeddability relation $\equiv_p$ is not Borel reducible to the isomorphism relation $\cong_p$.

Proof.

- $D_\infty = \{ A \in \mathcal{A}_p \mid \mathrm{rk}(A^{\tau(A)}) = \omega \}$ is a single $\equiv_p$-class.
- Every $\cong_p$-class is Borel.

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The bi-embeddability relation $\equiv_p$ is not Borel reducible to the isomorphism relation $\cong_p$.

Proof.

- $D_\infty = \{ A \in \mathcal{A}_p \mid \mathrm{rk}(A^{\tau(A)}) = \omega \}$ is a single $\equiv_p$-class.
- Every $\cong_p$-class is Borel.
- Thus it is enough to prove that D_∞ is not Borel.

□

D_∞ is a complete analytic subset of $\mathcal{A}_p$

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Definition (Feferman)

For each infinite tree $T \subseteq \omega^{<\omega}$, let $G_p(T)$ be the abelian p-group generated by the elements $\{ a_t \mid t \in T \}$ subject to the relations

- $p\, a_{t^\frown \ell} = a_t$
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- $G_p(T)$ is reduced iff T is well-founded.

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- $G_p(T)$ is reduced iff T is well-founded.
- Let $G_p(T)^{(\omega)}$ be the direct sum of ω copies of $G_p(T)$.

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Proof.

- $G_p(T)$ is reduced iff T is well-founded.
- Let $G_p(T)^{(\omega)}$ be the direct sum of ω copies of $G_p(T)$.
- $G_p(T)^{(\omega)} \in D_\infty$ iff T is not well-founded.

□

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$\cong_p$ *is not Borel reducible to* $\equiv_p$.

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- Suppose that $\cong_p$ is Borel reducible to $\equiv_p$.
- By Shoenfield Absoluteness, we can suppose that $2^\omega > \omega_1$.

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Proof.

- Suppose that $\cong_p$ is Borel reducible to $\equiv_p$.
- By Shoenfield Absoluteness, we can suppose that $2^\omega > \omega_1$.
- But there are 2^ω many $\cong_p$-classes and only ω_1 many $\equiv_p$-classes.

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□

Theorem (*RC*)

$\cong_p$ is not $\boldsymbol{\Delta}^1_2$ reducible to $\equiv_p$.

Proof.

By Martin-Solovay Absoluteness. □

Countable torsion abelian groups

Fact

If A is a countable torsion abelian group, then $A = \bigoplus_{p \in P} A_p$ is the direct sum of its (possibly finite) p-primary components

$$A_p = \{ a \in A \mid (\exists n \geq 0)\ p^n a = 0 \}.$$

Fact

Furthermore, if $B = \bigoplus_{p \in P} B_p$ is a second countable torsion abelian group, then:

- $A \cong B \iff A_p \cong_p B_p$ for every prime p;
- $A \equiv B \iff A_p \equiv_p B_p$ for every prime p.

Countable torsion abelian groups

Notation

- $\mathcal{A}_{tor}$ is the space of countable torsion abelian groups;
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Theorem (*RC*)

$\cong_{tor}$ is strictly more complex than $\equiv_{tor}$ with respect to $\mathbf{\Delta}^1_2$ reducibility.

The same old same old ...

Lemma

The bi-embeddability relation $\equiv_p$ is not Borel reducible to the isomorphism relation $\cong_{tor}$.

Proof.

$D_\infty = \{ A \in \mathcal{A}_p \mid \mathrm{rk}(A^{\tau(A)}) = \omega \}$ is a complete analytic $\equiv_p$-class. □

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Lemma

The bi-embeddability relation $\equiv_{tor}$ is $\mathbf{\Delta}^1_2$ reducible to the isomorphism relation $\cong_{tor}$.

Proof.

There exists a $\mathbf{\Delta}^1_2$ map which selects an isomorphism class within each bi-embeddability class. □

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Theorem (*RC*)

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Proof.

Otherwise, we can pass to a suitable forcing extension

$$V[G] \models \omega_1^\omega < (2^\omega)^{<\omega_1}$$

and then apply absoluteness. □

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Theorem (*RC*)

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Proof.

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$$V[G] \models \omega_2^\omega < 2^{\omega_1}$$

and then apply absoluteness. □

Pinned names

Definition (Kanovei)

Let E be an analytic equivalence relation on the Polish space X and let $\mathbb{P}$ be a forcing notion. Then a $\mathbb{P}$-name τ is E-pinned if:

- $\Vdash_{\mathbb{P}} \ \tau \in X^{V^{\mathbb{P}}}$
- $\Vdash_{\mathbb{P}\times\mathbb{P}} \ \tau_{left} \ E^{V^{\mathbb{P}\times\mathbb{P}}} \ \tau_{right}$

Here τ_{left}, τ_{right} are the $(\mathbb{P}\times\mathbb{P})$-names such that if $G \times H$ is $(\mathbb{P}\times\mathbb{P})$-generic, then $\tau_{left}[G \times H] = \tau[G]$ and $\tau_{right}[G \times H] = \tau[H]$.

Pinned names

Example

- Let E_{cntble} be the Borel equivalence relation on $\mathbb{R}^\omega$ defined by

$$z \; E_{cntble} \; z' \iff \{z(n) \mid n \in \omega\} = \{z'(n) \mid n \in \omega\}.$$

Pinned names

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$$z \; E_{cntble} \; z' \quad \Longleftrightarrow \quad \{z(n) \mid n \in \omega\} = \{z'(n) \mid n \in \omega\}.$$

- Let $\mathbb{P}$ consist of all finite partial functions $p : \omega \to \mathbb{R}$.

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$$z \; E_{cntble} \; z' \iff \{z(n) \mid n \in \omega\} = \{z'(n) \mid n \in \omega\}.$$

- Let $\mathbb{P}$ consist of all finite partial functions $p : \omega \to \mathbb{R}$.
- Let $G \subseteq \mathbb{P}$ be generic and let $g = \bigcup G$.

Pinned names

Example

- Let E_{cntble} be the Borel equivalence relation on $\mathbb{R}^\omega$ defined by

$$z \; E_{cntble} \; z' \iff \{z(n) \mid n \in \omega\} = \{z'(n) \mid n \in \omega\}.$$

- Let $\mathbb{P}$ consist of all finite partial functions $p : \omega \to \mathbb{R}$.
- Let $G \subseteq \mathbb{P}$ be generic and let $g = \bigcup G$.
- If τ is the canonical $\mathbb{P}$-name of g, then τ is E_{cntble}-pinned.

Pinned names

Definition (Zapletal)

Let E be an analytic equivalence relation on the Polish space X and let $\mathbb{P}$ be a forcing notion. Then we can extend E to the class $X(\mathbb{P}, E)$ of E-pinned $\mathbb{P}$-names by defining

$$\sigma \; E \; \sigma' \quad \Longleftrightarrow \quad \Vdash_{\mathbb{P}\times\mathbb{P}} \; \sigma_{\,left} \; E \; \sigma'_{right}$$

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Theorem

If E, F are analytic equivalence relations and $E \leq_B F$, then $\lambda_{\mathbb{P}}(E) \leq \lambda_{\mathbb{P}}(F)$.

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Let $\lambda_{\mathbb{P}}(E)$ be the number of E-pinned $\mathbb{P}$-names up to E-equivalence.

Theorem

Suppose that κ is a Ramsey cardinal and that $|\mathbb{P}| < \kappa$. If E, F are analytic equivalence relations and E is $\mathbf{\Delta}^1_2$ reducible to F, then $\lambda_{\mathbb{P}}(E) \leq \lambda_{\mathbb{P}}(F)$.

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The End